Slides for Math 334, Fall 2008

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What follows is as close to a transcription as I could make of what I put on the board every day during the fall term of 2008. These notes are provided as a study aid. Beware of typos.

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Lecture 1
What are signals?

- A “signal” is really just a function, but we will use the word “signal” since it is suggestive of some physical meaning.
- In this course, we will consider only single-variable signals, i.e., signals of a single variable. We shall typically think of the single variable as being “time,” although there is no mathematical significance to this terminology. Multi-variable signals are also possible.
- In this course, signals will generally be considered to be real- or complex-valued. Signals taking values in more general spaces do arise.
- We shall use the symbol “$\mathbb{F}$” to stand for either “$\mathbb{R}$” or “$\mathbb{C}$”. Correspondingly, if $x \in \mathbb{F}$, then we denote

$$\bar{x} = \begin{cases} x, & \mathbb{F} = \mathbb{R}, \\ \text{complex conjugate of } x, & \mathbb{F} = \mathbb{C}, \end{cases} \qquad |x| = \begin{cases} \text{absolute value of } x, & \mathbb{F} = \mathbb{R}, \\ \text{complex modulus of } x, & \mathbb{F} = \mathbb{C}. \end{cases}$$

Why do we need structure for sets of signals?

- If signals are just functions, then why not just deal with functions and move on?
- If we are thinking about functions of time—as we are—then time can be thought of as being parameterised by $\mathbb{R}$, and so a general signal is simply a general function. As shall maybe become clear as we go along, general functions are simply too general to be able to do anything with.
- Maybe we can restrict to a useful class of signals like, say, continuous signals?
- Well, there are lots of interesting and useful signals that are not continuous, so this is too much of a restriction.
- Are there useful classes of signals that are not completely general, but not as restrictive as continuous signals?

Why do we need structure for sets of signals?

- We also need structure for signals based on the chores we wish to do with signals.

Example

Consider a sort of vague example. A commonly encountered situation is where we have recorded a signal, but our recording carries some noise. We would like to eliminate the noise. We can use some sort of technique for doing this, but there will exist no technique that will eliminate a general sort of noise from a general sort of signal. (Why do you think this is?) Therefore, to devise a scheme for eliminating noise from a signal, we should deal with signals with certain properties and noise with certain properties. What might these properties be?

Why do we need structure for sets of signals?

- Let us also consider the notion of a system (a notion to be defined precisely in Math 335) as a motivation for introducing structural properties of signals.
- A system is a "black box" taking as input an input signal and producing as output an output signal:

$$\text{inputs} \longrightarrow \boxed{\text{System}} \longrightarrow \text{outputs}$$

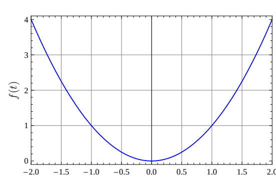
- What properties might a system possess?
  - It may be linear (as a map from the set of inputs to the set of outputs), in which case the space of inputs and space of outputs should have the structure of vector spaces. Some physical systems can be reasonably approximated by linear systems.
  - Systems are also often "continuous" in some way. For example, one may ask that if we have a converging sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ of inputs, the resulting sequence $(g_j)_{j\in\mathbb{Z}_{>0}}$ of outputs also converges. (You should recognise this as a form of continuity.)

Why do we need structure for sets of signals?

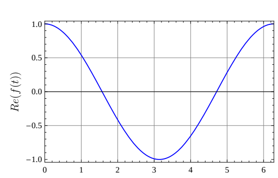
- Thus we see that we might need a vector space structure for sets of systems and also a means of saying that sequences of signals converge.
- The first part of the course will be devoted to exactly these sorts of structures.

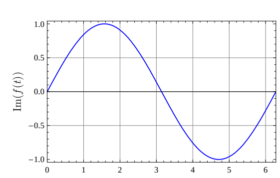
Representations of signals

- Given a $\mathbb{R}$-valued signal as a function of time, we can try to understand it by considering its graph which depicts how the values vary as a function of time.



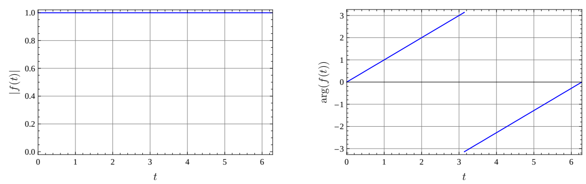
- If the signal is instead $\mathbb{C}$-valued, we may consider the graphs of its real and imaginary parts.





Representations of signals

- Sometimes one also plots the complex modulus and the complex argument to represent a $\mathbb{C}$-valued signal.



(What is the $\mathbb{C}$-valued signal being represented above?)

Representations of signals

- Is a time representation of a signal always best?
- This is a very deep question, actually. One way to think of the question a little more precisely is this: Given a space $\mathscr{T}$ of signals that are functions of time, is there a transformation $F\colon \mathscr{T} \to \mathscr{W}$ into some other space $\mathscr{W}$ which has useful properties?
- What are "useful" properties?
  - The map F should be such that, if $f \in \mathscr{T}$, then $F(f)$ has desirable attributes. This idea is vague and problem-dependent.
  - One should be able to recover the signal $f \in \mathscr{T}$ from its transformation $F(f) \in \mathscr{W}$.

Frequency-domain representations of signals

- In this course we shall study so-called “Fourier transforms.”
- A Fourier transform can be thought of as taking a time-domain representation of a signal and returning a frequency-domain representation of a signal.
- The idea of such a transform is that a highly oscillatory signal will have a frequency-domain representation with a lot of content at high frequencies and that a more “gentle” signal will have a frequency-domain representation with a lot of content at low frequencies.
- How does one measure the “frequency content” of a signal? Well, that is what we do in the second part of the course.

Reading for Lecture 1

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-1.1.1 and IV-1.1.7;
2. Sections IV-2.1, IV-2.2, IV-2.3, IV-2.4, and IV-2.5 (this material may be best read later in the course).

Lecture 2
Time

- An ***interval*** is a subset of $\mathbb{R}$ of one of the following forms:

$$[a,b], \quad [a,b), \quad (a,b], \quad (a,b),$$
$$(-\infty,b], \quad (-\infty,b), \quad [a,\infty), \quad (a,\infty), \quad (-\infty,\infty).$$

- Recall that $(\mathbb{R},+)$, the set of real numbers equipped with the operation of addition, is a group. We are interested in subgroups of $\mathbb{R}$. Some of these are:
  1. the set $\mathbb{Z}$ of integers;
  2. the set $\mathbb{Z}(\Delta) = \{k\Delta \mid k \in \mathbb{Z}\}$ of integer multiples of $\Delta \in \mathbb{R}_{>0}$;
  3. the set $\mathbb{Q}$ of rational numbers.
- We wish to allow time to be either continuous or discrete.
- Discrete time should be sampled at regular nonzero intervals, i.e., precluding the use of $\mathbb{Q}$.
- We wish to allow time to be bounded in either direction, and infinite in either or both directions.

Time

Definition (Time-domain)

A ***time-domain*** is a subset of $\mathbb{R}$ of the form $\mathbb{S} \cap I$ where $\mathbb{S} \subseteq \mathbb{R}$ is a subgroup in $(\mathbb{R},+)$ and $I \subseteq \mathbb{R}$ is an interval. A time-domain is

(i) ***continuous*** if $\mathbb{S} = \mathbb{R}$,

(ii) ***discrete*** if $\mathbb{S} = \mathbb{Z}(\Delta)$ for some $\Delta > 0$ called the ***sampling interval***,

(iii) ***finite*** if $\mathrm{cl}(I)$ is compact,

(iv) ***infinite*** if it is not finite,

(v) ***positively infinite*** if $\sup I = \infty$,

(vi) ***negatively infinite*** if $\inf I = -\infty$, and

(vii) ***totally infinite*** if $I = \mathbb{R}$.

- One can generalise this definition by using semigroups instead of subgroups. However, we will not use this generality in this course.

Operations on time-domains

- Sometimes one wishes to alter a time-domain.

Definition (Reparameterisation)

For time-domains $\mathbb{T}_1$ and $\mathbb{T}_2$, a ***reparameterisation*** of $\mathbb{T}_1$ to $\mathbb{T}_2$ is a bijection $\tau\colon \mathbb{T}_2 \to \mathbb{T}_1$ that is either monotonically increasing or monotonically decreasing.

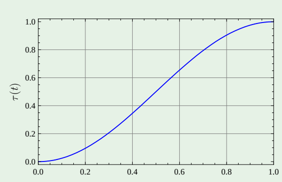
Examples (Reparameterisations)

1. For $a \in \mathbb{R}$, the ***shift*** of a time-domain $\mathbb{T}_1$ by a is defined by taking the time-domain
$$\mathbb{T}_2 = \{t + a \mid t \in \mathbb{T}\}$$
and the reparameterisation $\tau_a\colon \mathbb{T}_2 \to \mathbb{T}_1$ of $\mathbb{T}_1$ defined by $\tau_a(t) = t - a$.
2. For a time-domain $\mathbb{T}_1$, the ***transposition*** of $\mathbb{T}_1$ is defined by taking the time-domain
$$\mathbb{T}_2 = \{-t \mid t \in \mathbb{T}_1\}$$
and the reparameterisation $\sigma\colon \mathbb{T}_2 \to \mathbb{T}_1$ defined by $\sigma(t) = -t$. Often we will use the reparameterisation in the case when $\sigma(\mathbb{T}_1) = \mathbb{T}_1$.

Operations on time-domains

Examples (Reparameterisations (cont'd))

3. For a time-domain $\mathbb{T}_1$ and for $\lambda \in \mathbb{R}_{>0}$, the ***dilation*** of $\mathbb{T}_1$ by λ is defined by taking the time-domain
$$\mathbb{T}_2 = \{\lambda t \mid t \in \mathbb{T}_1\}$$
and the reparameterisation $\rho_\lambda\colon \mathbb{T}_2 \to \mathbb{T}_1$ defined by $\rho_\lambda(t) = \lambda^{-1} t$.
4. Here we take $\mathbb{T}_1 = \mathbb{T}_2 = [0, 1]$ and define a reparameterisation $\tau\colon \mathbb{T}_2 \to \mathbb{T}_1$ of $\mathbb{T}_1$ by $\tau(t) = \frac{1}{2}(1 - \cos(\pi t))$.



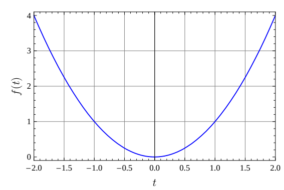
Time-domain signals

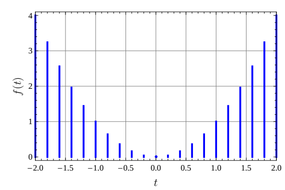
- A signal is simply an $\mathbb{F}$-valued function of time.

Definition (Time-domain signal)

Let $\mathbb{T} = \mathbb{S} \cap I$ be a time-domain and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. An ***$\mathbb{F}$-valued time-domain signal*** on $\mathbb{T}$ is a map $f\colon \mathbb{T} \to \mathbb{F}$. If $\mathbb{T}$ is continuous then f is a ***continuous-time*** signal and if $\mathbb{T}$ is discrete then f is a ***discrete-time*** signal.

- We represent signals by their graphs as follows:





Representation of continuous-time signal (left) and discrete-time signal (right)

Time-domain signals

Examples (Signals)

1. The signal
$$\mathsf{1}_{\geq 0}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$
is called the ***unit step signal*** and is a continuous-time signal defined on a totally infinite time-domain.
2. The signal
$$\mathsf{R}(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$
is called the ***unit ramp signal*** and again is a continuous-time signal defined on a totally infinite time-domain.
3. A ***binary data stream*** is a discrete-time signal defined on $\mathbb{T} = \mathbb{Z}$ and taking values in the set $\{0, 1\}$.

Time-domain signals

Examples (Signals)

4. Consider the special binary data stream $\mathsf{P}\colon \mathbb{Z} \to \{0,1\}$ defined by

$$\mathsf{P}(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

   This is called the ***unit pulse***.

5. On $[0,1)$ define a $\mathbb{R}$-valued signal g by

$$g(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ 0, & t \in (\frac{1}{2}, 1). \end{cases}$$

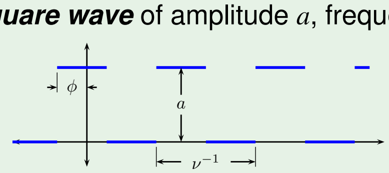
Time-domain signals

Examples (Signals (cont'd))

Now for $a,f \in \mathbb{R}_{>0}$ and $\phi \in \mathbb{R}$ define a signal

$$\square_{a,\nu,\phi}(t) = \sum_{n\in\mathbb{Z}} ag(\nu t + \phi),$$

which we call the ***square wave*** of amplitude a, frequency ν, and phase ϕ.



Note that as we have defined it, $\square_{a,\nu,\phi}$ is a continuous-time signal defined on a totally infinite time-domain.

Time-domain signals

Examples (Signals (cont'd))

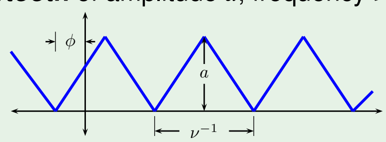
6. We proceed as in the preceding example, but now take

$$g(t) = \begin{cases} 2t, & t \in [0, \frac{1}{2}], \\ 2 - 2t, & t \in (\frac{1}{2}, 1), \end{cases}$$

and define

$$\triangle_{a,\nu,\phi}(t) = \sum_{n \in \mathbb{Z}} a g(\nu t + \phi),$$

which we call the ***sawtooth*** of amplitude a, frequency ν, and phase shift ϕ.



As with the square wave defined above, this is a continuous-time signal defined on a totally infinite time-domain.

Elementary operations on signals

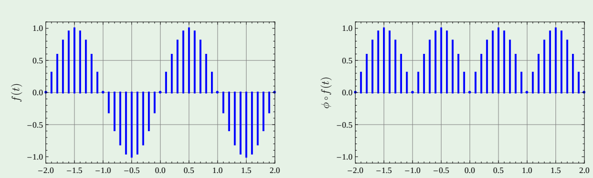
- There are various sorts of elementary ways of transforming a signal into another signal.

Definition (Codomain transformation)

If $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, if $\mathbb{T}$ is a time-domain, if $f : \mathbb{T} \to \mathbb{F}$ is a signal, and if $\phi : \mathbb{F} \to \mathbb{F}$ is a map, the ***codomain transformation*** of f by ϕ is the signal $\phi \circ f : \mathbb{T} \to \mathbb{F}$.

Examples (Codomain transformations)

1. We define $\phi : \mathbb{F} \to \mathbb{F}$ by $\phi(x) = \bar{x}$. Then the codomain transformed signal $\phi \circ f$ we denote by $\bar{f}$.
2. Let $\mathbb{F} = \mathbb{R}$ and define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(x) = |x|$. Then, for a signal $f : \mathbb{T} \to \mathbb{R}$ the codomain transformed signal $\phi \circ f$ is the ***full-wave rectification*** of f.



Elementary operations on signals

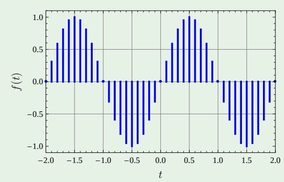
Examples (Codomain transformations (cont'd))

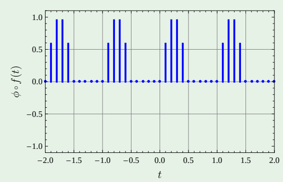
Of course the same ideas apply to continuous-time signals.

3. We again let $\mathbb{F} = \mathbb{R}$ and now we consider $\phi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} 0, & x < 0, \\ x, & x \geq 0. \end{cases}$$

In this case, for a signal $f\colon \mathbb{T} \to \mathbb{R}$ the codomain transformed signal $\phi \circ f$ is the ***half-wave rectification*** of f.





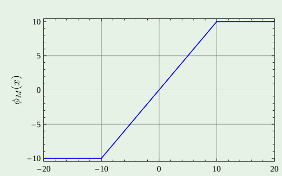
Elementary operations on signals

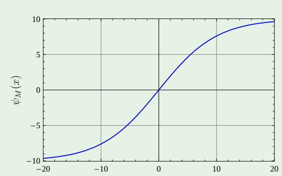
Examples (Codomain transformations (cont'd))

4. We take $\mathbb{F} = \mathbb{R}$ and for $M \in \mathbb{R}_{>0}$ consider the functions $\phi_M\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_M(x) = \begin{cases} x, & x \in [-M, M], \\ -M, & x < -M, \\ M, & x > M \end{cases}$$

and $\psi_M(x) = M \tanh(\frac{x}{M})$. We give the graphs of these functions below.

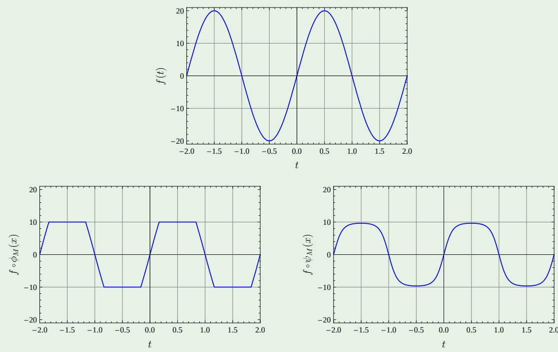




Elementary operations on signals

Examples (Codomain transformations (cont'd))

The idea of this codomain transformation is that it truncates the values of a signal to have a maximum absolute value of M. Such a codomain transformation is called a ***saturation function***. Sometimes it is advisable to use a smooth saturation function, and an example of one such is whose graph we show on the right above.

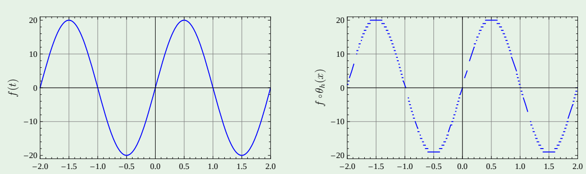


we show the two saturation functions applied to a continuous-time signal. Of course, one can as well apply the idea to a discrete-time signal.

Elementary operations on signals

Examples (Codomain transformations (cont'd))

5. Particularly in our world where almost everything is managed by digital computers, signals with continuous values are not often what one deals with in practice. Instead, what one actually has at hand is a signal whose values live in a discrete set. Thus one would like to convert a signal with continuous values to one with discrete values. This general process is known as ***quantisation***. A simple way to quantise a signal is via the codomain transformation $\theta_h \colon \mathbb{R} \to \mathbb{R}$ defined by $\theta_h(x) = h\lceil \frac{x}{h} \rceil$, where we recall the definition of the ceiling function $x \mapsto \lceil x \rceil$ as giving the largest integer less than or equal to x. The graph of the function is depicted in Figure I-2.1. The quantisation θ_h is called the ***uniform h-quantisation***.



we depict the uniform quantisation of a continuous-time signal. The same idea applies, and indeed is more natural, for discrete-time signals.

Elementary operations on signals

Definition (Domain transformation)

If $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, if $\mathbb{T}_1$ and $\mathbb{T}_2$ are time-domains, if $f\colon \mathbb{T}_1 \to \mathbb{F}$ is a signal, and if $\tau\colon \mathbb{T}_2 \to \mathbb{T}_1$ is a reparameterisation of $\mathbb{T}_1$, the ***domain transformation*** of f by τ is the signal $\tau^* f\colon \mathbb{T}_2 \to \mathbb{F}$ defined by $\tau^* f(t) = f \circ \tau(t)$.

Examples (Domain transformations)

1. For $a \in \mathbb{R}$ let us consider the shift $\tau_a\colon \mathbb{T}_2 \to \mathbb{T}_1$ of $\mathbb{T}_1$. For a signal $f\colon \mathbb{T}_1 \to \mathbb{F}$, the corresponding domain transformed signal is defined by $\tau_a^* f(t) = f(t-a)$ for every $t \in \mathbb{T}_2$.
2. Let us consider the transposition $\sigma\colon \mathbb{T}_2 \to \mathbb{T}_1$. For a signal $f\colon \mathbb{T}_1 \to \mathbb{F}$, the corresponding domain transformed signal is defined by $\sigma^* f(t) = f(-t)$ for every $t \in \mathbb{T}_2$.
3. For $\lambda \in \mathbb{R}_{>0}$, let us consider the dilation $\rho_\lambda\colon \mathbb{T}_2 \to \mathbb{T}_1$. For a signal $f\colon \mathbb{T}_1 \to \mathbb{F}$, the corresponding domain transformed signal is defined by $\rho_\lambda^* f(t) = f(\lambda^{-1} t)$.

Elementary operations on signals

- There are other interesting, but rather trivial, operations one can perform on signals.
  1. *Sampling:* This operation converts a continuous-time signal to a discrete-time signal in the more or less obvious way: At the times lying in the discrete time-domain, one takes the value of the sampled signal to be the corresponding value of the continuous-time signal.
  2. *Interpolation:* This is the "reverse" operation of sampling, taking a discrete-time signal and returning a continuous-time signal. There is no unique way to do this.
- Please see Section IV-1.1.4 of the course text for details of these straightforward concepts.

Reading for Lecture 2

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-1.1.2, IV-1.1.3, and IV-1.1.4.

Lecture 3
Elementary signal classification

- After we have finished with our elementary discussion of signals, we shall devote significant effort to classification of signals corresponding to their analytical properties. For now, we have some fairly elementary signal properties.
  1. *Periodic:* There exists $T \in \mathbb{R}_{>0}$ such that $f(t+T) = f(t)$ for every $t \in \mathbb{T}$. We say f is ***T-periodic*** in this case.
  2. *Harmonic:* A signal $f\colon \mathbb{T} \to \mathbb{F}$ is ***harmonic*** with ***frequency*** $\nu \in \mathbb{R}_{>0}$, ***amplitude*** $a \in \mathbb{R}_{>0}$, and ***phase*** $\phi \in \mathbb{R}$ if

$$f(t) = \begin{cases} a\mathrm{e}^{\mathrm{i}(2\pi\nu t+\phi)}, & \mathbb{F} = \mathbb{C}, \\ a\cos(2\pi\nu t + \phi), & \mathbb{F} = \mathbb{R}, \end{cases}$$

for all $t \in \mathbb{T}$. The ***angular frequency*** for the harmonic signal is $\omega = 2\pi\nu$.

Signal structure

- After the fairly elementary constructions and definitions of the preceding lecture, we now begin to do some more or less serious mathematics.
- Our objective is to understand the possible structures for sets of signals. We shall see that a typical signal space for us will be an “infinite-dimensional normed vector space.” Our objective now is to understand the meaning of this.
- We shall study:
  1. the algebraic structure of infinite-dimensional vector spaces;
  2. the analytical (okay, topological is really the right word) structure of normed vector spaces.

Vector spaces

- It is assumed you know what a vector space is: It is a set with two operations, addition and scalar multiplication, satisfying a list of natural associativity and distributivity axioms; see Section I-4.5.1 in the course text if you need a review of the definition.
- In this course we will talk about $\mathbb{F}$-vector spaces, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Examples (Vector spaces)

1. The prototypical n-dimensional $\mathbb{F}$-vector space is the set $\mathbb{F}^n$ of n-tuples of numbers in $\mathbb{F}$. The vector space structure is

$$\begin{aligned}(u_1, \dots, u_n) + (v_1, \dots, v_n) &= (u_1 + v_1, \dots, u_n + v_n),\\ a(u_1, \dots, u_n) &= (au_1, \dots, au_n).\end{aligned}$$

Vector spaces

Examples (Vector spaces (cont'd))

2. Let $C^0([0,1];\mathbb{F})$ denote the set of continuous $\mathbb{F}$-valued functions on the interval $[0,1]$. The vector space structure is defined by

$$(f+g)(x) = f(x) + g(x), \quad (af)(x) = a(f(x)).$$

Make sure you understand that this is a definition. Note that these operations make sense since the sum of continuous functions is continuous and a multiple of a continuous function is a continuous function.

3. A good example of a vector space for us, and an example that may well be new, is the following. Let J be an arbitrary set. Denote by $\mathbb{F}^J$ the set of maps $\phi\colon J \to \mathbb{F}$. Define a vector space structure on $\mathbb{F}^J$ by

$$(\phi_1 + \phi_2)(j) = \phi_1(j) + \phi_2(j), \quad (a\phi)(j) = a(\phi(j)).$$

Again, be sure you understand that these are actually definitions.

Vector spaces

Examples (Vector spaces (cont'd))

4. To better understand the preceding example, let us consider $J = \{1,\dots,n\}$. Note that if $\phi\colon \{1,\dots,n\} \to \mathbb{F}$ is an element of $\mathbb{F}^{\{1,\dots,n\}}$ then ϕ is defined by the n-tuple

$$(\phi(1),\dots,\phi(n)).$$

Thus we have a natural map

$$\mathbb{F}^{\{1,\dots,n\}} \ni \phi \mapsto (\phi(1),\dots,\phi(n)) \in \mathbb{F}^n,$$

and this map is easily verified to be linear and a bijection, i.e., an isomorphism of $\mathbb{F}$-vector spaces.

5. In like manner, if $J = \mathbb{Z}_{>0}$ then $\mathbb{F}^{\mathbb{Z}_{>0}}$ is to be thought of as the set of sequences in $\mathbb{F}$. Indeed, if $\phi\colon \mathbb{Z}_{>0} \to \mathbb{F}$ is in $\mathbb{F}^{\mathbb{Z}_{>0}}$, then we associate to this the sequence $(\phi(j))_{j\in\mathbb{Z}_{>0}}$. This assignment is easily seen to be an isomorphism as well.
It is important to understand this example, although it is sort of trivial.

Vector spaces

Examples (Vector spaces (cont'd))

6. We denote by $\mathbb{F}_0^J$ the subset of $\mathbb{F}^J$ consisting of those maps $\phi\colon J \to \mathbb{F}$ such that the set $\{j \in J \mid \phi(j) \neq 0\}$ is finite. It is readily seen that $\mathbb{F}_0^J$ is a subspace of $\mathbb{F}^J$; see Section I-4.5.2 in the course notes for a discussion of subspaces.
7. Note that $\mathbb{F}_0^{\{1,\dots,n\}} = \mathbb{F}^{\{1,\dots,n\}}$. More generally, and obviously, if and only if J is finite we have $\mathbb{F}_0^J = \mathbb{F}^J$.
8. Note that if we think of $\mathbb{F}^{\mathbb{Z}_{>0}}$ as being the set of sequences, as above, then $\mathbb{F}_0^{\mathbb{Z}_{>0}}$ is the set of sequences that are eventually zero.

Linear independence

- It is assumed that you know about linear independence for finite-dimensional vector spaces. For general vector spaces the discussion is not too much different.

Definition (Linear independence)

Let V be an $\mathbb{F}$-vector space.

(i) A finite set $\{v_1, \dots, v_k\}$ of vectors in V is ***linearly independent*** if the equality

$$c_1 v_1 + \cdots + c_k v_k = 0_{\mathsf{V}}, \qquad c_1, \dots, c_k \in \mathbb{F},$$

is satisfied only if $c_1 = \cdots = c_k = 0$.

(ii) A nonempty subset $S \subseteq$ V is ***linearly independent*** if every nonempty finite subset of S is linearly independent.

(iii) A nonempty subset $S \subseteq$ V is ***linearly dependent*** if it is not linearly independent.

Bases

- It is assumed that you know about bases for finite-dimensional vector spaces.

Definition (Span)

Let V be an $\mathbb{F}$-vector space and let $S \subseteq$ V.

(i) A ***linear combination*** from S is a vector of the form

$$a_1 v_1 + \cdots + a_k v_k,$$

for $k \in \mathbb{Z}_{>0}$, $a_1, \ldots, a_k \in \mathbb{F}$, and $v_1, \ldots, v_k \in S$.

(ii) The set of linear combinations from S is the ***span*** of S and is denoted by $\mathrm{span}_{\mathbb{F}}(S)$.

- Note that elements of $\mathrm{span}_{\mathbb{F}}(S)$ are *finite* linear combinations of vectors from S. Indeed, infinite linear combinations need care in their interpretation. This subtlety might confuse you later in the course when we *do* consider infinite sums of vectors.
- The set $\mathrm{span}_{\mathbb{F}}(S)$ is a subspace, and S is a subspace if and only if $\mathrm{span}_{\mathbb{F}}(S) = S$.

Bases

Definition (Basis)

Let V be a $\mathbb{F}$-vector space. A subset $\mathscr{B} \subseteq$ V is a ***basis*** for V if

(i) $\mathscr{B}$ is linearly independent and

(ii) $\mathrm{span}_{\mathbb{F}}(\mathscr{B}) =$ V.

- This sort of basis is often specifically called a ***Hamel basis***.

Reading for Lecture 3

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-1.1.6 and IV-1.1.7;
2. Section I-4.5.1 (especially the first part);
3. Section I-4.5.2 (especially the first part);
4. Section I-4.5.3 (especially the first part);
5. Section I-4.5.4 (especially the first part).

Lecture 4
Bases

Examples (Bases)

1. Take $\mathsf{V} = \mathbb{F}^n$ and, for $j \in \{1, \dots, n\}$, define $\boldsymbol{e}_j \in \mathbb{F}^n$ by

$$\boldsymbol{e}_j = \underbrace{(0, \dots, 1, \dots, 0)}_{1 \text{ in } j\text{th-position}}.$$

We claim that $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is a basis. To prove linear independence:

$$\begin{aligned} & c_1\boldsymbol{e}_1 + \cdots + c_n\boldsymbol{e}_n = \boldsymbol{0} \\ \implies \quad & (c_1, \dots, c_n) = (0, \dots, 0) \\ \implies \quad & c_1 = \cdots = c_n = 0, \end{aligned}$$

giving linear independence. To prove that $\operatorname{span}_{\mathbb{F}}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n) = \mathbb{F}^n$, let $\boldsymbol{v} \in \mathbb{F}^n$ and note that

$$(v_1, \dots, v_n) = v_1\boldsymbol{e}_1 + \cdots + v_n\boldsymbol{e}_n.$$

Thus $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is indeed a basis.

Bases

Examples (Bases (cont'd))

2. Let us redo the preceding example, but now with $\mathsf{V} = \mathbb{F}^{\{1,\dots,n\}}$. Here, for $j \in \{1,\dots,n\}$ we define $\boldsymbol{e}_j \in \mathbb{F}^{\{1,\dots,n\}}$ by

$$\boldsymbol{e}_j(k) = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

We claim that $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\}$ is a basis. To show linear independence:

$$\begin{aligned} & c_1\boldsymbol{e}_1 + \cdots + c_n\boldsymbol{e}_n = \boldsymbol{0} \\ \implies \quad & (c_1\boldsymbol{e}_1 + \cdots + c_n\boldsymbol{e}_n)(j) = 0, \qquad j \in \{1,\dots,n\} \\ \implies \quad & c_1\boldsymbol{e}_1(j) + \cdots + c_n\boldsymbol{e}_n(j) = 0, \qquad j \in \{1,\dots,n\} \\ \implies \quad & c_j = 0, \qquad j \in \{1,\dots,n\}. \end{aligned}$$

To prove that $\operatorname{span}_{\mathbb{F}}(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n) = \mathbb{F}^{\{1,\dots,n\}}$, let $\phi \in \mathbb{F}^{\{1,\dots,n\}}$ and note that, for any $j \in \{1,\dots,n\}$,

$$\phi(j) = \phi(1)\boldsymbol{e}_1(j) + \cdots + \phi(n)\boldsymbol{e}_n(j) = (\phi(1)\boldsymbol{e}_1 + \cdots + \phi(n)\boldsymbol{e}_n)(j),$$

and so $\phi = \phi(1)\boldsymbol{e}_1 + \cdots + \phi(n)\boldsymbol{e}_n$. Thus $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\}$ is a basis.

Bases

Examples (Bases (cont'd))

3. Now we let J be an arbitrary infinite set and consider $\mathsf{V} = \mathbb{F}^J$. For $j \in J$ define $\boldsymbol{e}_j \in \mathbb{F}^J$ by

$$\boldsymbol{e}_j(k) = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

We claim that $\mathscr{B} = \{\boldsymbol{e}_j \mid j \in J\}$ is linearly independent. We must show that any *finite* subset of $\mathscr{B}$ is linearly independent. Thus let $j_1,\dots,j_n \in J$ and note that

$$\begin{aligned} & c_1\boldsymbol{e}_{j_1} + \cdots + c_n\boldsymbol{e}_{j_n} = 0 \\ \implies \quad & (c_1\boldsymbol{e}_{j_1} + \cdots + c_n\boldsymbol{e}_{j_n})(j) = 0, \qquad j \in J \\ \implies \quad & c_1\boldsymbol{e}_{j_1}(j) + \cdots + c_n\boldsymbol{e}_{j_n}(j) = 0, \qquad j \in J \\ \implies \quad & c_k = 0, \qquad k \in \{1,\dots,n\}, \end{aligned}$$

giving linear independence. Is $\mathscr{B}$ a basis? No. Consider $\phi \in \mathbb{F}^J$ defined by $\phi(j) = 1, j \in J$. Then ϕ cannot be a *finite* linear combination of elements from $\mathscr{B}$.

Bases

Examples (Bases (cont'd))

4. Let us now restrict the preceding example by considering not $\mathbb{F}^J$, but $\mathbb{F}_0^J$. Let $\mathscr{B}$ be defined as in the preceding example; clearly $\mathscr{B} \subseteq \mathbb{F}_0^J$. As we have already shown that $\mathscr{B}$ is linearly independent, let us show that $\operatorname{span}_{\mathbb{F}}(\mathscr{B}) = \mathbb{F}_0^J$. Let $\phi \in \mathbb{F}_0^J$. By definition, there exists a finite distinct set $\{j_1, \dots, j_k\} \subseteq J$ such that $\{j \in J \mid \phi(j) \neq 0\} = \{j_1, \dots, j_k\}$. Then, for $j \in J$,

$$\phi(j) = (\phi(j_1)\boldsymbol{e}_{j_1}(j) + \cdots + \phi(j_k)\boldsymbol{e}_{j_k}(j) = (\phi(j_1)\boldsymbol{e}_{j_1} + \cdots + \phi(j_k)\boldsymbol{e}_{j_k})(j).$$

Thus $\phi = \phi(j_1)\boldsymbol{e}_{j_1} + \cdots + \phi(j_k)\boldsymbol{e}_{j_k} \in \operatorname{span}_{\mathbb{F}}(\mathscr{B})$. Thus $\mathscr{B}$ is indeed a basis.

Reading for Lecture 4

Material related to this lecture can be found in the following sections of the course notes:

1. Section I-4.5.4 (especially the first part).

Lecture 5
Bases

- Interesting factoids that are not part of this course.
  1. By our definition of a basis, every vector space has a basis. You know this for finite-dimensional vector spaces, but it is true in general.
  2. If V is a $\mathbb{F}$-vector space, there is a set J such that V is isomorphic to $\mathbb{F}_0^J$. This is analogous to the fact that every finite-dimensional $\mathbb{F}$-vector space is isomorphic to $\mathbb{F}^n$ for some suitable n.

Dimension

- It is helpful for a second or two to have at hand the notion of the cardinality of a set.[1] For a set S, let us denote by $\mathrm{card}(S)$, the ***cardinality*** of S, the equivalence class of sets T such that there is a bijection from S to T. One should think of $\mathrm{card}(S)$ as a "number" measuring the "size" of the set S. If S is a finite set, $S = \{x_1, \dots, x_n\}$, then one trivially identifies $\mathrm{card}(S)$ with the number n. If S is countable, then $\mathrm{card}(S) = \mathrm{card}(\mathbb{Z}_{>0})$.

Definition (Dimension)

If V is a $\mathbb{F}$-vector space with basis $\mathscr{B}$, the ***dimension*** of V is $\dim_{\mathbb{F}}(\mathsf{V}) = \mathrm{card}(\mathscr{B})$.

- Some factoids about dimension.
  1. It turns out that two bases always have the same cardinality. You know this already for finite-dimensional vector spaces, but it is true in general. Thus dimension is well-defined.
  2. If V is n-dimensional according to the definition of dimension with which you are already familiar, then according to the definition above, $\dim_{\mathbb{F}}(\mathsf{V}) = n$. Thus our definition above extends the one with which you are already familiar.

[1] The ensuing words are flawed, but will suffice for us for the moment.

Dimension

Examples (Dimension)

1. $\dim_{\mathbb{F}}(\mathbb{F}^n) = n$.
2. $\dim_{\mathbb{F}}(\mathbb{F}^{\{1,\dots,n\}}) = n$.
3. $\dim_{\mathbb{F}}(\mathbb{F}_0^J) = \mathrm{card}(J)$.
4. If J is a finite set then $\dim_{\mathbb{F}}(\mathbb{F}^J) = \mathrm{card}(J)$.
5. If J is not finite then $\dim_{\mathbb{F}}(\mathbb{F}^J)$ can be shown to be equal to $2^{\mathrm{card}(J)}$, whatever that is.[a]

[a]Statements like "whatever that is" are intended to tip you off that the corresponding discussion is not part of the course.

- *Summary:* We shall not make great use of the general notion of dimension. Mainly, we shall be interested in the case when $\mathrm{card}(\mathscr{B})$ is finite, in which case we say that V is ***finite-dimensional***, and the case where $\mathrm{card}(\mathscr{B})$ is not finite, in which case we shall say that V is ***infinite-dimensional***. Differing flavours in infinity will not be so much of a concern.

Discrete-time signals

- With a little linear algebra at hand, we now say some things about signals. It is easiest to start with discrete-time signals.
- Let us begin with the most general set of signals defined on a discrete time-domain.

Definition ($\mathbb{F}^{\mathbb{T}}$)

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{T}$ be a discrete time-domain. We denote by $\mathbb{F}^{\mathbb{T}}$ the set of maps $f: \mathbb{T} \to \mathbb{F}$. The $\mathbb{F}$-vector space structure on $\mathbb{F}^{\mathbb{T}}$ is given by

$$(f_1 + f_2)(t) = f_1(t) + f_2(t), \quad (\alpha f)(t) = \alpha(f(t)),$$

for $f, f_1, f_2 \in \mathbb{F}^{\mathbb{T}}$ and for $\alpha \in \mathbb{F}$.

Discrete-time signals

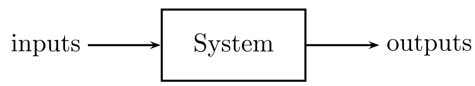
Proposition (Dimension of $\mathbb{F}^{\mathbb{T}}$)

If $\mathbb{T}$ is a discrete time-domain, then $\mathbb{F}^{\mathbb{T}}$ is finite-dimensional if and only if $\mathbb{T}$ is finite.

Proof.

Homework. □

- There is not much more that we can say algebraically.
- What about other signal structure? The following issue is a standard one in system theory. Consider a system:



The system is ***bounded-input, bounded-output stable*** if every bounded input produces a bounded output. What might a "bounded" signal be?

Normed vector spaces

- To talk about the "size" of an element in a vector space, particularly a vector space of signals, one can use the notion of a norm.

Definition (Norm)

Let V be a $\mathbb{F}$-vector space. A ***norm*** on V assigns to each vector $v \in \mathsf{V}$ the number $\|v\| \in \mathbb{R}_{\geq 0}$, and the assignment satisfies the following rules:

(i) $\|av\| = |a|\|v\|$ for $a \in \mathbb{F}$ and $v \in \mathsf{V}$ (***homogeneity***);

(ii) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ for $v_1, v_2 \in \mathsf{V}$ (***triangle inequality***);

(iii) $\|v\| = 0$ only if $v = 0_{\mathsf{V}}$ (***positive-definiteness***).

- In the course text, the notion of a seminorm is also defined, this by omitting the property of positive-definiteness. We shall only briefly need the notion of a seminorm, so will not dwell on properties of norms that are or are not true for seminorms.

Normed vector spaces

Examples (Norms)

1. On $\mathbb{F}^n$ define
$$\|\boldsymbol{v}\|_2 = \left(|v_1|^2 + \cdots + |v_n|^2\right)^{1/2}.$$
This norm defines the usual notion of length of a vector in $\mathbb{F}^n$, i.e., $\|\boldsymbol{v}\|$ is the distance from $\mathbf{0}_{\mathbb{F}^n}$ to $\boldsymbol{v}$. We shall also sometimes call it the ***2-norm*** on $\mathbb{F}^n$ or the ***standard norm***. It is pretty evident that $\|\cdot\|_2$ satisfies the homogeneity and positive-definiteness properties required of a norm. It is also true that $\|\cdot\|_2$ satisfies the triangle inequality. The proof of this relies on the so-called "Cauchy–Bunyakovsky–Schwarz Inequality." This inequality holds because $\|\cdot\|_2$ is the norm derived from an inner product on $\mathbb{F}^n$. Thus we shall see how $\|\cdot\|_2$ satisfies the triangle inequality when we discuss inner products.
2. Let us consider another norm on $\mathbb{F}^n$ which differs from the standard norm. For $\boldsymbol{v} = (v_1, \dots, v_n) \in \mathbb{F}^n$ define
$$\|\boldsymbol{v}\|_1 = |v_1| + \cdots + |v_n|.$$

Normed vector spaces

Examples (Norms (cont'd))

All properties of the norm are readily verified, including the triangle inequality, as this now follows from the triangle inequality for $|\cdot|$. This norm is called the ***1-norm***.

3. Let us consider another norm on $\mathbb{F}^n$ given by
$$\|\boldsymbol{v}\|_\infty = \max\{|v_j| \mid j \in \{1, \dots, n\}\}.$$
This is in fact a norm, called the ***∞-norm***. The only not entirely trivial norm property to verify is the triangle inequality. For this, let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{F}^n$ and let $j, k, \ell \in \{1, \dots, n\}$ have the property that $\|\boldsymbol{u}\|_\infty = |u_j|$, $\|\boldsymbol{v}\|_\infty = |v_k|$, and $\|\boldsymbol{u} + \boldsymbol{v}\|_\infty = |u_\ell + v_\ell|$. We then have
$$\|\boldsymbol{u} + \boldsymbol{v}\|_\infty = |u_\ell + v_\ell| \leq |u_\ell| + |v_\ell| \leq |u_j| + |v_k| = \|\boldsymbol{u}\|_\infty + \|\boldsymbol{v}\|_\infty.$$

Reading for Lecture 5

Material related to this lecture can be found in the following sections of the course notes:

1. Section I-4.5.4 (especially the first part);
2. Section IV-1.1.7;
3. Section IV-1.2.1;
4. Section III-3.1.1 (the definition of norm and norm examples); you can replace every occurrence of "seminorm" with "norm," for the purposes of this course.

Lecture 6
Normed vector spaces

Examples (Norms (cont'd))

4. Let us abbreviate $\mathbb{F}^{\mathbb{Z}_{>0}}$ by $\mathbb{F}^\infty$. Recall then that $\mathbb{F}_0^\infty$ (essentially) denotes the sequences (i.e., maps with domain $\mathbb{Z}_{>0}$) $(v_j)_{j\in\mathbb{Z}_{>0}}$ for which the set $\{j \in \mathbb{Z}_{>0} \mid v_j \neq 0\}$ is finite. Thus sequences in $\mathbb{F}_0^\infty$ are eventually zero. We define
$$\|(v_j)_{j\in\mathbb{Z}_{>0}}\|_2 = \left(\sum_{j=1}^{\infty}|v_j|^2\right)^{1/2},$$
noting that the sum makes sense since it is actually finite. That $\|\cdot\|_2$ satisfies the properties of a norm is straightforward, using the triangle inequality for the 2-norm on $\mathbb{F}^n$. This norm is called the ***2-norm*** on $\mathbb{F}_0^\infty$.
5. We again consider the vector space $\mathbb{F}_0^\infty$ and now define
$$\|(v_j)_{j\in\mathbb{Z}_{>0}}\|_1 = \sum_{j=1}^{\infty}|v_j|,$$

Normed vector spaces

Examples (Norms (cont'd))

this sum again making sense since it is finite. It is easy to verify, just as we did for the 2-norm above, that $\|\cdot\|_1$ is a norm, and we call it the $\mathbf{1}$***-norm***.

6. As a final norm on $\mathbb{F}_0^\infty$ we define

$$\|(v_j)_{j\in\mathbb{Z}_{>0}}\|_\infty = \sup\{|v_j| \mid j \in \mathbb{Z}_{>0}\}.$$

Because the sequence $(v_j)_{j\in\mathbb{Z}_{>0}}$ is finite, it is certainly bounded, and so the definition makes sense. Moreover, the norm properties follow, essentially from those of $\|\cdot\|_\infty$ on $\mathbb{F}^n$. This norm we call, of course, the ∞***-norm***.

7. We consider the $\mathbb{F}$-vector space $\mathsf{C}^0([a,b];\mathbb{F})$ of continuous $\mathbb{F}$-valued functions on the compact interval $[a,b]$. Provided that $b > a$ this is an infinite-dimensional vector space, cf. Example I-4.5.18–6. On this vector space we define

$$\|f\|_2 = \left(\int_a^b |f(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

Normed vector spaces

Examples (Norms (cont'd))

Note that continuous functions (and therefore their squares) on compact intervals are always Riemann integrable by Corollary I-3.4.12. It is easy to see that this possible norm satisfies the homogeneity and positive-definiteness properties of a norm (see Exercise I-3.4.1 for positive-definiteness). Thus, like its 2-norm brothers on $\mathbb{F}^n$ and $\mathbb{F}_0^\infty$, the difficult norm property to verify is the triangle inequality. We shall do this subsequently. This norm will be called the $\mathbf{2}$***-norm*** on $\mathsf{C}^0([a,b];\mathbb{F})$.

8. On $\mathsf{C}^0([a,b];\mathbb{F})$ define

$$\|f\|_1 = \int_a^b |f(x)|\,\mathrm{d}x.$$

Again, the integral here is the Riemann integral. The three norm properties are easily verified. Only the triangle inequality is possibly nontrivial:

Normed vector spaces

Examples (Norms (cont'd))

$$\begin{aligned}\|f+g\|_1 &= \int_a^b |f(x)+g(x)|\,\mathrm{d}x \le \int_a^b \big(|f(x)|+|g(x)|\big)\,\mathrm{d}x \\ &= \int_a^b |f(x)|\,\mathrm{d}x + \int_a^b |g(x)|\,\mathrm{d}x = \|f\|_1 + \|g\|_1.\end{aligned}$$

This norm, called the ***1-norm***, is different than the 2-norm.

9. As a final norm on $\mathsf{C}^0([a,b],\mathbb{F})$ we take

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in [a,b]\}.$$

Again, the triangle inequality is the troublesome property to verify. In this case the verification goes as follows:

Normed vector spaces

Examples (Norms (cont'd))

$$\begin{aligned}\|f+g\|_\infty &= \sup\{|f(x)+g(x)| \mid x \in [0,1]\} \\ &\le \sup\{|f(x)|+|g(x)| \mid x \in [0,1]\} \\ &\le \sup\{|f(x)|+|g(y)| \mid x,y \in [0,1]\} \\ &\le \sup\{|f(x)| \mid x \in [0,1]\} + \sup\{|g(y)| \mid y \in [0,1]\} \\ &= \|f\|_\infty + \|g\|_\infty.\end{aligned}$$

This norm is yet again different than the 1- and 2-norms.

- The pair $(\mathsf{V}, \|\cdot\|)$, where V is an $\mathbb{F}$-vector space and $\|\cdot\|$ is a norm on V, is called a ***normed $\mathbb{F}$-vector space***.

Normed vector spaces

- We have introduced three norms, $\|\cdot\|_2$, $\|\cdot\|_1$, and $\|\cdot\|_\infty$, for $\mathbb{F}^n$. These norms are different in that they will generally yield a different number for the norm of the same vector. However, it is a fact that different norms on finite-dimensional vector spaces are ***equivalent*** in a sense that is defined in Definition III-3.1.13. You will explore this in a homework problem.
- Unlike in the finite-dimensional case, the three norms, still denoted by $\|\cdot\|_2$, $\|\cdot\|_1$, and $\|\cdot\|_\infty$, for $\mathbb{F}_0^\infty$ and $\mathrm{C}^0([a,b];\mathbb{F})$ are genuinely different. In terms of Definition III-3.1.13, they are not equivalent. This is sort of subtle, and will not be really comprehensible for us until we discuss completeness, where we will see that the completions of $\mathbb{F}_0^\infty$ and $\mathrm{C}^0([a,b];\mathbb{F})$ are different for the three norms.

Completeness

- We now come to a difficult to understand concept whose importance cannot be overstated. Many of the important concepts that follow subsequently in the course have as their basis the notion of "completeness" which we now discuss.
- The following definitions should be familiar to you from your real analysis course, although our presentation here is more general.

Definition (Convergent sequence, Cauchy sequence)

Let $(\mathsf{V},\|\cdot\|)$ be a normed $\mathbb{F}$-vector space. A sequence $(v_j)_{j\in\mathbb{Z}_{>0}}$

(i) ***converges*** to $v_0 \in \mathsf{V}$ if, for every $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that $\|v_0 - v_j\| < \epsilon$ for every $j \geq N$;

(ii) is a ***Cauchy sequence*** if, for every $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that $\|v_j - v_k\| < \epsilon$ for every $j, k \geq N$.

Completeness

Proposition

Convergent sequences are Cauchy.

Proof.

Let $(v_j)_{j\in\mathbb{Z}_{>0}}$ converge to $v_0 \in \mathsf{V}$. Let $\epsilon \in \mathbb{R}_{>0}$ and let $N \in \mathbb{Z}_{>0}$ be such that $\|v_0 - v_j\| < \frac{\epsilon}{2}$ for all $j \geq N$. Then, for $j, k \geq N$,

$$\|v_j - v_k\| = \|v_j - v_0 + v_0 - v_k\| \leq \|v_0 - v_j\| + \|v_0 - v_k\| < \epsilon,$$

as desired. □

- Fact: Cauchy sequences in finite-dimensional normed vector spaces converge. I expect you have seen this for the normed vector space $(\mathbb{R}, |\cdot|)$.

Reading for Lecture 6

Material related to this lecture can be found in the following sections of the course notes:

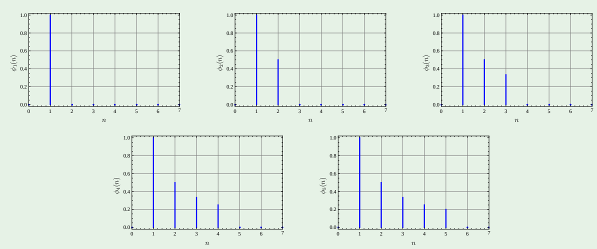
1. Section III-3.1.1 (the definition of norm and norm examples); you can replace every occurrence of “seminorm” with “norm,” for the purposes of this course.
2. Sections III-3.2.1 and III-3.3.1.

Lecture 7
Completeness

Example (Cauchy sequences need not converge)

In this example we shall construct a nonconvergent Cauchy sequence. The normed vector space must necessarily be infinite-dimensional. We shall use $(\mathbb{F}_0^\infty, \|\cdot\|_\infty)$. It is convenient to think now of elements of $\mathbb{F}_0^\infty$ as being maps $\phi\colon \mathbb{Z}_{>0} \to \mathbb{F}$ such that $\{j \in \mathbb{Z}_{>0} \mid \phi(j) \neq 0\}$ is finite. As we have seen, this is the same as the set of sequences in $\mathbb{F}$. We consider the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{F}_0^\infty$ defined by

$$\phi_j(n) = \begin{cases} \frac{1}{n}, & n \leq j, \\ 0, & n > j. \end{cases}$$



Completeness

Example (cont'd)

Claim

The sequence is Cauchy.

Proof.

Let $\epsilon \in \mathbb{R}_{>0}$. Take $N \in \mathbb{Z}_{>0}$ such that $\frac{1}{N} < \epsilon$ and let $j, k \geq N$ with $k < j$. Then

$$(\phi_j - \phi_k)(n) = \begin{cases} \frac{1}{n}, & n \in \{k+1, \dots, j\}, \\ 0, & \text{otherwise}. \end{cases}$$

It then immediately follows that

$$\|\phi_j - \phi_k\|_\infty = \sup\{|(\phi_j - \phi_k)(n)| \mid n \in \mathbb{Z}_{>0}\} < \epsilon,$$

giving the claim. □

Completeness

Example (cont'd)

Claim

The sequence does not converge.

Proof.

We claim that if the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to ϕ, then $\phi(n)=\frac{1}{n}$ for every $n\in\mathbb{Z}_{>0}$. Indeed, suppose that $\phi\in\mathbb{F}_0^\infty$ satisfies $|\phi(n_0)-\frac{1}{n_0}|=\alpha\in\mathbb{R}_{>0}$ for some $n_0\in\mathbb{Z}$. Now let $\epsilon<\alpha$. Then, for every $N>n_0$ we have, for $j\geq N$,

$$\|\phi-\phi_j\|_\infty=\sup\{|\phi(n)-\phi_j(n)|\mid n\in\mathbb{Z}_{>0}\}\geq|\phi(n_0)-\tfrac{1}{n_0}|>\epsilon.$$

Thus it must be the case that if $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to ϕ, then $\phi(n)=\frac{1}{n}$ for every $n\in\mathbb{Z}_{>0}$, as claimed. □

Thus the sequence is indeed a Cauchy sequence that does not converge. One does believe, however, that the sequence converges, and to ϕ satisfying $\phi(n)=\frac{1}{n}$ for every $n\in\mathbb{Z}_{>0}$. However, $\phi\notin\mathbb{F}_0^\infty$. Thus $\mathbb{F}_0^\infty$ is not "big" enough. More precisely, it is not complete.

Completeness

Definition (Completeness)

A normed $\mathbb{F}$-vector space $(\mathsf{V},\|\cdot\|)$ is ***complete*** if every Cauchy sequence in V converges.

- A question that often arises with completeness is: "Must convergence of the Cauchy sequences be to something in V?" Note that the question strictly does not make sense since the only thing in the universe of V is V. Convergence to something that is somewhere else makes no sense.

Completions

- If a normed vector space is not complete, one can "complete" it.

Definition (Completion)

A ***completion*** of a normed $\mathbb{F}$-vector space $(\mathsf{V}, \|\cdot\|)$ is a complete normed $\mathbb{F}$-vector space $(\overline{\mathsf{V}}, \overline{\|\cdot\|})$ such that there exists a linear injection $\iota_{\mathsf{V}} \colon \mathsf{V} \to \overline{\mathsf{V}}$ with the following properties:

(i) $\overline{\|\iota_{\mathsf{V}}(v)\|} = \|v\|$ for every $v \in \mathsf{V}$;

(ii) if $\overline{v} \in \overline{\mathsf{V}}$, then there exists a sequence $(v_j)_{j\in\mathbb{Z}_{>0}}$ for which the sequence $(\iota_{\mathsf{V}}(v_j))_{j\in\mathbb{Z}_{>0}}$ converges to $\overline{v}$.

- The following result and the sketch of the proof provided are important. Moreover, the proof is both constructive and instructive.

Theorem (Completions exist)

Every normed $\mathbb{F}$-vector space possesses a completion.

Completions

Outline of proof.

Let us call two Cauchy sequences $(u_j)_{j\in\mathbb{Z}_{>0}}$ and $(v_j)_{j\in\mathbb{Z}_{>0}}$ is V ***equivalent*** if, for every $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that $\|u_j - v_k\| < \epsilon$ for $j, k \geq N$. One can verify that this defines an equivalence relation on the set of Cauchy sequences in V. Let $\overline{\mathsf{V}}$ be the set of equivalence classes. Let us denote a point in $\overline{\mathsf{V}}$ by the usual equivalence class notation: $[(v_j)_{j\in\mathbb{Z}_{>0}}]$. Define $\overline{\|\cdot\|}$ by

$$\overline{\|[(v_j)_{j\in\mathbb{Z}_{>0}}]\|} = \lim_{j\to\infty} \|v_j\|.$$

Define a map $\iota_{\mathsf{V}} \colon \mathsf{V} \to \overline{\mathsf{V}}$ by asking that ι_{V} be the equivalence class containing the Cauchy sequence $(v_j = v)_{j\in\mathbb{Z}_{>0}}$. The remainder of the proof is a verification that this construction does indeed define a completion. Steps include the following.

1. Show that $\overline{\mathsf{V}}$ is an $\mathbb{F}$-vector space. The vector space operations are

$$[(u_j)_{j\in\mathbb{Z}_{>0}}] + [(v_j)_{j\in\mathbb{Z}_{>0}}] = [(u_j + v_j)_{j\in\mathbb{Z}_{>0}}], \quad a[(v_j)_{j\in\mathbb{Z}_{>0}}] = [(av_j)_{j\in\mathbb{Z}_{>0}}].$$

Completions

Outline of proof (cont'd).

Besides showing that these operations obey the vector space axioms, one must also verify that their definitions are independent of the choice of representative from the equivalence class.

2. Show that $\overline{\|\cdot\|}$ is well-defined and defines a norm.
3. Show that $(\overline{\mathsf{V}}, \overline{\|\cdot\|})$ is a completion.

□

- To understand the preceding proof, it is worth thinking about an analogous construction: the real numbers are the completion of the rational numbers.
- To define a real number, one can define it as an equivalence class of Cauchy sequences of rational numbers. For example, for the irrational number e we have

$$\mathrm{e} = \sum_{j=0}^{\infty} \frac{1}{j!} = \lim_{j\to\infty} \left(1 + \tfrac{1}{j}\right)^j = \lim_{j\to\infty} \left(1 + \tfrac{1}{j}\right)^{j+1},$$

Completions

giving three different sequences of rational numbers converging to the real number e. Thus each of these three sequences can be used to represent the equivalence class of Cauchy sequences that stand for the real number e.

Reading for Lecture 7

Material related to this lecture can be found in the following sections of the course notes:

1. Sections III-3.3.1 and III-3.3.4.

Lecture 8
Completions

- Apart from the assertion above about the existence of completions, one can also show that completions are "essentially" unique. By this we mean that if we have two completions $(\overline{\mathsf{V}}_1, \overline{\|\cdot\|}_1)$ and $(\overline{\mathsf{V}}_2, \overline{\|\cdot\|}_2)$ of $(\mathsf{V}, \|\cdot\|)$, then there exists an isomorphism $\phi\colon \overline{\mathsf{V}}_1 \to \overline{\mathsf{V}}_2$ such that $\overline{\|\overline{v}\|}_2 = \overline{\|\overline{v}\|}_1$ for all $\overline{v} \in \overline{\mathsf{V}}_1$. Thus the two normed vector spaces $(\overline{\mathsf{V}}_1, \overline{\|\cdot\|}_1)$ and $(\overline{\mathsf{V}}_2, \overline{\|\cdot\|}_2)$ are "the same."
- The matter of concretely describing the completion of a normed vector space is sometimes nontrivial. We shall see this for continuous-time signals.

Completions

- Equipped with the notions of completeness and completion, let us consider some examples.

Examples (Completions)

1. We resume our consideration of the normed vector space $(\mathbb{F}_0^\infty, \|\cdot\|_\infty)$. We have already shown that this normed vector space is not complete. What is its completion? There is not necessarily a systematic way of deducing what the completion is. However, a moment's reflection might lead one to conclude that a completion is the set of sequences which converge to zero. Let us denote these sequences by $\mathsf{c}_0(\mathbb{F})$, the notation hopefully suggesting the set of $\mathbb{F}$-valued sequences converging to 0. The norm we consider is the ∞-norm:

$$\|\phi\|_\infty = \sup\{|\phi(n)| \mid n \in \mathbb{Z}_{>0}\}, \qquad \phi \in \mathsf{c}_0(\mathbb{F}).$$

After making this educated guess, one can then ask whether the guess is correct. Let us verify this.

Completions

Examples (Completions (cont'd))

First of all, note that $\mathbb{F}_0^\infty$ is a subspace of $\mathsf{c}_0(\mathbb{F})$, so the natural map from $\mathbb{F}_0^\infty$ to its completion is just the inclusion. This clearly preserves the norm. We do not use notation for this, but simply regard elements of $\mathbb{F}_0^\infty$ naturally as lying in $\mathsf{c}_0(\mathbb{F})$. Thus we do not worry about the map ι_{V} from the definition of the completion.

We need now only verify that if $\phi \in \mathsf{c}_0(\mathbb{F})$ then there exists a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ such that $\phi = \lim_{j\to\infty} \phi_j$. We define this sequence by

$$\phi_j(n) = \begin{cases} \phi(n), & n \le j, \\ 0, & n > j. \end{cases}$$

To check that it converges to ϕ, let $\epsilon \in \mathbb{R}_{>0}$. Since the sequence $(\phi(n))_{n\in\mathbb{Z}_{>0}}$ converges to zero, let N be sufficiently large that $|\phi(n)| < \epsilon$ for $n \ge N$. Then, for $j \ge N$, we have

Completions

Examples (Completions (cont'd))

$$\|\phi - \phi_j\|_\infty = \sup\{|\phi(n) - \phi_j(n)| \mid n \in \mathbb{Z}_{>0}\} = \sup\{|\phi(n)| \mid n > j\} < \epsilon,$$

which shows that $\phi = \lim_{j\to\infty} \phi_j$, as desired. Thus $(\mathsf{c}_0(\mathbb{F}), \|\cdot\|_\infty)$ is indeed a completion of $(\mathbb{F}_0^\infty, \|\cdot\|_\infty)$.[a]

2. Now we consider the same vector space, $\mathbb{F}_0^\infty$, but with a different norm, the 2-norm:

$$\|\phi\|_2 = \left(\sum_{n=1}^{\infty} |\phi(n)|^2\right)^{1/2}.$$

We also consider the same sequence of functions $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ considered above when showing that $\mathbb{F}_0^\infty$ was not complete with the ∞-norm:

$$\phi_j(n) = \begin{cases} \frac{1}{n}, & n \le j, \\ 0, & n > j. \end{cases}$$

[a] Actually, we still need to prove completeness of $\mathsf{c}_0(\mathbb{F})$; we shall address this shortly.

Completions

Examples (Completions (cont'd))

We wish to show that this sequence is Cauchy, but does not converge.

Claim

The sequence is Cauchy.

Proof.

Let $\epsilon \in \mathbb{R}_{>0}$. Note that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, meaning that its sequence of partial sums converges. Therefore, its sequence of partial sum is Cauchy. Therefore, there exists $N \in \mathbb{Z}_{>0}$ such that

$$\sum_{n=j+1}^{k} \frac{1}{n^2} < \epsilon^2$$

Completions

Examples (Completions (cont'd))

Proof (cont'd).

for $j, k \geq N$ with $k > j$ (note that the above sum is the difference between the jth and kth partial sum). Now, for $j, k \geq N$ with $j < k$ we have

$$\|\phi_j - \phi_k\|_2 = \left(\sum_{n=1}^{\infty} |\phi_j(n) - \phi_k(n)|^2\right)^{1/2} = \left(\sum_{n=j+1}^{k} |\phi_j(n) - \phi_k(n)|^2\right)^{1/2} < \epsilon,$$

giving that $(\phi_j)_{j \in \mathbb{Z}_{>0}}$ is Cauchy. □

Claim

The sequence does not converge.

Completions

Examples (Completions (cont'd))

Proof.

Suppose that the sequence $(\phi_j)_{j \in \mathbb{Z}_{>0}}$ converges to $\phi \in \mathbb{F}_0^\infty$ and let $n_0 \in \mathbb{Z}_{>0}$. We claim that $\phi(n_0) = \frac{1}{n_0}$. Suppose otherwise, and that $|\phi(n_0) - \frac{1}{n_0}| = \alpha \in \mathbb{R}_{>0}$. Then, for every $N > n_0$ we have, for $j \geq N$.

$$\|\phi - \phi_j\|_2 = \left(\sum_{n=1}^{\infty} |\phi(n) - \phi_j(n)|^2\right)^{1/2} \geq |\phi(n_0) - \tfrac{1}{n_0}| = \alpha > 0.$$

Therefore, since we have $\phi(n) = \frac{1}{n}$ for every $n \in \mathbb{Z}_{>0}$, it follows that $\phi \notin \mathbb{F}_0^\infty$. Thus $(\phi_j)_{j \in \mathbb{Z}_{>0}}$ does not converge. □

Thus $(\mathbb{F}_0^\infty, \|\cdot\|_2)$ is not complete.

Reading for Lecture 8

Material related to this lecture can be found in the following sections of the course notes:

1. Sections III-3.3.4 and III-3.8.2.

Lecture 9
Completions

Examples (Completions (cont'd))

Now we can ponder on what its completion is. Again, there is no recipe for easily understanding this. However, a moment's thought might lead you to conclude that a completion is the set of maps $\phi\colon \mathbb{Z}_{>0} \to \mathbb{F}$ for which

$$\Big(\sum_{n=1}^{\infty} |\phi(n)|^2\Big)^{1/2} < \infty \quad \iff \quad \sum_{n=1}^{\infty} |\phi(n)|^2 < \infty.$$

Let us denote such maps by $\ell^2(\mathbb{F})$. (This sort of notation will be generalised later. The point is that the subscript "2" means "square" summable sequences.) The norm on the completion would be

$$\|\phi\|_2 = \Big(\sum_{n=1}^{\infty} |\phi(n)|^2\Big)^{1/2}.$$

Let us verify that this indeed a completion of $(\mathbb{F}_0^{\infty}, \|\cdot\|_2)$. Since $\mathbb{F}_0^{\infty} \subseteq \ell^2(\mathbb{F})$, we can sidestep the map ι_{V} in the definition of the completion.

Completions

Examples (Completions (cont'd))

We must show that if $\phi \in \ell^2(\mathbb{F})$ then there exists a sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathbb{F}_0^\infty$ which converges to ϕ using the norm $\|\cdot\|_2$. Let us define

$$\phi_j(n) = \begin{cases} \phi(n), & n \le j, \\ 0, & n > j. \end{cases}$$

To show that $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to ϕ, let $\epsilon \in \mathbb{R}_{>0}$. Since $\phi \in \ell^2(\mathbb{F})$ there exists $N \in \mathbb{Z}_{>0}$ such that

$$\sum_{n=N+1}^{\infty} |\phi(n)|^2 < \epsilon^2.$$

For $j \ge N$ we then have

$$\|\phi - \phi_j\|_2 = \Big(\sum_{n=1}^{\infty} |\phi(n) - \phi_j(n)|^2\Big)^{1/2} = \Big(\sum_{n=j+1}^{\infty} |\phi(n)|^2\Big)^{1/2} < \epsilon.$$

Completions

Examples (Completions (cont'd))

This give the desired convergence, and shows that $(\ell^2(\mathbb{F}), \|\cdot\|_2)$ is the completion of $(\mathbb{F}_0^\infty, \|\cdot\|_2)$.[a]

3. As our final example, we again consider the vector space $\mathbb{F}_0^\infty$, but now with the 1-norm:

$$\|\phi\|_1 = \sum_{n=1}^{\infty} |\phi(n)|.$$

We claim that, as in the cases above, this normed vector space is not complete. Let us endeavour to prove this with the sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ considered above in the previous two cases. Thus we take

$$\phi_j(n) = \begin{cases} \frac{1}{n}, & n \le j, \\ 0, & n > j. \end{cases}$$

[a] Actually, we still need to prove completeness of $\ell^2(\mathbb{F})$; we shall address this shortly.

Completions

Examples (Completions (cont'd))

Claim

This sequence is not even Cauchy.

Proof.

We must show that there exists $\alpha \in \mathbb{R}_{>0}$ such that, for every $N \in \mathbb{Z}_{>0}$, there are some $j, k \geq N$ such that

$$\|\phi_j - \phi_k\|_1 = \sum_{n=1}^{\infty} |\phi_j(n) - \phi_k(n)| > \alpha.$$

Let us take α to be *any* positive number. Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, for every $N \in \mathbb{Z}_{>0}$ we have $\sum_{n=N+1}^{\infty} \frac{1}{n} = \infty$. Therefore, there exists some finite set $\{j+1, \dots, k\}$ of integers greater than N such that

Completions

Examples (Completeness (cont'd))

Proof (cont'd).

$$\sum_{n=j+1}^{k} \frac{1}{n} > \alpha.$$

For this choice of N, j, and k we have

$$\|\phi_k - \phi_j\|_1 = \sum_{n=j+1}^{k} \frac{1}{n} > \alpha.$$

This gives the desired conclusion. $\square$

Thus, to show that $(\mathbb{F}_0^{\infty}, \|\cdot\|_1)$ is not complete, we should select come other candidate sequence to be a nonconvergent Cauchy sequence.

Completions

Examples (Completions (cont'd))

One can show that such a sequence is the sequence $(\psi_j)_{j\in\mathbb{Z}_{>0}}$ defined by

$$\psi_j(n) = \begin{cases} \frac{1}{n^2}, & n \le j, \\ 0, & n > j. \end{cases}$$

The proof that this sequence is Cauchy but nonconvergent follows along the same lines as the similar claims above. I leave you to sort through the details.

We should also describe the completion of $(\mathbb{F}_0^\infty, \|\cdot\|_1)$.

(Thinking. . . thinking. . .) It seems like a reasonable choice would be the set of functions $\phi\colon \mathbb{Z}_{>0} \to \mathbb{F}$ such that

$$\sum_{n=1}^{\infty} |\phi(n)| < \infty.$$

Let us denote the set of such functions by $\ell^1(\mathbb{F})$ and use the norm

Completions

Examples (Completions (cont'd))

$$\|\phi\|_1 = \sum_{n=1}^{\infty} |\phi(n)|$$

on the completion. To show that $(\ell^1(\mathbb{F}), \|\cdot\|_1)$ is a completion of $(\mathbb{F}_0^\infty, \|\cdot\|_1)$ is done pretty much exactly as in the previous cases. We leave you to do the straightforward computations.

Remark (Unfinished business)

We have not actually addressed the fact that $\mathsf{c}_0(\mathbb{F})$, $\ell^2(\mathbb{F})$, and $\ell^1(\mathbb{F})$ must be, in fact, complete in order to qualify as completions. The strategy to prove that all of these normed vector spaces is complete is the same in all cases. We suppose that we have a Cauchy sequence $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ and do the following.

1. Using the fact that $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ is Cauchy, one can show that $(\phi_j(n))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathbb{R}$ for each $n \in \mathbb{Z}_{>0}$. Thus, by completeness of $\mathbb{R}$, one can define $\phi(n) = \lim_{j\to\infty} \phi_j(n)$.

Completions

Remark

2. With $\phi \in \mathbb{F}^\infty$ defined as above, one can show that $\phi \in \mathsf{c}_0(\mathbb{F})$, or $\phi \in \ell^2(\mathbb{F})$, or $\phi \in \ell^1(\mathbb{F})$, as the case may be.
3. Finally, one shows that $(\phi_j)_{j\in\mathbb{Z}_{>0}}$ converges to ϕ in the norm $\|\cdot\|_\infty$, $\|\cdot\|_2$, or $\|\cdot\|_1$, as the case may be.

- The notions of completeness and completions are not always easy to grasp. The preceding examples are about the simplest illustrations of the phenomenon, so they are the best place to start gaining understanding.
- In the preceding examples we could “guess” a completion. We shall see that this is not always the case.

Banach spaces

Definition (Banach space)

An $\mathbb{F}$***-Banach space*** is a complete normed $\mathbb{F}$-vector space.

- Based on the observations in the preceding slides, we are interested in Banach spaces, and not so much in complete normed vector spaces that are not complete.
- We have seen three Banach spaces:
  1. $(\mathsf{c}_0(\mathbb{F}); \|\cdot\|_\infty)$;
  2. $(\ell^2(\mathbb{F}), \|\cdot\|_2)$;
  3. $(\ell^1(\mathbb{F}), \|\cdot\|_1)$.
- These are all completions of $\mathbb{F}_0^\infty$.
- We see clearly the effects of the three different norms on $\mathbb{F}_0^\infty$: they each have a different completion. Indeed, we have

$$\ell^1(\mathbb{F}) \subset \ell^2(\mathbb{F}) \subset \mathsf{c}_0(\mathbb{F})$$

(all inclusions are strict).

Reading for Lecture 9

Material related to this lecture can be found in the following sections of the course notes:

1. Sections III-3.3.1, III-3.3.4, and III-3.8.2.

Lecture 10
Why are completions important?

- At this point, completions may not seem so complicated since, in the examples above, the completion arose in a fairly natural way. However, we will see later that a terrible price must be paid to understand the completions of spaces of continuous functions with respect to some norms.
- One will then wonder, "Why do we care deeply about completions?"
- A good place to understand this sort of question is with the real numbers since the real numbers are the completion of the rational numbers.
- Computationally, the rational numbers are "sufficient" since computers, and possibly humans, are only capable of understanding things with finite representations: like rational numbers. If one is only interested in computation, then rational numbers are all one sees.
- But should we then dispense with real numbers? Perhaps we could, but we would lose much that gives us comfort.

Why are completions important?

- Philosophically, we would have to sacrifice models of the physical world which make use of a spatio-temporal continuum. This would then eliminate almost all current mathematical models of the physical world, like the Newton–Euler equations in mechanics, Maxwell's equations for electro-magnetism (from which are derived the basic properties of circuits about which you learn), the Navier–Stokes equations for fluid mechanics, and so on and so on.
- Mathematically, we would have to forgo calculus as we know it. We would also have to allow things like a bounded monotonically increasing sequence not having a limit.
- If you think such sacrifices are worth making merely for the efficacy of numerical computation, then good luck to you in your "rational" world.
- In general, completions can be viewed in the above light. One might have a normed vector space whose elements are natural and easy to understand (cf. the rational numbers), but for which the completion is complicated and difficult to understand (cf. the real numbers).
- But the property of completeness is itself so useful (cf. the usefulness of the completeness of $\mathbb{R}$) that one simply cannot function properly without it.

Topology of normed vector spaces

- In normed vector spaces one can talk easily about many of the topics of real analysis. The starting point is the following.

Definition (Open ball)

Let $(\mathsf{V}, \|\cdot\|)$ be a normed $\mathbb{F}$-vector space. For $r \in \mathbb{R}_{>0}$ and $v_0 \in \mathsf{V}$, the ***open ball*** of radius r and centre v_0 is $\mathsf{B}(r, v_0) = \{v \in \mathsf{V} \mid \|v - v_0\| < r\}$ and the ***closed ball*** of radius r and centre v_0 is $\overline{\mathsf{B}}(r, v_0) = \{v \in \mathsf{V} \mid \|v - v_0\| \leq r\}$.

- The notion of an open ball in a normed vector space generalises the usual notion of a "round" ball in Euclidean space. However, balls are not always "round," as you know from a homework problem.
- With the notion of a ball, one defines the well-known properties for subsets of normed vector spaces.

Topology of normed vector spaces

Definition

Let $(\mathsf{V}, \|\cdot\|)$ be a normed $\mathbb{F}$-vector space.

(i) A subset $U \subseteq \mathsf{V}$ is ***open*** if, for every $v_0 \in U$, there exists $\epsilon \in \mathbb{R}_{>0}$ such that $\mathsf{B}(\epsilon, v_0) \subseteq U$.

(ii) A subset $A \subseteq \mathsf{V}$ is ***closed*** if $\mathsf{V} \setminus A$ is open.

(iii) A subset $B \subseteq \mathsf{V}$ is ***bounded*** if $B \subseteq \mathsf{B}(R, 0_{\mathsf{V}})$ for some $R \in \mathbb{R}_{>0}$.

(iv) A subset $K \subseteq \mathsf{V}$ is ***compact*** if, for every family $(U_i)_{i \in I}$ of open subsets of V for which $K \subseteq \cup_{i \in I} U_i$, there exists a finite subset $\{i_1, \dots, i_k\} \subseteq I$ such that $K \subseteq \cup_{j=1}^{k} U_{i_j}$.

- Maybe the definition of compactness we give is not familiar to you. Here are some facts about this definition that are not part of this course.
  1. A subset $K \subseteq \mathsf{V}$ of a finite-dimensional normed vector space is compact if and only if it is closed and bounded. Thus the definition we give agrees with the one you already know.
  2. For infinite-dimensional normed vector space, “compact” and “closed and bounded” do not necessarily agree. For example, in an infinite-dimensional normed vector space, closed balls are closed and bounded but not compact.

Topology of normed vector spaces

- Convergent and Cauchy sequences can be characterised using balls.

Definition

Let $(\mathsf{V}, \|\cdot\|)$ be a normed vector space and let $(v_j)_{j \in \mathbb{Z}_{>0}}$ be a sequence in V.

(i) The sequence ***converges*** to $v_0 \in \mathsf{V}$ if, for every $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that $v_j \in \mathsf{B}(\epsilon, v_0)$ for $j \geq N$.

(ii) The sequence is ***Cauchy*** if, for every $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that $v_j, v_k \in \mathsf{B}(\epsilon, v)$ for $j, k \geq N$ and for some $v \in \mathsf{V}$.

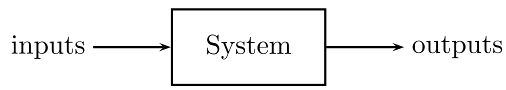
- We can also talk about continuous functions.

Definition

Let $(\mathsf{U}, \|\cdot\|_{\mathsf{U}})$ and $(\mathsf{V}, \|\cdot\|_{\mathsf{V}})$ be normed $\mathbb{F}$-vector spaces and let $S \subseteq \mathsf{U}$. A map $\phi \colon S \to \mathsf{V}$ is ***continuous*** at $u_0 \in S$ if, for every $\epsilon \in \mathbb{R}_{>0}$, there exists $\delta \in \mathbb{R}_{>0}$ such that $\phi(u) \in \mathsf{B}(\epsilon, \phi(u_0))$ for every $u \in S \cap \mathsf{B}(\delta, u_0)$.

Topology of normed vector spaces

- Continuous linear maps from $(\mathsf{U}, \|\cdot\|_{\mathsf{U}})$ to $(\mathsf{V}, \|\cdot\|_{\mathsf{V}})$ are important, so here are some facts:
  1. if U is finite-dimensional then any linear map from U to V is continuous;
  2. if U is infinite-dimensional, then there are discontinuous linear maps from U to V.
- Continuous linear maps arise in system theory. Consider a system:



  We have already claimed that it is useful to a system to be linear. Well, very often a system is required to be continuous.

Inner products

- We now discuss a particular sort of normed vector space.

Definition (Inner product)

An ***inner product*** on an $\mathbb{F}$-vector space V assigns to vectors $v_1, v_2 \in \mathsf{V}$ the number $\langle v_1, v_2\rangle \in \mathbb{F}$, and the assignment satisfies the following rules:

(i) $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}$ for $u, v \in \mathsf{V}$ (***symmetry***);

(ii) $\langle a_1 v_1 + a_2 v_2, v\rangle = a_1\langle v_1, v\rangle + a_2\langle v_2, v\rangle$ for $a_1, a_2 \in \mathbb{F}$ and $v_1, v_2 \in \mathsf{V}$ (***linearity***);

(iii) $\langle v, v\rangle \geq 0$ for $v \in \mathsf{V}$, (***positivity***);

(iv) $\langle v, v\rangle = 0$ only if $v = 0_{\mathsf{V}}$ (***definiteness***).

- Note that $\langle v, v\rangle \in \mathbb{R}$ by the symmetry property.
- Note that $\langle v_1, av_2\rangle = \overline{\langle av_2, v_1\rangle} = \overline{a}\overline{\langle v_2, v_1\rangle} = \overline{a}\langle v_1, v_2\rangle$.

Inner products

Definition (Inner product space)

An $\mathbb{F}$***-inner product space*** is a pair $(\mathsf{V}, \langle \cdot, \cdot \rangle)$ with V an $\mathbb{F}$-vector space and $\langle \cdot, \cdot \rangle$ an inner product on V.

- From inner products come norms. Indeed, we shall show that for an inner product $\langle \cdot, \cdot \rangle$, the assignment

$$v \mapsto \|v\| \triangleq \sqrt{\langle v, v \rangle}$$

defines a norm. (We use the norm notation $\|\cdot\|$, although the function has yet to be shown to be a norm.)

Theorem (Cauchy–Bunyakovsky–Schwarz inequality)

For an $\mathbb{F}$*-inner product space* $(\mathsf{V}, \langle \cdot, \cdot \rangle)$ *we have*

$$|\langle v_1, v_2 \rangle| \leq \|v_1\| \, \|v_2\|, \qquad v_1, v_2 \in \mathsf{V}.$$

Inner products

Proof.

The result is obviously true for $v_2 = 0$, so we shall suppose that $v_2 \neq 0$. We first prove the result for $\|v_2\| = 1$. In this case we have

$$\begin{aligned}
0 &\leq \|v_1 - \langle v_1, v_2 \rangle v_2\|^2 \\
&= \langle v_1 - \langle v_1, v_2 \rangle v_2, v_1 - \langle v_1, v_2 \rangle v_2 \rangle \\
&= \langle v_1, v_1 \rangle - \langle v_1, v_2 \rangle \langle v_2, v_1 \rangle - \overline{\langle v_1, v_2 \rangle} \langle v_1, v_2 \rangle + \langle v_1, v_2 \rangle \overline{\langle v_1, v_2 \rangle} \langle v_2, v_2 \rangle \\
&= \|v_1\|^2 - |\langle v_1, v_2 \rangle|^2,
\end{aligned}$$

using properties of inner products. Thus we have shown that, provided $\|v_2\| = 1$,

$$|\langle v_1, v_2 \rangle|^2 \leq \|v_1\|^2.$$

Taking square roots yields the result in this case. For $\|v_2\| \neq 1$ we define $v_3 = \frac{v_2}{\|v_2\|}$ so that $\|v_3\| = 1$. In this case

Inner products

Proof (cont'd).

$$\begin{aligned} & |\langle v_1, v_3\rangle| \leq \|v_1\| \\ \Longrightarrow \quad & \frac{|\langle v_1, v_2\rangle|}{\|v_2\|} \leq \|v_1\|, \end{aligned}$$

and so the inequality in the theorem holds. □

Reading for Lecture 10

Material related to this lecture can be found in the following sections of the course notes:

1. Section III-3.1.2 (especially the first part);
2. Section III-3.2.1;
3. Section III-3.3.2;
4. Sections III-3.5.1 and III-3.5.2.
5. Section III-3.6.1 (especially the first part);
6. Section III-3.6.3;
7. Sections III-4.1.1 and III-4.1.2.

Lecture 11
Inner products

Theorem

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(\mathsf{V}, \langle \cdot, \cdot \rangle)$ be an $\mathbb{F}$-inner product space, and define $\mathsf{V} \ni v \mapsto \|v\| \in \mathbb{R}_{\geq 0}$ be defined by $\|v\| = \sqrt{\langle v, v \rangle}$. Then $(\mathsf{V}, \|\cdot\|)$ is a normed vector space.

Proof.

All norm properties except the triangle inequality are easily verified. To verify the triangle inequality, for $v_1, v_2 \in \mathsf{V}$, we compute

$$\begin{aligned}
\|v_1 + v_2\|^2 = \langle v_1 + v_2, v_1 + v_2 \rangle &= \|v_1\|^2 + \langle v_1, v_2 \rangle + \langle v_2, v_1 \rangle + \|v_2\|^2 \\
&= \|v_1\|^2 + \langle v_1, v_2 \rangle + \overline{\langle v_1, v_2 \rangle} + \|v_2\|^2 = \|v_1\|^2 + 2\,\mathrm{Re}(\langle v_1, v_2 \rangle) + \|v_2\|^2 \\
&\leq \|v_1\|^2 + 2|\mathrm{Re}(\langle v_1, v_2 \rangle)| + \|v_2\|^2 \leq \|v_1\|^2 + 2|\langle v_1, v_2 \rangle| + \|v_2\|^2 \\
&\leq \|v_1\|^2 + 2\|v_1\|\|v_2\| + \|v_2\|^2 = (\|v_1\| + \|v_2\|)^2,
\end{aligned}$$

using the Cauchy–Bunyakovsky–Schwarz inequality. Taking square roots gives the result. □

Inner products

Definition (Hilbert space)

An $\mathbb{F}$***-Hilbert space*** is an $\mathbb{F}$-inner product space that is a complete normed vector space.

- Do *all* norms come from inner products? Not hardly.

Theorem

If $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and if $(\mathsf{V}, \|\cdot\|)$ is a normed $\mathbb{F}$-vector space, then the following statements are equivalent:

(i) there exists an inner product $\langle \cdot, \cdot \rangle$ on V such that $\|v\| = \sqrt{\langle v, v \rangle}$ for all $v \in \mathsf{V}$;

*(ii) $\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2\big(\|v_1\|^2 + \|v_2\|^2\big)$ for every $v_1, v_2 \in \mathsf{V}$ (**parallelogram law**).*

- The proof that (i) implies (ii) is a direct computation. The converse implication is rather difficult. This is Theorem III-4.1.9 in the course notes.

Examples of inner product spaces

Examples (Inner products)

1. Let $\mathsf{V} = \mathbb{F}^n$ and define
$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1\bar{v}_1 + \cdots + u_n\bar{v}_n.$$
This is easily verified to be an inner product. The corresponding norm is
$$\boldsymbol{v} \mapsto (|v_1|^2 + \cdots + |v_n|^2)^{1/2},$$
i.e., the 2-norm on $\mathbb{F}^n$. Since $\mathbb{F}^n$ is finite-dimensional, $(\mathbb{F}^n, \langle \cdot, \cdot \rangle)$ is a Hilbert space.
2. Let $\mathsf{V} = \mathbb{F}_0^\infty$ and define
$$\langle \phi, \psi \rangle = \sum_{n=1}^{\infty} \phi(n)\overline{\psi(n)}.$$
This is easily verified to be an inner product. The corresponding norm is

Examples of inner product spaces

Examples (Inner products (cont'd))

$$\phi \mapsto \Big(\sum_{n=1}^{\infty} |\phi(n)|^2\Big)^{1/2},$$
i.e., the 2-norm on $\mathbb{F}_0^\infty$. Note that $(\mathbb{F}_0^\infty, \|\cdot\|_2)$ is not complete so $(\mathbb{F}_0^\infty, \langle \cdot, \cdot \rangle)$ is not a Hilbert space.
However, if $\phi, \psi \in \ell^2(\mathbb{F})$ then note that, by Cauchy–Bunyakovsky–Schwarz,
$$\Big|\sum_{n=1}^{\infty} |\phi(n)\overline{\psi(n)}|\Big| \leq \||\phi|\|_2 \||\psi|\|_2 = \|\phi\|_2 \|\psi\|_2 < \infty.$$
Thus
$$\langle \phi, \psi \rangle = \sum_{n=1}^{\infty} \phi(n)\overline{\psi(n)}$$
makes sense (the sum on the right is absolutely convergent, and so convergent) for $\phi, \psi \in \ell^2(\mathbb{F})$, and $(\ell^2(\mathbb{F}), \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Examples of inner product spaces

Examples (Inner product spaces (cont'd))

3. Take $\mathsf{V} = \mathsf{C}^0([a,b];\mathbb{F})$ and define

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x.$$

This is easily verified to be an inner product whose norm is

$$f \mapsto \Big(\int_a^b |f(x)|^2\,\mathrm{d}x\Big)^{1/2},$$

i.e., the 2-norm. It is a fact that we have not yet explored that $(\mathsf{C}^0([a,b];\mathbb{F}), \langle\cdot,\cdot\rangle)$ is not a Hilbert space. We shall examine this, along with the corresponding completion, later in the course.

Discrete-time signal spaces

- We now have some structure that we can impose on spaces of signals, the structure of a normed vector space or an inner product space.
- Let $\mathbb{T}$ be a discrete time-domain with sampling interval Δ and note that $\mathbb{F}^{\mathbb{T}}$ denotes the set of $\mathbb{F}$-valued signals on $\mathbb{T}$.
- Define

$$\begin{aligned}
\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}) &= \{f \in \mathbb{F}^{\mathbb{T}} \mid f(t) = 0 \text{ for all but finitely many } t \in \mathbb{T}\};\\
\mathsf{c}_0(\mathbb{T};\mathbb{F}) &= \{f \in \mathbb{F}^{\mathbb{T}} \mid \text{ for each } \epsilon \in \mathbb{R}_{>0} \text{ there exists a finite subset } S \subseteq \mathbb{T}\\
&\qquad \text{such that } |f(t)| > \epsilon \text{ iff } t \in S\};\\
\ell^\infty(\mathbb{T};\mathbb{F}) &= \{f \in \mathbb{F}^{\mathbb{T}} \mid \sup\{|f(t)| \mid t \in \mathbb{T}\} < \infty\};\\
\ell^p(\mathbb{T};\mathbb{F}) &= \Big\{f \in \mathbb{F}^{\mathbb{T}} \;\Big|\; \sum_{t\in\mathbb{T}} |f(t)|^p < \infty\Big\},
\end{aligned}$$

where $p \in [1,\infty)$.

Discrete-time signal spaces

- On $\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F})$, $\mathsf{c}_0(\mathbb{T};\mathbb{F})$, and $\ell^\infty(\mathbb{T};\mathbb{F})$ we use the ∞-norm:
$$\|f\|_\infty = \sup\{|f(t)| \mid t \in \mathbb{T}\}$$
and on $\ell^p(\mathbb{T};\mathbb{F})$, $p \in [1,\infty)$, we use the p-norm:
$$\|f\|_p = \Big(\Delta \sum_{t\in\mathbb{T}} |f(t)|^p\Big)^{1/p}.$$
- Let us record some useful facts. Some of these we have essentially proved, others we state as facts whose proof is given in the course text.
  1. If $\mathbb{T}$ is finite then
  $$\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}) = \mathsf{c}_0(\mathbb{T};\mathbb{F}) = \ell^\infty(\mathbb{T};\mathbb{F}) = \ell^p(\mathbb{T};\mathbb{F}) = \mathbb{F}^{\mathbb{T}},$$
  and so the vector spaces are all finite-dimensional in this case. Because of this, the use of the norm $\|\cdot\|_\infty$ or $\|\cdot\|_p$ is not significant in that the "topology" on the spaces will be the same, no matter what norm is used; this is Theorem III-3.1.15.
  2. If $\mathbb{T}$ is infinite then
  $$\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}) \subset \ell^p(\mathbb{T};\mathbb{F}) \subset \mathsf{c}_0(\mathbb{T};\mathbb{F}) \subset \ell^\infty(\mathbb{T};\mathbb{F}) \subset \mathbb{F}^{\mathbb{T}}.$$
  This is obvious.

Discrete-time signal spaces

3. If $\mathbb{T}$ is infinite then
$$\mathsf{c}_0(\mathbb{T};\mathbb{F}) = \Big\{f \in \mathbb{F}^{\mathbb{T}} \;\Big|\; \lim_{|t|\to\infty} f(t) = 0\Big\}.$$
4. We have previously considered the vector space $\mathbb{F}_0^\infty$ which, in our present language, is simply $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{>0};\mathbb{F})$. For any infinite discrete time-domain $\mathbb{T}$ there exists an isomorphism of normed vector spaces between $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ and $(\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{>0};\mathbb{F}), \|\cdot\|_\infty)$. This isomorphism may not really be natural; for example there is no really natural way to construct an isomorphism from $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{>0};\mathbb{F})$ to $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z};\mathbb{F})$. However, the mere existence of an isomorphism of normed vector spaces allows us to deduce for $\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F})$ certain of the properties we have deduced for $\mathbb{F}_0^\infty$. In particular, if $\mathbb{T}$ is infinite then the normed vector space $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ is not complete.
5. We had defined the vector space $\mathsf{c}_0(\mathbb{F})$ which, in our present notation, is precisely $\mathsf{c}_0(\mathbb{Z}_{>0};\mathbb{F})$. As in the preceding paragraph, there exists an isomorphism of normed vector spaces between $\mathsf{c}_0(\mathbb{Z}_{>0};\mathbb{F})$ and $\mathsf{c}_0(\mathbb{T};\mathbb{F})$ for any infinite discrete time-domain $\mathbb{T}$. Thus certain of the conclusions we have deduced for $\mathsf{c}_0(\mathbb{F})$ hold for $\mathsf{c}_0(\mathbb{T};\mathbb{F})$ in this case. In particular, $(\mathsf{c}_0(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ is an $\mathbb{F}$-Banach space and is the completion of $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$.

Discrete-time signal spaces

6. We have considered the normed vector spaces $(\ell^p(\mathbb{F}), \|\cdot\|_p)$ for $p \in \{1, 2\}$. We can also define

$$\ell^p(\mathbb{F}) = \Big\{\phi\colon \mathbb{Z}_{>0} \to \mathbb{T} \;\Big|\; \sum_{n=1}^{\infty} |\phi(n)|^p < \infty\Big\}.$$

This can be shown to be a subspace of $\mathbb{F}^\infty$. Moreover,

$$\|\phi\|_p = \Big(\sum_{n=1}^{\infty} \Delta|\phi(n)|^p\Big)^{1/p}$$

defines a norm on $\ell^p(\mathbb{F})$ and the resulting normed vector space is a Banach space.
In terms of our present notation we have $\ell^p(\mathbb{F}) = \ell^p(\mathbb{Z}_{>0}; \mathbb{F})$. It is not difficult to show that, in fact, the normed vector spaces $(\ell^p(\mathbb{T}; \mathbb{F}), \|\cdot\|_p)$ and $(\ell^p(\mathbb{Z}_{>0}; \mathbb{F}), \|\cdot\|_p)$ are isomorphic (up to a constant factor of $\Delta^{1/p}$ for the norm) for any infinite discrete time-domain $\mathbb{T}$. This allows us to draw conclusions for $\ell^p(\mathbb{T}; \mathbb{F})$ based on conclusions we have already drawn for $\ell^p(\mathbb{F})$. For example $(\ell^p(\mathbb{T}; \mathbb{F}), \|\cdot\|_p)$ is an $\mathbb{F}$-Banach space and is the completion of $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T}; \mathbb{F}), \|\cdot\|_p)$. This Banach space is a Hilbert space if and only if $p = 2$.

Discrete-time signal spaces

- The inclusion relations for the most important (for us) discrete-time signal spaces is as follows, the left diagram for bounded time-domains and the right diagram for unbounded time-domains.

Reading for Lecture 11

Material related to this lecture can be found in the following sections of the course notes:

1. Sections III-4.1.1 and III-4.1.2;
2. Sections IV-1.2.2, IV-1.2.3, and IV-1.2.7.

Lecture 12
Continuous-time signal spaces

- Continuous-time signals are *much* more complicated than discrete-time signals.
- Let $\mathbb{T}$ be a continuous time-domain, i.e., an interval. We may still denote $\mathbb{F}^{\mathbb{T}}$ as the vector space of $\mathbb{F}$-valued signals on $\mathbb{T}$. For reasons that are not perfectly clear presently, $\mathbb{F}^{\mathbb{T}}$ is too large to be of much use. A hint as to why would be that arbitrary signals are not integrable, and so the definitions of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on, for example, $\mathrm{C}^0([a,b];\mathbb{F})$ are not defined for general signals.
- *Question:* What is the smallest class of signals one might wish to include in the collection of all signals?
- *Possible answer:* All continuous signals, denoted by $\mathrm{C}^0(\mathbb{T};\mathbb{F})$.
- *Objection:* The set of all continuous signals may be too large if one wants to use the norms $\|\cdot\|_1$ or $\|\cdot\|_2$. For example, if $\mathbb{T}=\mathbb{R}$, then the continuous signal $t \mapsto 1$ does not have its 1- or 2-norm defined.
- The ***support*** of a continuous signal $f\colon \mathbb{T} \to \mathbb{F}$ is

$$\mathrm{supp}(f) = \mathrm{cl}(\{t \in \mathbb{T} \mid f(t) \neq 0\}) \cap \mathbb{T}.$$

Continuous-time signal spaces

- For a continuous time-domain $\mathbb{T}$, let $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F})$ be the set of continuous signals $f\colon \mathbb{T}\to\mathbb{F}$ such that $\mathrm{supp}(f)$ is compact.
- *Actual answer:* We ask that all signal spaces on a continuous time-domain $\mathbb{T}$ contain $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F})$.

Proposition

If $\mathrm{int}(\mathbb{T})\neq\varnothing$ then $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F})$ is infinite-dimensional.

Proof.

We first show that $\mathsf{C}^0([0,1];\mathbb{R})$ is infinite-dimensional. We consider the family of signals $\mathscr{F}=(f_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{C}^0([0,1];\mathbb{R})$ given by $f_j(t)=\sin(2\pi jt)$. We claim that this set is linearly independent. Let $j_1,\dots,j_k\in\mathbb{Z}_{>0}$ and suppose that

$$c_1 f_{j_1}+\cdots+c_k f_{j_k}=0.$$

Then

Continuous-time signal spaces

Proof (cont'd).

$$\begin{aligned}
& c_1\sin(2\pi j_1 t)+\cdots+c_k\sin(2\pi j_k t)=0, \qquad t\in[0,1]\\
\implies\quad & \int_0^1 (c_1\sin(2\pi j_1 t)+\cdots+c_k\sin(2\pi j_k t))\sin(2\pi j_m t)\,\mathrm{d}t, \qquad m\in\{1,\dots,k\}\\
\implies\quad & 2c_{j_m}=0, \qquad m\in\{1,\dots,k\},
\end{aligned}$$

using the fact that

$$\int_0^1 \sin(2\pi mt)\sin(2\pi nt)\,\mathrm{d}t=\begin{cases}2, & m=n,\\ 0, & m\neq n.\end{cases}$$

Thus every finite subset of $\mathscr{F}$ is linearly independent, and so $\mathscr{F}$ is linearly independent.
Now, if $\mathbb{T}$ is an arbitrary continuous time-domain with a nonempty interior, there is some interval $[a,b]\subseteq\mathbb{T}$. The sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ defined by $f_j(t)=\sin(\frac{2\pi j(t-a)}{b-a})$ is linearly independent. $\square$

Norms on continuous-time signal spaces

- Now let us investigate the topological structure of continuous-time signal spaces.
- Note that if $f \in \mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T},\mathbb{F})$ then we can define

$$\|f\|_\infty = \sup\{|f(t)| \mid t \in \mathbb{T}\}$$

  and

$$\|f\|_p = \Big(\int_{\mathbb{T}} |f(t)|^p \,\mathrm{d}t\Big)^{1/p}, \qquad p \in [1,\infty).$$

  Moreover, $\|\cdot\|_p$, $p \in [1,\infty]$ is a norm.
- *Question:* What are the properties of the normed vector space $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$?
- *Useful analogy:* One can think of $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F})$ as playing the same rôle for continuous-time signals as $\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F})$ plays for discrete-time signals.

Continuous-time signals with the ∞-norm

- We begin by studying the ∞-norm. It turns out that convergence in this norm is known to you from your past life.

Definition (Pointwise and uniform convergence)

Let $\mathbb{T}$ be a continuous time-domain and let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of $\mathbb{F}$-valued signals on $\mathbb{T}$.

(i) The sequence ***converges pointwise*** to $f\colon \mathbb{T} \to \mathbb{F}$ if, for each $t \in \mathbb{T}$ and for each $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that $|f(t) - f_j(t)| < \epsilon$ for all $j \geq N$.

(ii) The sequence ***converges uniformly*** to $f\colon \mathbb{T} \to \mathbb{F}$ if, for each $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ such that $|f(t) - f_j(t)| < \epsilon$ for all $t \in \mathbb{T}$ and for all $j \geq N$.

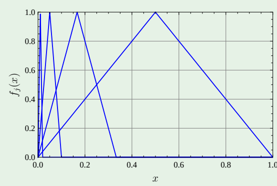
Continuous-time signals with the ∞-norm

Example

On $\mathbb{T} = [0, 1]$ we consider the sequence of $\mathbb{R}$-valued signals defined by

$$f_j(t) = \begin{cases} 2jt, & t \in [0, \frac{1}{2j}], \\ -2jt + 2, & t \in (\frac{1}{2j}, \frac{1}{j}], \\ 0, & t \in (\frac{1}{j}, 1]. \end{cases}$$

The sequence looks like this:



Continuous-time signals with the ∞-norm

Example (cont'd)

We claim that the sequence converges pointwise to the limit signal $f(t) = 0$, $t \in \mathbb{T}$. Since $f_j(0) = 0$ for all $j \in \mathbb{Z}_{>0}$, obviously the sequence converges to 0 at $t = 0$. For $t \in (0, 1]$, if $N \in \mathbb{Z}_{>0}$ satisfies $\frac{1}{N} < t$ then we have $f_j(t) = 0$ for $j \geq N$. Thus we do indeed have pointwise convergence to f.

We claim that the sequence does not converge uniformly. Indeed, for any positive $\epsilon < 1$, we see that $f_j(\frac{1}{2j}) = 1 > \epsilon$ for every $j \in \mathbb{Z}_{>0}$. This prohibits our asserting the existence of $N \in \mathbb{Z}_{>0}$ such that $|f(t) - f_j(t)| < \epsilon$ for every $t \in [0, 1]$, provided that $j \geq N$. Thus convergence is indeed not uniform.

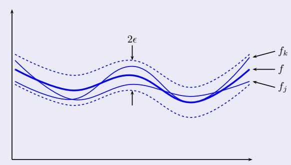
Proposition

For a continuous time-domain $\mathbb{T}$, a sequence $(f_j)_{j \in \mathbb{Z}_{>0}}$ of bounded signals converges uniformly to f if and only if the sequence converges to a bounded signal f in the norm $\|\cdot\|_\infty$.

Continuous-time signals with the ∞-norm

"Proof".

This is more or less simply a matter of working through the definitions. Consider the following picture as a guide:



The point is that both uniform convergence and convergence in the ∞-norm require that the signals in the sequence get close to the limit signal *over the entire domain*. □

- So convergence in the norm $\|\cdot\|_\infty$ is "simple."

Reading for Lecture 12

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-1.3.1, IV-1.3.2, and IV-1.3.3.
2. Sections III-3.3.1 and III-3.8.5.
3. Sections I-3.6.1 and I-3.6.2.

Lecture 13
Continuous-time signals with the ∞-norm

- Next let us consider completeness using the ∞-norm. To do this it is convenient to define

$$\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}) = \{f \in \mathsf{C}^0(\mathbb{T};\mathbb{F}) \mid f \text{ is bounded}\}.$$

- We have $\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}) = \mathsf{C}^0(\mathbb{T};\mathbb{F})$ if and only if $\mathbb{T}$ is compact. Some intuition for this...
  1. If $\mathbb{T}$ is not bounded, then a continuous signal can grow attain arbitrarily large values for large times. Consider, for example, $\mathbb{T} = \mathbb{R}_{\geq 0}$ and $f(t) = t$.
  2. If $\mathbb{T}$ is bounded but not closed, then at an open boundary of $\mathbb{T}$, a continuous signal can attain arbitrarily large values. Consider, for example, $\mathbb{T} = (0, 1]$ and $f(t) = \frac{1}{t}$.
- *Useful analogy:* One can think of $\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F})$ as playing the same rôle for continuous-time signals as $\ell^\infty(\mathbb{T};\mathbb{F})$ plays for discrete-time signals. We shall not deal too often with either class of signals, but they do provide some helpful structure.

Continuous-time signals with the ∞-norm

Theorem

For a continuous time-domain $\mathbb{T}$, $(\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ is a Banach space.

Proof.

Let $(f_j)_{j\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in $(\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$.

1. *The sequence converges pointwise:* Let $t \in \mathbb{T}$ and let $\epsilon \in \mathbb{R}_{>0}$. Since $(f_j)_{j\in\mathbb{Z}_{>0}}$ is Cauchy, there exists $N \in \mathbb{Z}_{>0}$ such that $\|f_j - f_k\|_\infty < \epsilon$ for $j, k \geq N$. We then have

$$|f_j(t) - f_k(t)| \leq \|f_j - f_k\|_\infty < \epsilon$$

   for $j, k \geq N$. This shows that the sequence $(f_j(t))_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathbb{F}$. Thus it converges to some limit which we denote by $f(t)$. This defines a signal $f\colon \mathbb{T} \to \mathbb{F}$.
2. *The sequence converges uniformly:* Let $\epsilon \in \mathbb{R}_{>0}$ and let $N_1 \in \mathbb{Z}_{>0}$ have the property that $\|f_j - f_k\|_\infty < \frac{\epsilon}{2}$ for $j, k \geq N_1$. Then $|f_j(t) - f_k(t)| < \frac{\epsilon}{2}$ for $j, k \geq N_1$ and for each $t \in \mathbb{T}$. Now let $t \in \mathbb{T}$ and let $N_2 \in \mathbb{Z}_{>0}$ have the

Continuous-time signals with the ∞-norm

Proof (cont'd).

property that $|f(t) - f_k(t)| < \frac{\epsilon}{2}$ for $k \geq N_2$. Then, for $j \geq N_1$, we compute

$$|f(t) - f_j(t)| \leq |f_j(t) - f_k(t)| + |f(t) - f_k(t)| < \epsilon,$$

where $k \geq \max\{N_1, N_2\}$. Thus we have uniform convergence, as desired.

3. *The limit signal is bounded:* Let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $|f(t) - f_j(t)| < 1$ for $j \geq N$ and for all $t \in \mathbb{T}$, this being possible by uniform convergence. Then, for $t \in \mathbb{T}$,

$$|f(t)| \leq |f(t) - f_N(t)| + |f_N(t)| < \|f_N\|_\infty + 1,$$

giving boundedness of f.

4. *The limit signal is continuous:* Let $\epsilon \in \mathbb{R}_{>0}$. By uniform convergence, there exists $N \in \mathbb{Z}_{>0}$ such that $|f(t) - f_j(t)| < \frac{\epsilon}{3}$ for all $t \in \mathbb{T}$ and $j \geq N$. Now fix $t_0 \in \mathbb{T}$, and consider the $N \in \mathbb{Z}_{>0}$ just defined. By continuity of f_N, there exists $\delta \in \mathbb{R}_{>0}$ such that, if $t \in \mathbb{T}$ satisfies $|t - t_0| < \delta$, then $|f_N(t) - f_N(t_0)| < \frac{\epsilon}{3}$. Then, for $t \in I$ satisfying $|t - t_0| < \delta$, we have

Continuous-time signals with the ∞-norm

Proof (cont'd).

$$\begin{aligned}
|f(t) - f(t_0)| &= |(f(t) - f_N(t)) + (f_N(t) - f_N(t_0)) + (f_N(t_0) - f(t_0))| \\
&\leq |f(t) - f_N(t)| + |f_N(t) - f_N(t_0)| + |f_N(t_0) - f(t_0)| \\
&< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon,
\end{aligned}$$

where we have used the triangle inequality. Since this argument is valid for any $t_0 \in \mathbb{T}$, it follows that f is continuous.

This shows that the sequence $(f_j)_{j \in \mathbb{Z}_{>0}}$ converges to the bounded continuous signal f in the norm $\|\cdot\|_\infty$. □

- You may have seen the following result which was arrived at along the way.

Corollary

A uniformly convergent sequence of continuous bounded functions has a bounded continuous limit.

Reading for Lecture 13

Material related to this lecture can be found in the following sections of the course notes:

1. Section I-3.6.2.
2. Section III-3.8.4.
3. Section IV-1.3.2.

Lecture 14
Continuous-time signals with the ∞-norm

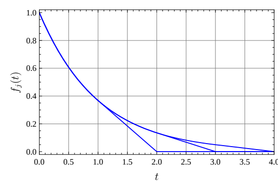
- If $\mathbb{T}$ is compact then
$$\mathsf{C}^0_{\mathsf{cpt}}(\mathbb{T};\mathbb{F}) = \mathsf{C}^0_{\mathsf{bdd}}(\mathbb{T};\mathbb{F}) = \mathsf{C}^0(\mathbb{T};\mathbb{F}).$$
Thus these three (identical) normed vector spaces are Banach spaces when equipped with the ∞-norm.
- What if $\mathbb{T}$ is not compact?
- We claim that if $\mathbb{T}$ is not compact, then $(\mathsf{C}^0_{\mathsf{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ is not a Banach space. Here is some intuition for this.

Continuous-time signals with the ∞-norm

1. If $\mathbb{T}$ is unbounded, then one can have a Cauchy sequence of signals whose domain grows in size so that the limit signal is not compact. For example, consider $\mathbb{T} = \mathbb{R}_{\geq 0}$ and a sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F})$ defined by

$$f_j(t) = \begin{cases} \mathrm{e}^{-t}, & t \in [0,j], \\ \mathrm{e}^{-j}(j+1-t), & t \in (j,j+1], \\ 0, & t \in (j+1,\infty). \end{cases}$$

Here are the first three terms in the sequence:



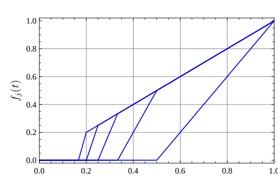
One can show that this is a nonconvergent Cauchy sequence; this is done rather in the same way we showed the existence of a nonconvergent Cauchy sequence in $(\mathbb{F}^\infty_0, \|\cdot\|_\infty)$. The main idea here is that the "limit signal," $f(t) = \mathrm{e}^{-t}$, does not have compact support.

Continuous-time signals with the ∞-norm

2. If $\mathbb{T}$ is bounded but not closed, then the construction is more subtle. Consider $\mathbb{T} = (0,1]$ and the sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ defined by

$$f_j(t) = \begin{cases} t, & t \in [\frac{1}{j}, 1], \\ (1+j)t - 1, & t \in [\frac{1}{j+1}, \frac{1}{j}), \\ 0, & t \in [0, \frac{1}{j+1}). \end{cases}$$

Here are the first five terms in the sequence:



One may show that this is a nonconvergent Cauchy sequence. The issue is that the "limit signal," $f(t) = t$, does not have compact support. Make sure you understand why this signal does not have compact support.

Continuous-time signals with the ∞-norm

- The preceding discussion hopefully makes the following assertion believable.

Theorem

The completion of $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ *is* $(\mathsf{C}^0_0(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$, *where*

$$\mathsf{C}^0_0(\mathbb{T};\mathbb{F}) = \{f \in \mathsf{C}^0(\mathbb{T};\mathbb{F}) \mid \textit{for every } \epsilon \in \mathbb{R}_{>0} \textit{ there exists a compact set } K \subseteq \mathbb{T} \textit{ such that } \{t \in \mathbb{T} \mid |f(t)| \geq \epsilon\} \subseteq K\}.$$

- This theorem is analogous to the statement for discrete-time signals that $(\mathsf{c}_0(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$ is the completion of $(\mathsf{c}_{\mathrm{fin}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$.
- If $\mathbb{T}$ is closed and infinite then

$$\mathsf{C}^0_0(\mathbb{T};\mathbb{F}) = \left\{f \in \mathsf{C}^0(\mathbb{T};\mathbb{F}) \;\middle|\; \lim_{|t|\to\infty} f(t) = 0\right\}.$$

- The point is that, when using the ∞-norm, completions are comprised of spaces of continuous signals.

Continuous-time signals with the p-norms

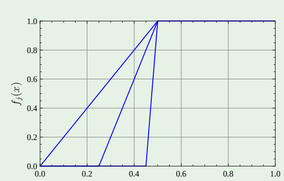
- Now we consider the completion of $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$ where $p \in [1,\infty)$.
- Even when $\mathbb{T}$ is compact, these spaces are not complete.

Example

Let $\mathbb{T} = [0,1]$ and consider the sequence of signals $(f_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{C}^0(\mathbb{T};\mathbb{R})$ defined by

$$f_j(t) = \begin{cases} 1, & t \in [\frac{1}{2}, 1], \\ 2jt + 1 - j, & t \in [\frac{1}{2} - \frac{1}{2j}, \frac{1}{2}), \\ 0, & t \in [0, \frac{1}{2} - \frac{1}{2j}). \end{cases}$$

Here are a few terms in the sequence:



Continuous-time signals with the p-norms

Example (cont'd)

Claim

The sequence is Cauchy with respect to the norm $\|\cdot\|_1$.

Proof.

Suppose that $k \geq j$ so that the signal $f_j - f_k$ is positive. A simple computation gives

$$\begin{aligned}\|f_j - f_k\|_1 &= \int_0^1 |f_j(t) - f_k(t)|\,\mathrm{d}t = \int_0^1 (f_j(t) - f_k(t))\,\mathrm{d}t = \int_0^1 f_j(t)\,\mathrm{d}t - \int_0^1 f_k(t)\,\mathrm{d}t \\ &= \frac{1}{2} + \frac{1}{4j} - \frac{1}{2} - \frac{1}{4k} = \frac{1}{4j} - \frac{1}{4k}.\end{aligned}$$

For N large enough that $\frac{1}{2N} < \epsilon$ we have $\|f_j - f_k\|_1 < \epsilon$ for $j, k \geq N$, showing that the sequence is Cauchy. □

Continuous-time signals with the p-norms

Example (cont'd)

Claim

The sequence does not converge in the norm $\|\cdot\|_1$.

Proof.

We claim that if $(f_j)_{j\in\mathbb{Z}_{>0}}$ converges to $g \in \mathrm{C}^0([0,1];\mathbb{R})$ then $g(t) = 0$ for $t \in [0, \frac{1}{2})$ and $g(t) = 1$ for $t \in (\frac{1}{2}, 1]$. To see that the first part of this assertion holds, suppose that $g(t_0) > 0$ for some $t_0 \in [0, \frac{1}{2})$. Then, by continuity of g, there exists a closed interval I such that $g(t) \geq \frac{1}{2}g(t_0)$ for all $t \in I$ and such that $I \subseteq [0, \frac{1}{2})$. Choose N sufficiently large that $\frac{1}{2} - \frac{1}{2j}$ lies to the right of I for $j \geq N$. Then $f_j(t) = 0$ for all $t \in I$ and for $j \geq N$. Therefore, for $j \geq N$,

$$\|g - f_j\|_1 = \int_0^1 |g(t) - f_j(t)|\,\mathrm{d}t \geq \int_I |g(t) - f_j(t)|\,\mathrm{d}t \geq \tfrac{1}{2}g(t_0)\ell(I),$$

Continuous-time signals with the p-norms

Example (cont'd)

Proof (cont'd).

where $\ell(I)$ is the length of I. This precludes $(f_j)_{j\in\mathbb{Z}_{>0}}$ from converging to g. We arrive at a similar conclusion in a similar manner if $g(t_0) < 0$ for some $t_0 \in [0, \frac{1}{2})$. Thus $g(t) = 0$ for $t \in [0, \frac{1}{2})$. The proof that $g(t) = 1$ for $t \in (\frac{1}{2}, 1]$ follows in a similar way.
The proof is concluded by noting that there is no continuous signal taking the values of g on $[0, 1] \setminus \{\frac{1}{2}\}$. □

Thus $(\mathsf{C}^0([0, 1]; \mathbb{R}), \|\cdot\|_1)$ is not complete.

- The same sequence in the example can be used to show that $(\mathsf{C}^0([0, 1]; \mathbb{R}), \|\cdot\|_p)$ is not complete for $p \in [1, \infty)$.

Reading for Lecture 14

Material related to this lecture can be found in the following sections of the course notes:

1. Sections III-3.3.1, III-3.8.4, and III-3.8.7.
2. Sections IV-1.3.2 and IV-1.3.3.

Lecture 15

Continuous-time signals with the p-norms

- *Question:* What is the completion of $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$ for a continuous time-domain $\mathbb{T}$ and for $p \in [1,\infty)$?
- To answer this question, we need to ascertain the sorts of signals that Cauchy sequences in $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$ are "trying" to converge to.
- Our example above suggests that we need to allow discontinuous signals as limits of Cauchy sequences. But "how discontinuous" can these limit signals be?
- *Possible answer:* Based on what we have seen for normed vector spaces of discrete-time signals, a perfectly reasonable guess for the completion of $(\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$ is

$$\mathsf{R}^{(p)}(\mathbb{T};\mathbb{F}) = \left\{ f\colon \mathbb{T} \to \mathbb{F} \;\middle|\; \int_{\mathbb{T}} |f(t)|^p \,\mathrm{d}t < \infty \right\}.$$

  Here "R" denotes "Riemann integrable."
- There are at least two problems with this guess:
  1. $\|\cdot\|_p$ does not define a norm on $\mathsf{R}^{(p)}(\mathbb{T};\mathbb{F})$ since there are nonzero signals with zero seminorm (this is easy to deal with);
  2. $\mathsf{R}^{(p)}(\mathbb{T};\mathbb{F})$ is not complete in the p-norm (dealing with this requires effort).

The Riemann integral

- We sketch the definition of the Riemann integral.
- We first consider a bounded function f defined on $[a,b]$.
  1. Partition $[a,b]$ into a disjoint collection of intervals $I_1,\dots,I_k$. Call this partition P.
  2. Define two functions $s_-(f,P)$ and $s_+(f,P)$ by asking that, if $x \in I_j$, then

$$s_-(f,P)(x) = c_{j,-} \triangleq \inf\{f(y) \mid y \in I_j\}, \quad s_+(f,P)(x) = c_{j,+} \triangleq \sup\{f(y) \mid y \in I_j\}.$$

     These are the ***lower step function*** and the ***upper step function*** for the function and the partition.
  3. Define

$$A_-(f,P) = \sum_{j=1}^{k} c_{-,j}\ell(I_j), \quad A_+(f,P) = \sum_{j=1}^{k} c_{+,j}\ell(I_j).$$

     These are the ***lower Riemann sum*** and the ***upper Riemann sum*** for the function and the partition.

The Riemann integral

4. The function f is ***Riemann integrable*** if, for every $\epsilon \in \mathbb{R}_{>0}$, there exists a partition P such that $A_+(f, P) - A_-(f, P) < \epsilon$.
5. If f is Riemann integrable and if P_n is a partition such that $A_+(f, P_n) - A_-(f, P_n) < \frac{1}{n}$, then the ***Riemann integral*** of f is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} A_-(f, P_n) = \lim_{n\to\infty} A_+(f, P_n),$$

and the two limits may be verified to indeed be equal, and moreover independent of the choice of sequence of partitions $(P_n)_{n\in\mathbb{Z}_{>0}}$.

Examples (Riemann integrable functions)

1. *Fact:* Continuous functions with compact support are Riemann integrable.
2. If $x_0 \in [a, b]$ consider $f\colon [a, b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}$$

To show that this function is Riemann integrable, consider the sequence

The Riemann integral

Examples (Riemann integrable functions (cont'd))

of upper and lower step functions

$$s_{-,n}(x) = 0, \quad s_{+,n}(x) = \begin{cases} 1, & |x - x_0| < \frac{1}{n}, \\ 0, & \text{otherwise}. \end{cases}$$

We obviously have

$$\lim_{n\to\infty} \int_a^b s_{-,n}(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b s_{+,n}(x)\,\mathrm{d}x = 0,$$

so the function is Riemann integrable with Riemann integral zero.

3. The Riemann integral is linear. Thus if f and g are Riemann integrable and if $c \in \mathbb{R}$ then $f + g$ and cf are Riemann integrable and

$$\int_I (f(x) + g(x))\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x + \int_I g(x)\,\mathrm{d}x, \quad \int_I cf(x)\,\mathrm{d}x = c\int_I f(x)\,\mathrm{d}x.$$

Reading for Lecture 15

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-1.3.2 and IV-1.3.3.
2. Sections I-3.4.1 and I-3.4.2.

Lecture 16
The Riemann integral

- Now let I be an arbitrary interval and first suppose that f is nonnegative-valued and such that $f|K$ is Riemann integrable for every compact interval $K \subseteq I$.
  1. If $I = [a, b]$ then the Riemann integral of f is as defined in the preceding section.
  2. If $I = (a, b]$ then define
  $$\int_a^b f(x)\,\mathrm{d}x = \lim_{r_a \downarrow a} \int_{r_a}^b f(x)\,\mathrm{d}x.$$
  3. If $I = [a, b)$ then define
  $$\int_a^b f(x)\,\mathrm{d}x = \lim_{r_b \uparrow b} \int_a^{r_b} f(x)\,\mathrm{d}x.$$

The Riemann integral

4. If $I = (a,b)$ then define
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{r_a \downarrow a} \int_{r_a}^c f(x)\,\mathrm{d}x + \lim_{r_b \uparrow b} \int_c^{r_b} f(x)\,\mathrm{d}x$$
for some $c \in (a,b)$.
5. If $I = (-\infty, b]$ then define
$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^b f(x)\,\mathrm{d}x.$$
6. If $I = (-\infty, b)$ then define
$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^c f(x)\,\mathrm{d}x + \lim_{r_b \uparrow b} \int_c^{r_b} f(x)\,\mathrm{d}x$$
for some $c \in (-\infty, b)$.

The Riemann integral

7. If $I = [a,\infty)$ then define
$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_a^R f(x)\,\mathrm{d}x.$$
8. If $I = (a,\infty)$ then define
$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{r_a \downarrow a} \int_{r_a}^c f(x)\,\mathrm{d}x + \lim_{R\to\infty} \int_c^R f(x)\,\mathrm{d}x$$
for some $c \in (a,\infty)$.
9. If $I = \mathbb{R}$ then define
$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^c f(x)\,\mathrm{d}x + \lim_{R\to\infty} \int_c^R f(x)\,\mathrm{d}x$$
for some $c \in \mathbb{R}$.

The function f is ***Riemann integrable*** if the appropriate of the above limits is finite. The ***Riemann integral*** is the value of the limit and denoted
$$\int_I f(x)\,\mathrm{d}x.$$

The Riemann integral

- If I is a general interval and if $f\colon I \to \mathbb{R}$ then define

$$f_-(x) = -\min\{f(x), 0\}, \quad f_+(x) = \max\{f(x), 0\}.$$

Then f is ***Riemann integrable*** if both f_- and f_+ are Riemann integrable and the ***Riemann integral*** of f is

$$\int_I f(x)\,\mathrm{d}x = \int_I f_+(x)\,\mathrm{d}x - \int_I f_-(x)\,\mathrm{d}x.$$

Example

Take $I = (0, 1]$ and $f(x) = x^{-1/2}$. Then

$$\lim_{r\downarrow 0} \int_r^1 f(x)\,\mathrm{d}x = \lim_{r\downarrow 0} 2\sqrt{x}\big|_r^1 = 2.$$

Thus f is an example of an unbounded function defined on a noncompact interval that is Riemann integrable.

The Riemann integral

Example (A function that is not Riemann integrable)

Let $(q_j)_{j\in\mathbb{Z}_{>0}}$ be an enumeration of the rational numbers in $[0, 1]$. Let $f_k\colon [0, 1] \to \mathbb{R}$ be defined by

$$f_k(x) = \begin{cases} 1, & x \in \{q_1, \dots, q_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that f_k is Riemann integrable. Indeed, it is a finite sum of functions, each of which takes the value 1 at a point in $[0, 1]$ and is zero elsewhere. Each such function is Riemann integrable, as we have seen.

The pointwise limit function, $f(x) = \lim_{k\to\infty} f_k(x)$, is not Riemann integrable. Indeed,

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

If s_- is a lower step function for f then we claim that $s_-(x) = 0$ for every $x \in [0, 1]$. Indeed, if I is any interval in the partition defining s_- then there is an

The Riemann integral

Example (A function that is not Riemann integrable (cont'd))

irrational number in I since there is a rational number in any nontrivial interval. Thus $s_-(x) \leq 0$ for $x \in I$. Moreover, since $f(x) \geq 0$ for every $x \in I$, we also have $s_-(x) \geq 0$ for all $x \in I$. Thus $s_-(x) = 0$ for all $x \in I$, and since this holds for every interval of the partition, $s_-(x) = 0$ for all $x \in [0,1]$. The same sort of argument, using the fact that every nontrivial interval contains an irrational number, shows that if s_+ is an upper step function for f, then $s_+(x) = 1$ for every $x \in [0,1]$. Therefore, $A_+(f,P) - A_-(f,P) = 1$ for every partition P. This precludes f from being Riemann integrable.

The Riemann integral

- Note that not only is f above not Riemann integrable, it is the pointwise limit of Riemann integrable functions. Thus we do not have

$$\lim_{k\to\infty} \int_0^1 f_k(x)\,\mathrm{d}x = \int_0^1 \lim_{k\to\infty} f_k(x)\,\mathrm{d}x,$$

since the left limit is zero, whereas the expression on the right makes no sense, since the integral is not defined. This is problematic. Moreover...

Proposition

For $p \in [1,\infty)$, there exists a Cauchy sequence in $(\mathsf{C}^0([0,1];\mathbb{R}); \|\cdot\|_p)$ for which there is no limit in $(\mathsf{R}^{(p)}([0,1];\mathbb{R}), \|\cdot\|_p)$.

See course notes for proof.

This is essentially Proposition III-2.1.12 in the course notes. The sequence of functions in the proof of that result are not continuous, but can be made continuous without changing the result. □

The Riemann integral

- The upshot is that there is an inherent defect in using the Riemann integral to define the p-norm. This can be rectified by using a more general notion of the integral, the "Lebesgue integral." This we now sketch.

Lebesgue outer measure

- The objective of measure theory is to measure the "size" of sets according to a rule which has useful properties.
- It is convenient to use the ***extended real numbers***, $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ on which we inherit the expected operations of addition, multiplication, absolute value, and order. Note that $\overline{\mathbb{R}}$ is not a field since, for example, $\infty + (-\infty)$ is not defined. We also denote $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$.
- The collection of subsets of a set X we denote by $\mathbf{2}^X$.

Definition (Outer measure)

An ***outer measure*** on a set X is a map $\mu^* \colon \mathbf{2}^X \to \overline{\mathbb{R}}_{\geq 0}$ with the following properties:

(i) $\mu^*(\varnothing) = 0$;

(ii) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$;

(iii) $\mu^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$ for $A_j \subseteq X$, $j \in \mathbb{Z}_{>0}$.

Lebesgue outer measure

- The idea is that an outer measure provides some rule for measuring the "size" of subsets of a set X. Note that the axioms of an outer measure all agree with one's intuition about how "size" should behave. What is not so clear is that these are the *right* axioms; other axioms might also seem just as reasonable. This is an important thing to think about, but the only (lame) answer we will give here is that these rules give the desired conclusions in examples, mainly in the only example we consider here.
- Speaking of this...

Definition (Lebesgue outer measure)

The ***Lebesgue outer measure*** of a set $A \subseteq \mathbb{R}$ is

$$\lambda^*(A) = \inf\left\{ \sum_{j=1}^{\infty}(b_j - a_j) \;\middle|\; A \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j) \right\}.$$

- In words, the Lebesgue outer measure of A is the infimum of the total length of any countable collection of open intervals which cover A.

Reading for Lecture 16

Material related to this lecture can be found in the following sections of the course notes:

1. Section I-3.4.4.
2. Section III-2.3.2 (parts of this).
3. Section III-2.4.1 (parts of this).

Lecture 17
Lebesgue outer measure

Theorem

The Lebesgue outer measure is an outer measure, and if $I \subseteq \mathbb{R}$ is an interval, then $\lambda^*(I)$ is the length of I.

Partial proof.

We will only prove the last part of this as it give some idea of the Lebesgue outer measure works.
We first take $I = [a, b]$. We may cover $[a, b]$ by $\{(a - \frac{\epsilon}{4}, b + \frac{\epsilon}{4})\} \cup ((0, \frac{\epsilon}{2^{j+1}}))_{j \in \mathbb{Z}_{>0}}$. Therefore,

$$\lambda^*([a, b]) \leq (b + \tfrac{\epsilon}{4} - a + \tfrac{\epsilon}{4}) + \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j+1}} = b - a + \epsilon,$$

where we use Example I-2.4.2–1. Since ϵ can be made arbitrarily small we have $\lambda^*([a, b]) \leq b - a$. Also, suppose that $((a_j, b_j))_{j \in \mathbb{Z}_{>0}}$ covers $[a, b]$. By the Heine–Borel Theorem, there exists $n \in \mathbb{Z}_{>0}$ such that $[a, b] \subseteq \cup_{j=1}^{n}(a_j, b_j)$. Among the intervals $((a_j, b_j))_{j=1}^{n}$ we can pick a subset $((a_{j_k}, b_{j_k}))_{k=1}^{m}$ with the

Lebesgue outer measure

Partial proof (cont'd).

properties that $a \in (a_{j_1}, b_{j_1})$, $b \in (a_{j_m}, b_{j_m})$, and $b_{j_k} \in (a_{j_{k+1}}, b_{j_{k+1}})$. (Do this by choosing (a_{j_1}, b_{j_1}) such that a is in this interval. Then choose (a_{j_2}, b_{j_2}) such that b_{j_1} is in this interval. Since there are only finitely many intervals covering $[a, b]$, this can be continued and will stop by finding an interval containing b.) These intervals then clearly cover $[a, b]$ and also clearly satisfy $\sum_{k=1}^{m}|b_{j_k} - a_{j_k}| \geq b - a$ since they overlap. Thus we have

$$b - a \leq \sum_{k=1}^{m}|b_{j_k} - a_{j_k}| \leq \sum_{j=1}^{\infty}|b_j - a_j|.$$

Thus $b - a$ is a lower bound for the set

$$\Big\{\sum_{j=1}^{\infty}|b_j - a_j| \;\Big|\; [a, b] \subseteq \bigcup_{j \in \mathbb{Z}_{>0}}(a_j, b_j)\Big\}.$$

Since $\lambda^*([a, b])$ is the greatest lower bound we have $\lambda^*([a, b]) \geq b - a$. Thus $\lambda^*([a, b]) = b - a$.

Lebesgue outer measure

Partial proof (cont'd).

Now let I be a bounded interval and denote $\mathrm{cl}(I) = [a, b]$. Since $I \subseteq [a, b]$ we have $\lambda^*(I) \leq b - a$ using monotonicity of λ^*. If $\epsilon \in \mathbb{R}_{>0}$ we may find a closed interval $J \subseteq I$ for which the length of I exceeds that of J by at most ϵ. Since $\lambda^*(J) \leq \lambda^*(I)$ by monotonicity of λ^*, it follows that $\lambda^*(I)$ differs from the length of I by at most ϵ. Thus

$$\lambda^*(I) \geq \lambda^*(J) = b - a - \epsilon.$$

Since $\epsilon \in \mathbb{R}_{>0}$ is arbitrary $\lambda^*(I) \geq b - a$, showing that $\lambda^*(I) = b - a$, as desired. Finally, if I is unbounded then for any $M \in \mathbb{R}_{>0}$ we may find a closed interval $J \subseteq I$ for which $\lambda^*(J) > M$. Since $\lambda^*(I) \geq \lambda^*(J)$ by monotonicity of λ^*, this means that $\lambda^*(I) = \infty$. □

- *The point:* The Lebesgue outer measure provides a way of measuring the "size" of a subset of $\mathbb{R}$. Moreover, it has the following properties:
  1. it is a fairly natural definition;
  2. it agrees with our preexisting notions of "size."

Measures and measures from outer measures

- The Lebesgue outer measure seems like such a natural construction that one might be tempted to stop in one's quest to measure "size." However, the Lebesgue outer measure has a problem. The problem is that there exists sets $A, B \subseteq \mathbb{R}$, with $A \cap B = \varnothing$, for which

$$\lambda^*(A \cup B) \neq \lambda^*(A) + \lambda^*(B).$$

  That is, it can happen that the sum of the outer measure of disjoint sets is not equal to the sum of the union of the sets. This is not a property which "size" should have. To rectify this is not so straightforward.
- The starting point is the following definition.

Definition (σ-algebra)

Let X be a set. A ***σ-algebra*** on X is a family $\mathscr{A}$ of subsets of X such that

(i) $X \in \mathscr{A}$,

(ii) $A \in \mathscr{A}$ implies $X \setminus A \in \mathscr{A}$, and

(iii) $\cup_{j \in \mathbb{Z}_{>0}} A_j \in \mathscr{A}$ for any countable family $(A_j)_{j \in \mathbb{Z}_{>0}}$ of subsets.

Measures and measures from outer measures

- *Exercise:* $\cap_{j\in\mathbb{Z}_{>0}} A_j \in \mathscr{A}$ for any countable family $(A_j)_{j\in\mathbb{Z}_{>0}}$ of subsets.
- The idea is that a σ-algebra will provide a collection of subsets whose size we wish to measure.
- It is true that $\mathbf{2}^X$ is a σ-algebra, but there may well be σ-algebras of interest that are smaller than this. Indeed, we shall be interested in one of these.
- A pair $(X, \mathscr{A})$ is a ***measurable space*** if $\mathscr{A}$ is a σ-algebra on X. Sets in $\mathscr{A}$ are called ***measurable***.

Definition (Measure)

For a set X and a σ-algebra $\mathscr{A}$ on X, a ***measure*** on $\mathscr{A}$ is a map $\mu\colon \mathscr{A} \to \overline{\mathbb{R}}$ such that

(i) $\mu(\varnothing) = 0$;

(ii) $\mu\Big(\bigcup_{j\in\mathbb{Z}_{>0}} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j)$ for every countable family $(A_j)_{j\in\mathbb{Z}_{>0}}$ of pairwise disjoint sets from $\mathscr{A}$.

- A triple $(X, \mathscr{A}, \mu)$ is a ***measure space*** if $\mathscr{A}$ is a σ-algebra on X and if μ is a measure on $\mathscr{A}$.

Measures and measures from outer measures

- Here are some useful properties of measures:
  1. $\mu(A) \leq \mu(B)$ if $A \subseteq B$;
  2. if $\mu(A) < \infty$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$;
  3. for every countable family of subsets $(A_j)_{j\in\mathbb{Z}_{>0}}$ from $\mathscr{A}$ for which $A_j \subseteq A_{j+1}$, $j \in \mathbb{Z}_{>0}$, $\mu\Big(\bigcup_{j\in\mathbb{Z}_{>0}} A_j\Big) = \lim_{j\to\infty} \mu(A_j)$;
  4. for every countable family of subsets $(A_j)_{j\in\mathbb{Z}_{>0}}$ from $\mathscr{A}$ for which $A_j \supseteq A_{j+1}$, $j \in \mathbb{Z}_{>0}$, for which $\mu(A_k) < \infty$ for some $k \in \mathbb{Z}_{>0}$, $\mu\Big(\bigcap_{j\in\mathbb{Z}_{>0}} A_j\Big) = \lim_{j\to\infty} \mu(A_j)$.
- A common way of arriving at a measure is to begin with an outer measure. The key is the following definition.

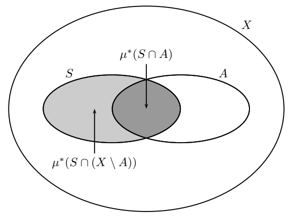
Definition (Measurable set)

Let μ^* be an outer measure on a set X. A subset $A \subseteq X$ is $\boldsymbol{\mu^*}$***-measurable*** if, for every subset $S \subseteq X$, we have

$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap (S \setminus A)).$$

Measures and measures from outer measures

- The idea behind a measurable set is depicted as follows:



- Because the defining property of a measurable set is so natural, what is not obvious is that nonmeasurable sets exist. But they do for the Lebesgue outer measure. But before we get to that. . .

Reading for Lecture 17

Material related to this lecture can be found in the following sections of the course notes:

1. Section III-2.3.2 (parts of this).
2. Section III-2.4.1 (parts of this).

Lecture 18
Measures and measures from outer measures

Theorem (Measures from outer measures)

Let μ^* be an outer measure on a set X, denote by $\mathscr{M}(X, \mu^*)$ the family of μ^* measurable sets, and denote $\mu = \mu^* | \mathscr{M}(X, \mu^*)$. Then $(X, \mathscr{M}(X, \mu^*), \mu)$ is a measure space.

- It may be the case that for a given outer measure, *all* subsets are μ^*-measurable. The slightly odd thing is when there are sets that are *not* μ^*-measurable. However, this can happen.

Lebesgue measure

- The family of λ^*-measurable subsets of $\mathbb{R}$ will be denoted by $\mathscr{L}(\mathbb{R})$. Subsets from $\mathscr{L}(\mathbb{R})$ will be said to be ***Lebesgue measurable*** and λ is the ***Lebesgue measure***.
- It is a fact that $\mathscr{L}(\mathbb{R}) \subset \mathbf{2}^{\mathbb{R}}$; that is to say, there are subsets of $\mathbb{R}$ that are not Lebesgue measurable. The construction of such a set is not obviously done, and indeed relies crucially on the Axiom of Choice. We will never see in this course an example of a set that is not Lebesgue measurable.
- Here is a large class of Lebesgue measurable sets.

Definition (Borel sets)

Denote by $\mathscr{B}(\mathbb{R})$ the smallest σ-algebra on $\mathbb{R}$ containing the open subsets of $\mathbb{R}$. Subsets from $\mathscr{B}(\mathbb{R})$ are called ***Borel sets***.

- $\mathscr{B}(\mathbb{R}) \subset \mathscr{L}(\mathbb{R})$ and the only subsets of $\mathbb{R}$ you will see in this course are Borel sets.
- To construct the Borel sets, one first takes complements and countable unions of open sets. Then one takes complements and countable unions of the resulting sets. One continues this process "transfinitely often" (this is a lot of times) to arrive at the Borel sets.

Lebesgue measure

- Interesting facts about $\mathscr{L}(\mathbb{R})$ and $\mathscr{B}(\mathbb{R})$.
  1. $\operatorname{card}(\mathscr{B}(\mathbb{R})) = \operatorname{card}(\mathbb{R})$ and $\operatorname{card}(\mathscr{L}(\mathbb{R})) = \operatorname{card}(\mathbf{2}^{\mathbb{R}})$. Thus there are *many* more Lebesgue measurable sets than Borel sets.
  2. If $A \in \mathscr{L}(\mathbb{R})$, then there exists $B \in \mathscr{B}(\mathbb{R})$ and $Z \in \mathscr{L}(\mathbb{R})$ such that $\lambda(Z) = 0$ and $A = B \cup Z$. Thus Lebesgue measurable sets are "almost" Borel.
  3. If $A \in \mathscr{L}(\mathbb{R})$ then there exists a sequence $(U_j)_{j\in\mathbb{Z}_{>0}}$ of open sets and a sequence $(K_j)_{j\in\mathbb{Z}_{>0}}$ of compact sets such that $A \subseteq U_j$ and $A \supseteq K_j$, $j \in \mathbb{Z}_{>0}$, and for which
  $$\lambda(A) = \lim_{j\to\infty} \lambda(U_j) = \lim_{j\to\infty} \lambda(K_j).$$
  Thus sets from $\mathscr{L}(\mathbb{R})$ are "approximated" both by open and compact sets.
  4. λ is "translation-invariant." Thus if A is Lebesgue measurable, then any translate of A is also Lebesgue measurable, and has the same Lebesgue measure as A.
- To illustrate how to use Lebesgue measure, we give an example.

Lebesgue measure

Example

We claim that $\mathbb{Q}$ is a Lebesgue measurable set having measure zero.
To see that the set is measurable, note that any singleton $\{x_0\} \subseteq \mathbb{R}$ is closed. Therefore, its complement is open. Therefore, it is a Borel set. Borel sets are Lebesgue measurable and so singletons are Lebesgue measurable. Since $\mathbb{Q}$ is countable, it is a disjoint countable union of singletons: $\mathbb{Q} = \cup_{j\in\mathbb{Z}_{>0}}\{q_j\}$. Since the countable union of Borel sets is a Borel set (the Borel sets form a σ-algebra), it follows that $\mathbb{Q}$ is a Borel set. Since Borel sets are Lebesgue measurable, $\mathbb{Q}$ is Lebesgue measurable.
We show that $\mathbb{Q}$ has measure zero in two ways. First of all, note that any singleton $\{x_0\}$ has measure zero. Indeed, for $\epsilon \in \mathbb{R}_{>0}$, $\{x_0\}$ can be covered by $(x_0 - \frac{\epsilon}{4}, x_0 + \frac{\epsilon}{4}) \cup \{(0, \frac{\epsilon}{2^{j+1}})\}_{j\in\mathbb{Z}_{>0}}$. The total length of these open intervals is

$$\frac{\epsilon}{2} + \frac{1}{2}\sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon.$$

Lebesgue measure

Example (cont'd)

Since ϵ is arbitrary the infimum of the lengths of the open intervals covering $\{x_0\}$ must be zero. That is $\lambda(\{x_0\}) = 0$. Now, using the fact that the measure of countable disjoint union is the sum of the measures of the sets in the union (i.e., property (ii) of measures), we have

$$\lambda(\mathbb{Q}) = \sum_{j=1}^{\infty} \lambda(\{q_j\}) = 0.$$

Let us also prove "directly" that $\lambda(\mathbb{Q}) = 0$. For $\epsilon \in \mathbb{R}_{>0}$, define $I_j = (q_j - \frac{\epsilon}{2^{j+1}}, q_j + \frac{\epsilon}{2^{j+1}})$. Thus I_j is an open interval of length $\frac{\epsilon}{2^j}$ centred at the jth rational number in our list. Therefore, $\mathbb{Q} \subseteq \cup_{j=1}^{\infty} I_j$. Now note that

$$\sum_{j=1}^{\infty} \ell(I_j) = \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon.$$

Since ϵ is arbitrary, it follows that the infimum of the lengths of the open intervals covering $\mathbb{Q}$ is zero. Thus $\lambda(\mathbb{Q}) = 0$.

Lebesgue measure

- *Useful notation:* If a property P of the real numbers has the property that there exists a set $A \subseteq X$ for which $\mu(A) = 0$, and such that P holds for all $x \in X \setminus A$, then property P holds ***almost everywhere*** or ***a.e.***.
- *The point of our excursion into measure theory:* There exists a σ-algebra $\mathscr{L}(\mathbb{R})$ on $\mathbb{R}$ whose sets can be usefully measured, the measure of these sets is prescribed by a function $\lambda \colon \mathscr{L}(\mathbb{R}) \to \overline{\mathbb{R}}_{\geq 0}$ having a natural definition which agrees with what we expect the size of a subset of $\mathbb{R}$ to be.

Reading for Lecture 18

Material related to this lecture can be found in the following sections of the course notes:

1. Section III-2.3.2 (parts of this).
2. Section III-2.4.1 (parts of this).
3. Section III-2.4.2 (parts of this).

Lecture 19
The Lebesgue integral

- We use the Lebesgue measure to define an integral. We first define the class of functions whose integrals we will define.
- If $I \subseteq \mathbb{R}$ is an interval, $\mathscr{L}(I) = \{A \cap I \mid A \in \mathscr{L}(\mathbb{R})\}$.

Definition (Measurable function)

For an interval $I \subseteq \mathbb{R}$, a function $f: I \to \overline{\mathbb{R}}$ is ***Lebesgue measurable***, or just ***measurable***, if

$$f^{-1}([-\infty, b]) = \{x \in I \mid f(x) \leq b\}$$

is Lebesgue measurable.

- A function $f: I \to \mathbb{R}$ can be shown to be measurable if and only if $f^{-1}(B) \in \mathscr{L}(I)$ for every Borel set B.
- Compare this to: A function $f: I \to \mathbb{R}$ is continuous if and only if $f^{-1}(U)$ is open for every open set U.

The Lebesgue integral

- Just as nonmeasurable sets are not so easy to come up with, nonmeasurable functions are not so easy to come up with.
- All continuous functions are measurable.
- For the purposes of this course, unless you are asked to explicitly show that a function is measurable, you may assume that all functions you see are measurable.
- Our definition of the integral now follows three stages.
  1. *Integral of simple functions:* For a set X and $A \subseteq X$, the ***characteristic function*** of S is $\chi_A \colon X \to \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

A function $f \colon I \to \mathbb{R}$ is ***simple*** if there exist pairwise disjoint sets $A_1, \dots, A_n \in \mathscr{L}(I)$ and $c_1, \dots, c_n \in \mathbb{R}$ such that $f = \sum_{j=1}^n c_j \chi_{A_j}$. The ***Lebesgue integral*** of such a simple function is

$$\int_I f \, \mathrm{d}\lambda = \sum_{j=1}^n c_j \lambda(A_j).$$

The Lebesgue integral

  2. *Integral of positive measurable functions:* Positive measurable functions are approximated by simple functions.

Proposition

If $f \colon I \to \overline{\mathbb{R}}_{\geq 0}$ is measurable then there exists a sequence $(g_j)_{j \in \mathbb{Z}_{>0}}$ is nonnegative-valued simple functions such that

(i) $g_{j+1}(x) \geq g_j(x)$ *for all* $x \in I$ *and* $j \in \mathbb{Z}_{>0}$ *and*

(ii) $\lim_{j \to \infty} g_j(x) = f(x)$.

Then, for $f \colon I \to \overline{\mathbb{R}}_{\geq 0}$ measurable, we define the ***Lebesgue integral*** of f by

$$\int_I f \, \mathrm{d}\lambda = \sup \left\{ \int_I g \, \mathrm{d}\lambda \;\middle|\; g \text{ a simple function with } g(x) \leq f(x),\ x \in I \right\},$$

this definition making sense by the preceding proposition.

The Lebesgue integral

3. *Integral of arbitrary measurable function:* If $f\colon I \to \overline{\mathbb{R}}$ is measurable, define

$$f_-(x) = -\min\{f(x), 0\}, \quad f_+(x) = \max\{f(x), 0\}.$$

Then

(i) the Lebesgue integral of f ***exists*** if either $\int_I f_+ \, d\lambda$ or $\int_I f_- \, d\lambda$ is finite,

(ii) f is ***Lebesgue integrable*** if both $\int_I f_+ \, d\lambda$ and $\int_I f_- \, d\lambda$ are finite, and

(iii) the Lebesgue integral of f ***does not exist*** if both $\int_I f_+ \, d\lambda$ and $\int_I f_- \, d\lambda$ are infinite.

If the Lebesgue integral of f exists then we define

$$\int_I f \, d\lambda = \int_I f_+ \, d\lambda - \int_I f_- \, d\lambda,$$

which is the ***Lebesgue integral*** of f.

The Lebesgue integral

- Here are some facts about the Lebesgue integral.
  1. If a function $f\colon I \to \mathbb{R}$ is Riemann integrable, then it is Lebesgue integrable, and, moreover,
  $$\int_I f(x) \, dx = \int_I f \, d\lambda.$$
  So all the integrals you know how to compute still have the same value.
  2. The characteristic function of the set of rationals in $[0, 1]$ (see Slide 143) is Lebesgue integrable but not Riemann integrable.
  3. It is hard to come up with a bounded function on a compact interval that is not Lebesgue integrable; this is connected with the fact that measurable sets are complicated.
  4. The Lebesgue integral is linear.
  5. There is a very powerful theorem, called the Dominated Convergence Theorem (see course notes), which allows the swapping of limits and integrals. Thus it gives conditions when we can write
  $$\lim_{j\to\infty} \int_I f_j \, d\lambda = \int_I \lim_{j\to\infty} f_j \, d\lambda.$$
  The example from Slide 143 shows that such a theorem does not hold for the Riemann integral.

Reading for Lecture 19

Material related to this lecture can be found in the following sections of the course notes:

1. Section III-2.6.1 (only small parts of this).
2. Sections III-2.6.5, III-2.7.1, III-2.9.2, and III-2.9.3.

Lecture 20
Continuous-time signals with the p-norms (cont'd)

- For a continuous time-domain $\mathbb{T}$ define

$$\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F}) = \{f\colon \mathbb{T}\to\mathbb{F} \mid t\mapsto |f(t)|^p \text{ is Lebesgue integrable}\}.$$

- For $f \in \mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})$ define

$$\|f\|_p = \left(\int_I |f|^p \,\mathrm{d}\lambda\right)^{1/p}.$$

- Note that $\|\cdot\|_p$ is not a norm since there are nonzero functions with zero "norm" (in fact, $\|\cdot\|_p$ is a seminorm).
- Define an equivalence relation on $\mathsf{L}^{(p)}(\mathbb{T};\mathbb{F})$ by saying that $f \sim g$ if

$$\lambda\{t \in \mathbb{T} \mid f(t) \neq g(t)\} = 0.$$

- One can show that $f \sim g$ if and only if $\|f - g\|_p$.
- Denote by $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ the set of equivalence classes under this equivalence relation and denote $\|[f]\|_p = \|f\|_p$.

Continuous-time signals with the p-norms (cont'd)

- For us, a factor of primary importance for the Lebesgue integral is the following result.

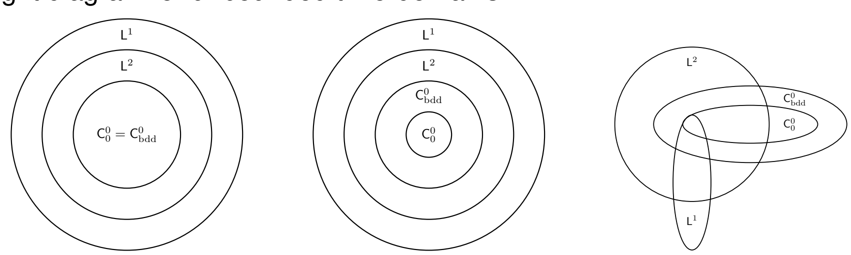
Theorem

$(\mathsf{L}^p(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$ is a Banach space, and is moreover the completion of $(\mathsf{C}^0_{\mathsf{cpt}}(\mathbb{T};\mathbb{F}), \|\cdot\|_p)$.

- Note that the completion of $(\mathsf{C}^0_{\mathsf{cpt}}(\mathbb{T};\mathbb{F}); \|\cdot\|_p)$ consists, not of signals, but of equivalence classes of signals. Nonetheless, we shall denote elements of $\mathsf{L}^p(\mathbb{T};\mathbb{F})$ by f and not by $[f]$.
- The above theorem is false if one uses the Riemann integral instead of the Lebesgue integral, cf. the proposition on Slide 146.

Continuous-time signals with the p-norms (cont'd)

- The inclusion relations for the most important (for us) continuous-time signal spaces is as follows, the left diagram for compact time-domains, the middle diagram for bounded but not compact time-domains, and the right diagram for unbounded time-domains.



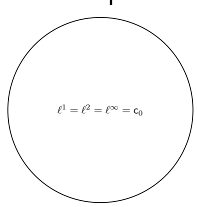
Summary of life until now

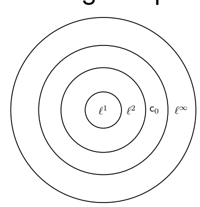
- This completes the first part of the course. Let us recap.
- We have introduced structure for spaces of discrete- and continuous-time signals. This structure is algebraic (a structure of a vector space) and topological (the structure of a norm).
  1. Algebraically: For all signal spaces, we know their dimension.
  2. Topologically:
     - ★ We carefully studied completeness and completions of all signal spaces. We have at the end of the day arrived at a collection of Banach spaces of signals that we will use for the rest of the course.
     - ★ Some of the norms we use are derived from inner products. We shall not say much about this here, but this will be followed up on in MATH 335.
- Here is a summary of signal spaces we will use in the remainder of the course.

Discrete-time	Continuous-time
$(\ell^1(\mathbb{T};\mathbb{F}), \|\cdot\|_1)$	$(\mathsf{L}^1(\mathbb{T};\mathbb{F}), \|\cdot\|_1)$
$(\ell^2(\mathbb{T};\mathbb{F}), \|\cdot\|_2)$	$(\mathsf{L}^2(\mathbb{T};\mathbb{F}), \|\cdot\|_2)$
$(\ell^\infty(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$	$(\mathsf{C}^0_{\mathrm{bdd}}(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$
$(\mathsf{c}_0(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$	$(\mathsf{C}^0_0(\mathbb{T};\mathbb{F}), \|\cdot\|_\infty)$

Summary of life until now

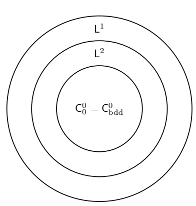
- Here are the relationships for the discrete-time signal spaces:

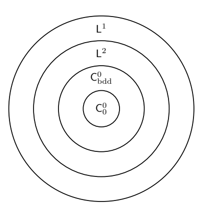


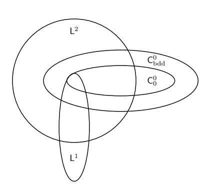


Bounded time-domain   Unbounded time-domain

- Here are the relationships for continuous-time signal spaces:







Compact time-domain   Bounded time-domain   Unbounded time-domain

Reading for Lecture 20

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-1.2.2, IV-1.2.3, IV-1.2.7, IV-1.3.2, IV-1.3.3, and IV-1.3.7.

Lecture 21
Introduction to Fourier transform theory

- Now that we have at our disposal a few classes of signal spaces, we discuss the theory of Fourier transforms for these spaces.
- By taking the Fourier transform of a signal, we change the signal from one where the independent variable is thought of as "time" to one where the independent variable is thought of as "frequency." The motivation for doing this is discussed in Sections IV-2.1–IV-2.4 in the course notes.
- In the discussion of Fourier transform theory we shall consider signals with both continuous and discrete domains. In all cases, we shall consider the time-domain as being totally infinite, and so will consider either $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}(\Delta)$.
- We will consider signals that are either periodic or aperiodic.
- Thus we consider Fourier transforms for four different classes of signals:
    1. continuous-time aperiodic signals;
    2. continuous-time periodic signals;
    3. discrete-time aperiodic signals;
    4. discrete-time periodic signals.

Introduction to Fourier transform theory

- We shall see that there are some sharp similarities with all of these cases, but also some sometimes subtle distinctions.
- The idea in every case is to transform the time-domain signal into a frequency-domain signal whose value at a certain frequency ν represents the "content" of the signal at that frequency.
- This requires that we have a "fundamental" signal at each of the frequencies we consider. The fundamental signals used in Fourier transform theory are the harmonic signals. For simplicity, we assume all signals to be complex (real signals are then a special case) and so the fundamental signal with frequency ν is $t \mapsto e^{2\pi i\nu t}$.
- For each of the four cases we consider different harmonic signals, depending on whether the signal is continuous- or discrete-time, and periodic or aperiodic.
- When the signal is aperiodic, we place no restrictions on the frequency of the harmonic signal. When the signal is T-periodic, we consider only harmonic signals with the same frequency as the signals in the class, i.e., with frequency equal to $\frac{1}{T}$. These latter are of the form $t \mapsto e^{2\pi i n \frac{t}{T}}$, $n \in \mathbb{Z}$.

Introduction to Fourier transform theory

- To understand this for discrete-time signals, note that a discrete-time signal takes values at the times $k\Delta$ for $k \in \mathbb{Z}$ and that a discrete-time signal of period T has the property that $T = N\Delta$ for some $N \in \mathbb{Z}_{>0}$.
- With this in mind, the four types of harmonic signals considered are summarised in the following table.

	Continuous-time	Discrete-time
Period T	$t \mapsto e^{2\pi i n \frac{t}{T}},\ n \in \mathbb{Z}$	$k\Delta \mapsto e^{2\pi i n \frac{k}{N}},\ n \in \mathbb{Z}$
Aperiodic	$t \mapsto e^{2\pi i\nu t},\ \nu \in \mathbb{R}$	$k\Delta \mapsto e^{2\pi i\nu k\Delta},\ \nu \in \mathbb{R}$

- The objective of Fourier transform theory is this.

Problem

Given a continuous or discrete time-domain $\mathbb{T}$ and a T-periodic or aperiodic signal $f\colon \mathbb{T} \to \mathbb{C}$, determine the "content" of the signal at each of the frequencies of the appropriate fundamental harmonic signals.

Introduction to Fourier transform theory

- The question now is, "How should one determine the 'content' of a signal at a prescribed frequency?"
- We first observe that the sets of signals
  1. $\{t \mapsto \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}} \mid n \in \mathbb{Z}\}$ and $\{t \mapsto \mathrm{e}^{2\pi\mathrm{i}\nu t} \mid \nu \in \mathbb{R}\}$ for continuous-time and
  2. $\{k\Delta \mapsto \mathrm{e}^{2\pi\mathrm{i}n\frac{k}{N}} \mid n \in \mathbb{Z}\}$ and $\{k\Delta \mapsto \mathrm{e}^{2\pi\mathrm{i}\nu k\Delta} \mid \nu \in \mathbb{R}\}$ for discrete-time

  are linearly independent. (This is easy to prove for the first set, and less easy for the second set.)
- Now, imagine the following simpler problem. Given a collection of linearly independent vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in \mathbb{F}^n$ and $\boldsymbol{v} \in \mathbb{F}^n$, how might one define the "content" of $\boldsymbol{v}$ in the direction of $\boldsymbol{v}_j$, $j \in \{1, \dots, k\}$.
- One might do this using the standard inner product, saying that the content of $\boldsymbol{v}$ in the direction of $\boldsymbol{v}_j$ is $\langle \boldsymbol{v}, \boldsymbol{v}_j \rangle$.
- One can do this for determining frequency content of a signal as well, taking the appropriate inner product of the signal f with the appropriate harmonic signal.

Introduction to Fourier transform theory

- The inner products we use are those seen previously for our signal spaces; see Slide 102. One has to be careful with periodic signals. Although the signals are defined on $\mathbb{R}$ or $\mathbb{Z}(\Delta)$, the integral or sum defining the inner product is not taken over the entire domain, but only over one period.
- Thus the inner products are:
  1. for continuous-time T periodic signals: $\langle f, g \rangle = \int_0^T f(t)\overline{g(t)}\,\mathrm{d}t$.
  2. for continuous-time aperiodic signals: $\langle f, g \rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}\,\mathrm{d}t$.
  3. for discrete-time $N\Delta$-periodic signals: $\langle f, g \rangle = \Delta \sum_{k=0}^{N-1} f(k\Delta)\overline{g(k\Delta)}$.
  4. for discrete-time aperiodic signals: $\langle f, g \rangle = \Delta \sum_{k\in\mathbb{Z}} f(k\Delta)\overline{g(k\Delta)}$.
- Of course, these inner products do not make sense for arbitrary signals, but we will worry about this later.

Introduction to Fourier transform theory

- Doing this gives:

Signal type	Transform	Name
Continuous-time T-periodic	$\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1}) = \int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t$	CDFT
Continuous-time aperiodic	$\mathscr{F}_{\mathsf{CC}}(f)(\nu) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t$	CCFT
Discrete-time $N\Delta$-periodic	$\mathscr{F}_{\mathsf{DD}}(f)(n(N\Delta)^{-1}) = \Delta\sum_{k=0}^{N-1} f(k\Delta)\mathrm{e}^{-2\pi\mathrm{i}n\frac{k}{N}}$	DDFT
Discrete-time aperiodic	$\mathscr{F}_{\mathsf{DC}}(f)(\nu) = \Delta\sum_{k\in\mathbb{Z}} f(k\Delta)\mathrm{e}^{-2\pi\mathrm{i}\nu k\Delta}$	DCFT

- The names of these transforms are derived as follows. The “CDFT” takes a **c**ontinuous function of time and produces a **d**iscrete function of frequency. Thus it is the “continuous discrete Fourier transform,” or “CDFT.” The reasoning is the same for the other three transforms.

Introduction to Fourier transform theory

- The preceding discussion is mere gibberish unless one addresses a few basic and important questions.
  1. Are the above “frequency content” formulae well-defined?
  2. If they are well-defined, what useful properties do these formulae have, if any?
  3. If one computes the “frequency content” representation of a signal, does this faithfully represent the signal? In particular, can a signal be recovered from its “frequency content” representation?
- We shall devote the remainder of the course to dealing with these questions.

Reading for Lecture 21

Material related to this lecture can be found in the following sections of the course notes:

1. Chapter IV-2.

Lecture 22
The L^1-CDFT (definitions)

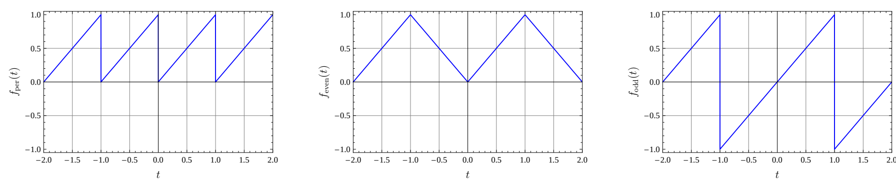
- The first Fourier transform we consider is that for periodic continuous-time signals.
- Let us introduce some notation:

$$\mathsf{L}^p_{\mathrm{per},T}(\mathbb{R};\mathbb{F}) = \{f\colon \mathbb{R}\to\mathbb{F} \mid f \text{ is } T\text{-periodic and } f|[0,T]\in\mathsf{L}^p([0,T];\mathbb{F})\};$$
$$\mathsf{C}^r_{\mathrm{per},T}(\mathbb{R};\mathbb{F}) = \{f\in\mathsf{C}^r(\mathbb{R};\mathbb{F}) \mid f \text{ is } T\text{-periodic}\}.$$

- On $\mathsf{L}^p_{\mathrm{per},T}(\mathbb{R};\mathbb{F})$ we have the norm $\|f\|_p = \left(\int_0^T |f(t)|^p\,\mathrm{d}t\right)^{1/p}$.
- Given a function $f\colon [0,T)\to\mathbb{F}$ we can extend this to a periodic signal in at least three ways.
  1. The ***periodic extension*** $f_{\mathrm{per}}\colon \mathbb{R}\to\mathbb{F}$ is defined by asking that $f_{\mathrm{per}}(t+kT) = f(t)$ for every $t\in[0,T)$ and $k\in\mathbb{Z}$.
  2. The ***even periodic extension*** $f_{\mathrm{even}}\colon \mathbb{R}\to\mathbb{F}$ is first defined on $(-T,T)$ by asking that it be even on this domain. Then it is extended to all of $\mathbb{R}$ by periodic extension.
  3. The ***odd periodic extension*** $f_{\mathrm{odd}}\colon \mathbb{R}\to\mathbb{F}$ is first defined on $(-T,T)$ by asking that it be odd on this domain. Then it is extended to all of $\mathbb{R}$ by periodic extension.

The L^1-CDFT (definitions)

- We depict the periodic, even periodic, and odd periodic extensions of the function defined on $[0,1]$ by $f(t) = t$.



Definition (CDFT)

The ***continuous-discrete Fourier transform*** or ***CDFT*** assigns to $f \in \mathrm{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the signal $\mathscr{F}_{\mathrm{CD}}(f)\colon \mathbb{Z}(T^{-1}) \to \mathbb{C}$ by

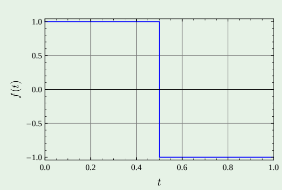
$$\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = \int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t, \qquad n \in \mathbb{Z}.$$

The L^1-CDFT (definitions)

- Note that the condition that $f \in \mathrm{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is precisely the condition that the CDFT can be computed using the given formula (this is because $|\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}| = 1$.
- We shall always think of signals as being $\mathbb{C}$-valued, and so $\mathbb{R}$-valued signals are a special case.

Examples

1. We consider the signal $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \square_{2,1,0}(t) - 1$ and depicted below.



Thus f is the 1-periodic extension of the signal defined on $[0,1]$ by

The L^1-CDFT (definitions)

Examples (cont'd)

$$(f|[0,1])(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ -1, & t \in (\frac{1}{2}, 1]. \end{cases}$$

We compute

$$\mathscr{F}_{\mathrm{CD}}(f)(0) = \int_0^1 f(t)\,\mathrm{d}t = 0.$$

For $n \neq 0$ we have

$$\mathscr{F}_{\mathrm{CD}}(f)(n) = \int_0^1 f(t)\mathrm{e}^{-2\pi \mathrm{i} nt}\,\mathrm{d}t = \int_0^{\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{i} nt}\,\mathrm{d}t - \int_{\frac{1}{2}}^1 \mathrm{e}^{-2\pi \mathrm{i} nt}\,\mathrm{d}t$$
$$= -\frac{\mathrm{e}^{-2\pi \mathrm{i} nt}}{2\pi \mathrm{i} n}\Big|_0^{\frac{1}{2}} + \frac{\mathrm{e}^{-2\pi \mathrm{i} nt}}{2\pi \mathrm{i} n}\Big|_{\frac{1}{2}}^1 = \frac{1 - \mathrm{e}^{\mathrm{i} n\pi}}{2\pi \mathrm{i} n} - \frac{\mathrm{e}^{\mathrm{i} n\pi} - 1}{2\pi \mathrm{i} n} = \mathrm{i}\frac{(-1)^n - 1}{n\pi},$$

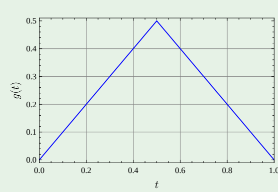
using the identity $\mathrm{e}^{\mathrm{i} n\pi} = (-1)^n$ for $n \in \mathbb{Z}$. Thus we have

The L^1-CDFT (definitions)

Examples (cont'd)

$$\mathscr{F}_{\mathrm{CD}}(f)(n) = \begin{cases} 0, & n = 0, \\ \mathrm{i}\frac{(-1)^n - 1}{n\pi} & \text{otherwise.} \end{cases}$$

2. Next we consider the signal $g = \triangle_{\frac{1}{2},1,0}$ depicted below.



Thus g is the 1-periodic extension of the signal

$$(g|[0,1])(t) = \begin{cases} t, & t \in [0, \frac{1}{2}], \\ 1 - t, & t \in (\frac{1}{2}, 1]. \end{cases}$$

We then compute

The L^1-CDFT (definitions)

Examples (cont'd)

$$\mathscr{F}_{\mathsf{CD}}(g)(0) = \int_0^1 g(t)\,\mathrm{d}t = \int_0^{\frac{1}{2}} t\,\mathrm{d}t + \int_{\frac{1}{2}}^1 (1-t)\,\mathrm{d}t = \frac{t^2}{2}\Big|_0^{\frac{1}{2}} + \Big(t - \frac{t^2}{2}\Big)\Big|_{\frac{1}{2}}^1 = \frac{1}{4},$$

and for $n \neq 0$,

$$\begin{aligned}
\mathscr{F}_{\mathsf{CD}}(g)(n) &= \int_0^1 g(t)\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t = \int_0^{\frac{1}{2}} t\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t + \int_{\frac{1}{2}}^1 (1-t)\mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t \\
&= -\frac{t\mathrm{e}^{-2\pi\mathrm{i}nt}}{2\pi\mathrm{i}n}\Big|_0^{\frac{1}{2}} + \frac{1}{2\pi\mathrm{i}n}\int_0^{\frac{1}{2}} \mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^1 \mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t + \frac{t\mathrm{e}^{-2\pi\mathrm{i}nt}}{2\pi\mathrm{i}n}\Big|_{\frac{1}{2}}^1 - \frac{1}{2\pi\mathrm{i}n}\int_{\frac{1}{2}}^1 \mathrm{e}^{-2\pi\mathrm{i}nt}\,\mathrm{d}t \\
&= -\frac{\mathrm{e}^{-\mathrm{i}n\pi}}{4\mathrm{i}n\pi} + \frac{\mathrm{e}^{-2\pi\mathrm{i}nt}}{4n^2\pi^2}\Big|_0^{\frac{1}{2}} - \frac{\mathrm{e}^{-2\pi\mathrm{i}nt}}{2\pi\mathrm{i}n}\Big|_{\frac{1}{2}}^1 + \frac{1}{2\pi\mathrm{i}n} - \frac{\mathrm{e}^{-\mathrm{i}n\pi}}{4\mathrm{i}n\pi} - \frac{\mathrm{e}^{-2\pi\mathrm{i}nt}}{4n^2\pi^2}\Big|_{\frac{1}{2}}^1
\end{aligned}$$

The L^1-CDFT (definitions)

Examples (cont'd)

$$\begin{aligned}
&= -\frac{\mathrm{e}^{-\mathrm{i}n\pi}}{4\mathrm{i}n\pi} + \frac{\mathrm{e}^{-\mathrm{i}n\pi}}{4n^2\pi^2} - \frac{1}{4n^2\pi^2} - \frac{1}{2\pi\mathrm{i}n} + \frac{\mathrm{e}^{-\mathrm{i}n\pi}}{2\pi\mathrm{i}n} + \frac{1}{2\pi\mathrm{i}n} \\
&\quad - \frac{\mathrm{e}^{-\mathrm{i}n\pi}}{4\mathrm{i}n\pi} - \frac{1}{4n^2\pi^2} + \frac{\mathrm{e}^{-\mathrm{i}n\pi}}{4n^2\pi^2} = \frac{\mathrm{e}^{-\mathrm{i}n\pi} - 1}{2n^2\pi^2} = \frac{(-1)^n - 1}{2n^2\pi^2}.
\end{aligned}$$

We have used the fact that $\mathrm{e}^{-\mathrm{i}n\pi} = (-1)^n$ for $n \in \mathbb{Z}$. Thus we have

$$\mathscr{F}_{\mathsf{CD}}(g)(n) = \begin{cases} \frac{1}{4}, & n = 0, \\ \frac{(-1)^n - 1}{2n^2\pi^2}, & \text{otherwise.} \end{cases}$$

- Sometimes for $\mathbb{R}$-valued signals, rather than using the complex exponential one uses sine's and cosine's. Thus one uses the ***continuous-discrete cosine transform*** or ***CDCT*** which assigns to $f \in \mathsf{L}^1_{\mathsf{per},T}(\mathbb{R};\mathbb{C})$ the signal $\mathscr{C}_{\mathsf{CD}}(f)\colon \mathbb{Z}_{\geq 0}(T^{-1}) \to \mathbb{C}$ by

The L^1-CDFT (definitions)

$$\mathscr{C}_{\mathsf{CD}}(f)(nT^{-1}) = \int_0^T f(t)\cos(2\pi n\tfrac{t}{T})\,\mathrm{d}t, \qquad n \in \mathbb{Z}_{\geq 0},$$

and the ***continuous-discrete sine transform*** or ***CDST*** which assigns to $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the signal $\mathscr{S}_{\mathsf{CD}}(f)\colon \mathbb{Z}_{>0}(T^{-1}) \to \mathbb{C}$ by

$$\mathscr{S}_{\mathsf{CD}}(f)(nT^{-1}) = 2\int_0^T f(t)\sin(2\pi n\tfrac{t}{T})\,\mathrm{d}t, \qquad n \in \mathbb{Z}_{>0}.$$

- By Euler's formula:

Proposition

(i) $\mathscr{F}_{\mathsf{CD}}(0) = \mathscr{C}_{\mathsf{CD}}(f)(0)$;

(ii) $\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1}) = \mathscr{C}_{\mathsf{CD}}(f)(nT^{-1}) - \mathrm{i}\mathscr{S}_{\mathsf{CD}}(f)(nT^{-1})$ *and* $\mathscr{F}_{\mathsf{CD}}(f)(-nT^{-1}) = \mathscr{C}_{\mathsf{CD}}(f)(nT^{-1}) + \mathrm{i}\mathscr{S}_{\mathsf{CD}}(f)(nT^{-1})$ *for every* $n \in \mathbb{Z}_{>0}$;

(iii) $\mathscr{C}_{\mathsf{CD}}(f)(nT^{-1}) = \frac{1}{2}(\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1}) + \mathscr{F}_{\mathsf{CD}}(f)(-nT^{-1}))$ *for every* $n \in \mathbb{Z}_{\geq 0}$;

(iv) $\mathscr{S}_{\mathsf{CD}}(f)(nT^{-1}) = \frac{1}{2\mathrm{i}}(\mathscr{F}_{\mathsf{CD}}(f)(-nT^{-1}) - \mathscr{F}_{\mathsf{CD}}(f)(nT^{-1}))$ *for every* $n \in \mathbb{Z}_{>0}$.

The L^1-CDFT (properties)

- Now let us turn to some of the properties of the CDFT.
- First some things more or less elementary. Recall from Slide 14 the reparameterisations τ_a and σ of the time-domain $\mathbb{R}$.
- Let us also define $\overline{\mathscr{F}}_{\mathsf{CD}}(f)(nT^{-1}) = \int_0^T f(t)\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t$.

Proposition

(i) $\overline{\mathscr{F}_{\mathsf{CD}}(f)} = \overline{\mathscr{F}}_{\mathsf{CD}}(\bar{f})$;

(ii) $\mathscr{F}_{\mathsf{CD}}(\sigma^* f) = \sigma^*(\mathscr{F}_{\mathsf{CD}}(f)) = \overline{\mathscr{F}}_{\mathsf{CD}}(f)$;

(iii) if f is even (resp. odd) then $\mathscr{F}_{\mathsf{CD}}(f)$ is even (resp. odd);

(iv) if f is real and even (resp. real and odd) then $\mathscr{F}_{\mathsf{CD}}(f)$ is real and even (resp. imaginary and odd);

(v) $\mathscr{F}_{\mathsf{CD}}(\tau_a^* f)(nT^{-1}) = \mathrm{e}^{-2\pi\mathrm{i}n\frac{a}{T}}\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1})$.

The L^1-CDFT (properties)

Theorem (Riemann–Lebesgue Lemma)

If $f \in \mathsf{L}^1([a,b];\mathbb{C})$ *then*

$$\lim_{|n|\to\infty} \int_a^b f(t)\mathrm{e}^{2\pi \mathrm{i} n \frac{t}{T}}\,\mathrm{d}t = 0.$$

In particular, if $(\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1}))_{n\in\mathbb{Z}}$ *are the values of the CDFT of* $f \in \mathsf{L}^1_{\mathsf{per},T}(\mathbb{R};\mathbb{C})$ *then* $\lim_{|n|\to\infty}|\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1})| = 0.$

- An immediate consequence of this is that the CDFT is a map $\mathscr{F}_{\mathsf{CD}} \colon \mathsf{L}^1_{\mathsf{per},T}(\mathbb{R};\mathbb{C}) \to \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$.
- In fact, one can easily show that the CDFT is a continuous linear map (see Slide 94) from $(\mathsf{L}^1_{\mathsf{per},T}(\mathbb{R};\mathbb{C}), \|\cdot\|_1)$ to $(\mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C}), \|\cdot\|_\infty)$; this is Corollary IV-5.1.10 in the course notes.

Reading for Lecture 22

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-1.1.6 and IV-1.3.4.
2. Sections IV-5.1.1 and IV-5.1.2.

Lecture 23
Differentiation and the CDFT

- There are some interesting and useful relationships between the character of a signal and the character of its CDFT.
- The simplest relationships involve the differentiability of f.
- Recall that a function $f\colon [a,b] \to \mathbb{R}$ is ***piecewise continuous*** if there exists a partition of $[a,b]$ into disjoint subintervals $I_1, \dots, I_k$ such that $f|I_j$ is continuous for each $j \in \{1, \dots, k\}$ and such that the limit of the values of the function is defined as one approaches the endpoints of each of the subintervals.

Proposition (The CDFT and differentiation)

Suppose that $f \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and suppose that there exists a piecewise continuous signal $f'\colon [0,T] \to \mathbb{C}$ with the property that

$$f(t) = f(0) + \int_0^t f'(\tau)\,\mathrm{d}\tau, \qquad t \in [0,T].$$

Then

$$\mathscr{F}_{\mathrm{CD}}(f'_{\mathrm{per}})(nT^{-1}) = \frac{2\pi \mathrm{i} n}{T}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}), \qquad n \in \mathbb{Z}.$$

Differentiation and the CDFT

Proof.

Let $(t_0, t_1, \dots, t_k)$ be the endpoints of a partition having the property that f' is continuous on each subinterval (t_j, t_{j-1}), $j = 1, \dots, k$. Integration by parts of the expression for $\mathscr{F}_{\mathrm{CD}}(f')(nT^{-1})$ on (t_j, t_{j-1}) gives

$$\int_{t_{j-1}}^{t_j} f'(t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t = f(t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\Big|_{t_{j-1}}^{t_j} + \frac{2\pi \mathrm{i} n}{T}\int_{t_{j-1}}^{t_j} f(t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t.$$

Over the entire interval $[0,T]$ we then have

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(f'_{\mathrm{per}})(nT^{-1}) &= \int_0^T f'(t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t = \sum_{j=1}^k \int_{t_{j-1}}^{t_j} f'(t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t \\ &= \frac{2\pi \mathrm{i} n}{T}\int_0^T f(t)\mathrm{e}^{-2\pi \mathrm{i} n\frac{t}{T}}\,\mathrm{d}t = \frac{2\pi \mathrm{i} n}{T}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}),\end{aligned}$$

using the fact that f is continuous and that $f(0) = f(T)$. $\square$

Differentiation and the CDFT

Example

Consider the signals f and g whose CDFT's we computed starting on Slide 187. Note that $g \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ satisfies

$$g(t) = g(0) + \int_0^t f(s)\,\mathrm{d}s, \qquad t \in [0,1].$$

Therefore, we must have

$$\mathscr{F}_{\mathsf{CD}}(f)(n) = \frac{2\pi\mathrm{i}n}{T}\mathscr{F}_{\mathsf{CD}}(g)(n), \qquad n \in \mathbb{Z}.$$

Recalling that

$$\mathscr{F}_{\mathsf{CD}}(f)(n) = \begin{cases} 0, & n=0, \\ \mathrm{i}\frac{(-1)^n-1}{n\pi} & \text{otherwise,} \end{cases} \qquad \mathscr{F}_{\mathsf{CD}}(g)(n) = \begin{cases} \frac{1}{4}, & n=0, \\ \frac{(-1)^n-1}{2n^2\pi^2}, & \text{otherwise,} \end{cases}$$

we see that the conclusions of the differentiation result are verified.

Differentiation and the CDFT

- Applying the preceding result iteratively for signals that are more than once differentiable gives the following.

Corollary (The CDFT and higher-order derivatives)

Suppose that $f \in \mathsf{C}^{r-1}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ for $r \in \mathbb{Z}_{>0}$ and suppose that there exists a piecewise continuous signal $f^{(r)}\colon [0,T] \to \mathbb{C}$ with the property that

$$f^{(r-1)}(t) = f^{(r-1)}(0) + \int_0^t f^{(r)}(\tau)\,\mathrm{d}\tau.$$

Then

$$\mathscr{F}_{\mathsf{CD}}(f^{(r)}_{\mathrm{per}})(nT^{-1}) = \left(\frac{2\pi\mathrm{i}n}{T}\right)^r \mathscr{F}_{\mathsf{CD}}(f)(nT^{-1}).$$

- We then have the following facts, some of which we will not get to for a few lectures.
  1. If $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then the Fourier coefficients satisfy

     $$\lim_{|n|\to\infty} |\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1})| = 0.$$

     This is the Riemann–Lebesgue Lemma.

Differentiation and the CDFT

2. If $f \in \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then $\mathscr{F}_{\mathrm{CD}}(f) \in \ell^2(\mathbb{Z}(T^{-1});\mathbb{C})$. This will be discussed further when we discuss the L^2-CDFT.
3. If f satisfies the conditions of the differentiation result above, then $(\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$ as we shall show when we discuss uniform convergence of Fourier series later. Note that this is a stronger condition on the coefficients than one gets from a signal simply being in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$.
4. If $f \in \mathsf{C}^r_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then the CDFT of f satisfies
$$\lim_{|n|\to\infty} |n^r \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| = 0.$$
5. If $f \in \mathsf{C}^\infty_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is infinitely differentiable then the Fourier coefficients satisfy
$$\lim_{|n|\to\infty} |n^k \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| = 0$$
for any $k \in \mathbb{Z}_{\geq 0}$.

CDFT inversion (warm up)

- Next we turn to a very important matter, that of understanding whether a signal is determined by its CDFT.
- More precisely, given two distinct signals $f, g \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, is it ensured that $\mathscr{F}_{\mathrm{CD}}(f) \neq \mathscr{F}_{\mathrm{CD}}(g)$? In other words, is $\mathscr{F}_{\mathrm{CD}}$ injective?

Theorem (CDFT is injective)

$\mathscr{F}_{\mathrm{CD}} \colon \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \to \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$ *is injective.*

Proof.

(Buckle up, it's going to be a hard one.) We give a lemma dealing with a rather strange looking T-periodic signal. We define $F^{\mathrm{per}}_{T,N} \colon \mathbb{R} \to \mathbb{C}$ by given by

$$F^{\mathrm{per}}_{T,N}(t) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{|n|\leq k} \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

Let us record some useful facts about the signal $F^{\mathrm{per}}_{T,N}$.

CDFT inversion (warm up)

Proof (cont'd).

Lemma

The following statements hold:

(i) we have

$$F^{\mathrm{per}}_{T,N}(t) = \begin{cases} \dfrac{1}{N}\dfrac{\sin^2\big(\pi N \frac{t}{T}\big)}{\sin^2(\pi\frac{t}{T})}, & t \neq 0, \\ N, & t = 0; \end{cases}$$

(ii) $F^{\mathrm{per}}_{T,N}$ has period T;

(iii) $F^{\mathrm{per}}_{T,N}(t) \in \mathbb{R}_{\geq 0}$ for $t \in \mathbb{R}$;

(iv) $\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F^{\mathrm{per}}_{T,N}(t)\,\mathrm{d}t = 1$;

(v) for each $\alpha \in \mathbb{R}_{>0}$ we have $\lim_{N\to\infty}\int_{\alpha \leq |t| \leq \frac{T}{2}} F^{\mathrm{per}}_{T,N}(t)\,\mathrm{d}t = 0$.

CDFT inversion (warm up)

Proof (cont'd).

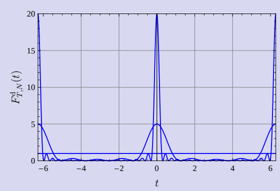
Sketch of proof.

(i) This is a tedious computation involving trigonometric and exponential manipulations; see the course text for the details.
(ii) This is an easy consequence of the definition of $F^{\mathrm{per}}_{T,N}$.
(iii) This too is an easy consequence of the definition of $F^{\mathrm{per}}_{T,N}$.
(iv) You can ask your favourite symbolic manipulation program to do this integral for you.
(v) One can do the estimates here to prove this rigorously, and this is done in the course notes. However, it is more insightful to look at the graph of $F^{\mathrm{per}}_{T,N}$ as $N \to \infty$. In the figure below we show the graph for $N \in \{1, 5, 20\}$, noting that the peak at $t = 0$ gets larger as N gets larger.

CDFT inversion (warm up)

Proof (cont'd).

Sketch of proof (cont'd).



The point is that as $N \to \infty$ the integral of $F^{\text{per}}_{T,N}$ over a period remains constant, with value 1, but the function gets more and more concentrated around $t = 0$. Thus the integral away from an interval around 0 goes to zero as $N \to \infty$. (If you find that as unsatisfying as I do, please look at the formal estimates in the course notes).
The signal $F^{\text{per}}_{T,N}$ is called the ***periodic Fejér kernel***. □

CDFT inversion (warm up)

Proof (cont'd).

Lemma

Let $f_1, f_2 \in \mathrm{C}^0_{\text{per},T}(\mathbb{R};\mathbb{F})$ and suppose that $\mathscr{F}_{\text{CD}}(f_1)(nT^{-1}) = \mathscr{F}_{\text{CD}}(f_2)(nT^{-1})$ for all $n \in \mathbb{Z}$. Then $f_1 = f_2$.

Proof.

By linearity, the lemma amounts to showing that, if $f \in \mathrm{C}^0_{\text{per},T}(\mathbb{R};\mathbb{F})$ and if $\mathscr{F}_{\text{CD}}(f)(nT^{-1}) = 0$ for $n \in \mathbb{Z}$, then $f = 0$. Also by linearity of the integral, we may as well suppose that $\mathbb{F} = \mathbb{R}$, as if this is not so, we may apply the theorem separately to the real and imaginary parts of f.
We suppose that $\mathscr{F}_{\text{CD}}(f)(nT^{-1}) = 0$ for $n \in \mathbb{Z}$ and that $f \neq 0$. By translation (cf. Proposition IV-5.1.6) and multiplication by -1 if necessary, we may suppose that $f(0) \in \mathbb{R}_{>0}$. Note that the relation

CDFT inversion (warm up)

Proof (cont'd).

Proof (cont'd).

$$\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1}) = \int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = 0$$

implies, by periodicity of f and of $\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}$, $n \in \mathbb{Z}$, that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = 0, \quad n \in \mathbb{Z}.$$

By linearity of the integral this means that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)g(t)\,\mathrm{d}t = 0 \tag{1}$$

where g is any finite linear combination of the harmonic signals $(\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}})_{n\in\mathbb{Z}}$.

CDFT inversion (warm up)

Proof (cont'd).

Proof (cont'd).

In particular, we can take $g = F^{\mathrm{per}}_{T,N}$. As f is continuous and $f(0) \neq 0$, we can choose $\alpha \in \mathbb{R}_{>0}$ so that $f(t) \geq \frac{1}{2}f(0)$ for all $t \in [-\alpha, \alpha]$. We then write

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)F^{\mathrm{per}}_{T,N}\,\mathrm{d}t = \int_{-\alpha}^{\alpha} f(t)F^{\mathrm{per}}_{T,N}(t)\,\mathrm{d}t + \int_{\alpha\leq|t|\leq\frac{T}{2}} f(t)F^{\mathrm{per}}_{T,N}(t)\,\mathrm{d}t. \tag{2}$$

It is now easy to see that, if we take the limit as $N \to \infty$, the second integral on the left goes to zero by property (v) of $F^{\mathrm{per}}_{T,N}$ above. The first integral, however, will be positive and bounded from below uniformly in N. This is less easy to see, but it is basically because f is positive and bounded away from zero on the domain of integration, while $F^{\mathrm{per}}_{T,N}$ is also nonnegative, and with an integral approaching 1 over the domain of integration as $N \to \infty$. Thus there is a sufficiently large N such that the integral in (2) is positive, so

CDFT inversion (warm up)

Proof (cont'd).

Proof (cont'd).

contradicting (1). Thus if f is continuous and nonzero, it must hold that $\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) \neq 0$ for some $n \in \mathbb{Z}$, so giving the result. (If you wish to see the detailed estimates, see the course notes.) □

Now let $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. It is sufficient to show that if $\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) = 0$ for every $n \in \mathbb{Z}$ then $f(t) = 0$ for almost every $t \in \mathbb{R}$. Define $F\colon [0,T] \to \mathbb{C}$ by

$$F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau.$$

Since F is the indefinite integral of an integrable function it is continuous (note that this is probably known to you for Riemann integrable functions, but it is also true for Lebesgue integrable functions). Moreover,

CDFT inversion (warm up)

Proof (cont'd).

$$F(T) = \int_0^T f(t)\,\mathrm{d}t = \mathscr{F}_{\mathrm{CD}}(f)(0) = 0 = F(0),$$

and so we conclude that F_{per} is continuous. For $n \neq 0$, using Fubini's Theorem we compute

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(F_{\mathrm{per}})(nT^{-1}) &= \int_0^T F(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = \int_0^T \Big(\int_0^t f(\tau)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}\tau\Big)\mathrm{d}t \\ &= \int_0^T f(\tau)\Big(\int_\tau^T \mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\Big)\,\mathrm{d}\tau \\ &= \frac{T}{2\pi\mathrm{i}n}\Big(\int_0^T f(\tau)\mathrm{e}^{-2\pi\mathrm{i}n\frac{\tau}{T}}\,\mathrm{d}\tau - \mathrm{e}^{-2\pi\mathrm{i}n}\int_0^T f(\tau)\,\mathrm{d}\tau\Big) \\ &= \frac{T}{2\pi\mathrm{i}n}(\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) - \mathscr{F}_{\mathrm{CD}}(f)(0)) = 0,\end{aligned}$$

since $\mathscr{F}_{\mathrm{CD}}(f) = 0$. Now consider the signal $G \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ defined by

CDFT inversion (warm up)

Proof (cont'd).

$G(t) = F_{\mathrm{per}}(t) - \frac{1}{T}\mathscr{F}_{\mathrm{CD}}(F_{\mathrm{per}})(0)$. We have

$$\mathscr{F}_{\mathrm{CD}}(G)(nT^{-1}) = \mathscr{F}_{\mathrm{CD}}(F_{\mathrm{per}})(nT^{-1}) - \frac{1}{T}\mathscr{F}_{\mathrm{CD}}(F_{\mathrm{per}})(0)\int_0^T \mathrm{e}^{-2\pi \mathrm{i} n \frac{t}{T}}\,\mathrm{d}t = 0, \qquad n \in \mathbb{Z}.$$

By our second lemma above, G is zero since it is continuous. Now, since f on the interval $[0, T]$ is the indefinite integral of F, it holds that $f(t) = F'_{\mathrm{per}}(t)$ for almost every $t \in [0, T]$. This will be familiar to you for the Riemann integral—it is the Fundamental Theorem of Calculus—but is also true for the Lebesgue integral. Moreover, since $F'(t) = G'(t) = 0$ for $t \in [0, T]$, we have $f(t) = 0$ for almost every $t \in [0, T]$, as desired. □

- The preceding proof is very detailed. This is not surprising since it is proving something rather important.

Reading for Lecture 23

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-5.1.3 and IV-5.2.1.

Lecture 24
CDFT inversion (warm up)

- One can also show that $\mathscr{F}_{\mathrm{CD}}\colon \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \to \mathsf{c}_0(\mathbb{Z}(T^{-1});\mathbb{C})$ is *not* surjective.
- Thus it is not invertible. However, since it is injective, it possesses a left-inverse $\mathscr{I}_{\mathrm{CD}}\colon \mathsf{c}_0(\mathbb{Z}_{>0}(T^{-1});\mathbb{C}) \to \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ satisfying $\mathscr{I}_{\mathrm{CD}} \circ \mathscr{F}_{\mathrm{CD}}(f) = f$. Thus the left-inverse recovers f from its CDFT.
- Now we might do one of two following things.
    1. Find an explicit left-inverse $\mathscr{I}_{\mathrm{CD}}$ for $\mathscr{F}_{\mathrm{CD}}$. If possible, find one that has useful properties.
    2. Propose a potential left-inverse $\mathscr{I}'_{\mathrm{CD}}$ for $\mathscr{F}_{\mathrm{CD}}$. This may not be an actual left-inverse, but nonetheless it may be the case that for a large class of signals $\mathscr{S}$ it holds that $\mathscr{I}'_{\mathrm{CD}} \circ \mathscr{F}_{\mathrm{CD}}(f) = f$ for $f \in \mathscr{S}$.
- We shall explore both possibilities, starting with the latter.

Fourier series

- If our initial idea for the CDFT is correct, and $\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})$ represents the "content" of f in the direction of the harmonic $t \mapsto \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}$, then perhaps we can write
$$f(t) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}$$
for some coefficients $c_n \in \mathbb{C}$, $n \in \mathbb{Z}$.
- Now, assuming that swapping sums and integrals is permitted (it is not, actually), we compute
$$\int_0^T f(t)\mathrm{e}^{-2\pi\mathrm{i}m\frac{t}{T}}\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} c_n \int_0^T \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}\mathrm{e}^{-2\pi\mathrm{i}m\frac{t}{T}}\,\mathrm{d}t = Tc_m,$$
using the identity
$$\int_0^T \mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}\mathrm{e}^{-2\pi\mathrm{i}m\frac{t}{T}}\,\mathrm{d}t = \begin{cases} T, & n = m, \\ 0, & n \neq m. \end{cases}$$

Fourier series

- Thus we have $c_n = \frac{1}{T}\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1})$.
- Motivated by this, we make the following definition.

Definition (Fourier series)

For $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the ***Fourier series*** of f is the series

$$\mathsf{FS}[f](t) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathsf{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi \mathrm{i} n\frac{t}{T}},$$

disregarding the convergence of this series. The ***real Fourier series*** of f is

$$\mathsf{FS}[f](t) = \frac{1}{2T}\mathscr{C}_{\mathsf{CD}}(0) + \frac{1}{T}\sum_{n=1}^{\infty}\big(\mathscr{C}_{\mathsf{CD}}(f)(nT^{-1})\sin(2\pi n\tfrac{t}{T}) + \mathscr{S}_{\mathsf{CD}}(f)(nT^{-1})\sin(2\pi n\tfrac{t}{T})\big),$$

again disregarding convergence of the series.

Reading for Lecture 24

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-5.2.1 and IV-5.2.2.

Lecture 25
Fourier series

- First of all, we can ask whether the Fourier series is a left-inverse for $\mathscr{F}_{\mathrm{CD}}$. That is, we can ask to what extent it is true that

$$f(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}. \tag{3}$$

  This was addressed definitively in 1926 by A. N. Kolmogorov: There exists $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ such that, for every $t \in \mathbb{R}$, the sequence

$$f_N(t) \triangleq \frac{1}{T}\sum_{n=-N}^{N} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}$$

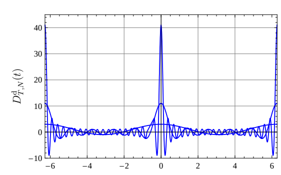
  of partial sums diverges.
- Note that it is not realistic to expect that (3) hold pointwise generally, since signals which differ on a set of zero measure will always have the same Fourier series. Do you see this?
- We wish to determine conditions on a signal f that ensure that, for $t \in \mathbb{R}$, the above sequence of partial sums converges. Moreover, we will examine the relationships between the limit signal and the signal f.

Fourier series

- Let us define the ***discrete Dirichlet kernel*** on $[-\frac{T}{2},\frac{T}{2}]$:

$$D^{\mathrm{per}}_{T,N}(t) = \begin{cases} \frac{\sin((2N+1)\pi\frac{t}{T})}{\sin(\pi\frac{t}{T})}, & t \neq 0, \\ 2N+1, & t = 0. \end{cases}$$

  Here is a sample of the discrete Dirichlet kernel for $N \in \{1, 5, 20\}$.



- The following formula for the partial sums is critical for assessing convergence.

Lemma

For $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *we have* $f_N(t) = \dfrac{1}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-\tau)D^{\mathrm{per}}_{T,N}(\tau)\,\mathrm{d}\tau$, $N \in \mathbb{Z}_{>0}$.

Fourier series

Proof.

The proof is a fairly uninteresting combination of exponential identities and swapping order of integration. We refer to the course notes for details. □

- The integral in the lemma ought to look familiar. It is a convolution, albeit a periodic convolution which you may not have seen. Thus, using convolution notation, $f_N(t) = \frac{1}{T} D^{\mathrm{per}}_{T,N} * f(t)$.
- We get rid of the "$*$" and the factor of $\frac{1}{T}$ and simply write

$$D^{\mathrm{per}}_{T,N} f(t) \triangleq \frac{1}{T} \sum_{n=-N}^{N} \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}) \mathrm{e}^{2\pi \mathrm{i} n \frac{t}{T}}.$$

- Thus $D^{\mathrm{per}}_{T,N} f(t)$ is simply funny notation for a partial sum, the notation being reasonable since it reflects the underlying convolution.

Pointwise convergence of Fourier series

- The key to understanding the pointwise convergence of Fourier series is the following result.

Theorem (Pointwise convergence of Fourier series)

Let $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$, let $t_0 \in \mathbb{R}$, and let $s \in \mathbb{C}$. The following statements are equivalent:

(i) $\lim_{N\to\infty} f_N(t_0) = s$;

(ii) $\displaystyle\lim_{N\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (f(t_0 - t) - s) D^{\mathrm{per}}_{T,N}(t)\, \mathrm{d}t = 0$;

(iii) for each $\epsilon \in (0, \frac{T}{2}]$ *we have* $\displaystyle\lim_{N\to\infty} \frac{1}{T} \int_{-\epsilon}^{\epsilon} (f(t_0 - t) - s) D^{\mathrm{per}}_{T,N}(t)\, \mathrm{d}t = 0$;

(iv) for each $\epsilon \in (0, \frac{T}{2}]$ *we have*

$$\lim_{N\to\infty} \frac{1}{\pi} \int_{-\epsilon}^{\epsilon} (f(t_0 - t) - s) \frac{\sin\big((2N+1)\pi \frac{t}{T}\big)}{t}\, \mathrm{d}t = 0.$$

Pointwise convergence of Fourier series

Proof.

To prove the equivalence of parts (i) and (ii) we note that by our previous lemma on the partial sums (f_N),

$$\lim_{N\to\infty} f_N(t_0) = s \quad \iff \quad \lim_{N\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t_0 - t) D^{\mathrm{per}}_{T,N}(t)\, \mathrm{d}t = s.$$

Applying the lemma when $f\colon t \mapsto 1$ gives

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} D^{\mathrm{per}}_{T,N}(t)\, \mathrm{d}t = 1.$$

Therefore,

$$\lim_{N\to\infty} f_N(t_0) = s \quad \iff \quad \lim_{N\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (f(t_0 - t) - s) D^{\mathrm{per}}_{T,N}(t)\, \mathrm{d}t = 0$$

as desired.

Pointwise convergence of Fourier series

Proof (cont'd).

To prove the equivalence of parts (ii) and (iii) we observe the character of $D^{\mathrm{per}}_{T,N}$ as $N \to \infty$ as depicted in our plots for the discrete Dirichlet kernel. The main point is that for $\epsilon \in (0, \frac{T}{2}]$ the signal

$$t \mapsto \frac{f(t_0 - t) - s}{\sin(\pi \frac{t}{T})}$$

is integrable on $[-\frac{T}{2}, -\epsilon] \cup [\epsilon, \frac{T}{2}]$ and independent of N. Therefore, by the Riemann–Lebesgue Lemma,

$$\lim_{N\to\infty} \int_{\epsilon \le |t| \le \frac{T}{2}} \frac{f(t_0 - t) - s}{\sin(\pi \frac{t}{T})} \sin((2N+1)\pi \frac{t}{T}) = 0.$$

This gives the desired result.

Pointwise convergence of Fourier series

Proof (cont'd).

The equivalence of parts (iii) and (iv) is proved by noting that

$$\frac{\sin((2N+1)\pi\frac{t}{T})}{\sin(\pi\frac{t}{T})} \approx \frac{T}{\pi}\frac{\sin((2N+1)\pi\frac{t}{T})}{t}$$

for t near zero. □

- An important observation is that the pointwise convergence at t_0 for the Fourier series of an L^1 signal depends only on the values of the signal in an arbitrarily small neighbourhood of t_0. This is called the ***localisation principle***.
- Our next chore is to come up with conditions on the signal f in a neighbourhood of t_0 that will ensure that the equivalent conditions of the preceding theorem are met. There are many such conditions, none of these being sharp.

Reading for Lecture 25

Material related to this lecture can be found in the following sections of the course notes:

1. Section IV-5.2.4.

Lecture 26
Pointwise convergence of Fourier series

Theorem (Dini's test)

Let $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *and let* $t_0 \in \mathbb{R}$*. If there exists* $\epsilon \in (0, \frac{T}{2}]$ *so that*

$$\int_{-\epsilon}^{\epsilon} \left| \frac{f(t_0 - t) - s}{t} \right| \mathrm{d}t < \infty,$$

then $\lim_{N\to\infty} D^{\mathrm{per}}_{T,N} f(t_0) = s$.

Proof.

If $t \mapsto \frac{f(t_0-t)-s}{t}$ is in $\mathsf{L}^1([-\epsilon,\epsilon];\mathbb{C})$ then, by the Riemann–Lebesgue Lemma,

$$\lim_{N\to\infty} \int_{-\epsilon}^{\epsilon} \frac{f(t_0 - t) - s}{t} \sin\big((2N+1)\pi\tfrac{t}{T}\big)\, \mathrm{d}t = 0.$$

By our pointwise convergence theorem, the result follows. □

Pointwise convergence of Fourier series

Corollary (Fourier series converge at points of differentiability)

Let $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *and let* $t_0 \in \mathbb{R}$*. If* f *is differentiable at* t_0 *then* $\lim_{N\to\infty} D^{\mathrm{per}}_{T,N} f(t_0) = f(t_0)$.

Proof.

If f is differentiable at t_0 then $\lim_{t\to t_0} \frac{f(t_0-t)-f(t_0)}{t}$ exists and so the function $t \mapsto \frac{f(t_0-t)-f(t_0)}{t}$ is bounded in a neighbourhood of t_0. From this is follows that $\frac{f(t_0-t)-f(t_0)}{t}$ is integrable in a neighbourhood of t_0, and so the corollary follows from Dini's test. □

Corollary (An alternative version of Dini's test)

Let $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ *and let* $t_0 \in \mathbb{R}$*. If there exists* $\epsilon \in (0, \frac{T}{2}]$ *such that*

$$\int_{0}^{\epsilon} \left| \frac{\frac{1}{2}(f(t_0+t) + f(t_0 - t)) - s}{t} \right| \mathrm{d}t < \infty,$$

then $\lim_{N\to\infty} D^{\mathrm{per}}_{T,N} f(t_0) = s$.

Pointwise convergence of Fourier series

- For $f\colon [a,b] \to \mathbb{R}$ and $t_0 \in (a,b)$ define the following limits, when they exist:

$$f(t_0+) = \lim_{\epsilon \downarrow 0} f(t_0+\epsilon), \quad f(t_0-) = \lim_{\epsilon \downarrow 0} f(t_0-\epsilon),$$
$$f'(t_0+) = \lim_{\epsilon \downarrow 0} \frac{f(t_0+\epsilon)-f(t_0)}{\epsilon}, \quad f'(t_0-) = \lim_{\epsilon \downarrow 0} -\frac{f(t_0-\epsilon)-f(t_0)}{\epsilon}.$$

Thus $f(t_0+)$ and $f(t_0-)$ are the left and right limits of the values of f around t_0, and $f'(t_0+)$ and $f'(t_0-)$ are the left and right limits of the values of the derivative of f around t_0.

Theorem (Dirichlet's test)

Let $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and suppose that the limits $f(t_0-)$, $f(t_0+)$, $f'(t_0-)$, and $f'(t_0+)$ exist for $t_0 \in \mathbb{R}$. Then

$$\lim_{N\to\infty} D^{\mathrm{per}}_{T,N} f(t_0) = \frac{1}{2}\big(f(t_0+)+f(t_0-)\big).$$

Pointwise convergence of Fourier series

Examples (Pointwise convergence of Fourier series)

1. We consider the square wave signal f first considered in Slide 187. To check pointwise convergence of the Fourier series for f, we first note that at points that are not of the form $n\frac{1}{2}$, $n \in \mathbb{Z}$, f is differentiable, and so its Fourier series converges to $f(t)$ at these points. The trouble points are, therefore, those of the form $n\frac{1}{2}$, $n \in \mathbb{Z}$. Let us consider $t_0 = 0$, for example.
   (a) *Dini's test:* For pointwise convergence, for $\epsilon \in \mathbb{R}_{>0}$ there must exist $s \in \mathbb{C}$ such that
   $$\int_{-\epsilon}^{\epsilon} \left|\frac{f(-t)-s}{t}\right| \mathrm{d}t < \infty$$
   We have
   $$\int_{-\epsilon}^{\epsilon} \left|\frac{f(-t)-s}{t}\right| \mathrm{d}t = \int_{-\epsilon}^{0} \left|\frac{1-s}{t}\right| \mathrm{d}t + \int_{0}^{\epsilon} \left|\frac{-1-s}{t}\right| \mathrm{d}t.$$
   The two integrals on the right are both finite if and only if $1-s=0$ and $-1-s=0$. There is no s meeting this requirement, so Dini's test does not apply.

Pointwise convergence of Fourier series

Examples (Pointwise convergence of Fourier series (cont'd))

(b) *Alternate Dini's test:* Here, for $\epsilon \in \mathbb{R}_{>0}$, there must exist $s \in \mathbb{C}$ such that

$$\int_0^{\epsilon} \frac{\frac{1}{2}(f(t)+f(-t)) - s}{t}\, dt < \infty.$$

Since $f(t)+f(-t) = 0$ (f is odd), this holds for $s = 0$. Thus the alternate Dini's test gives convergence of the Fourier series to 0 at $t_0 = 0$. This test can be applied at any time $n\frac{1}{2}$, $n \in \mathbb{Z}$, to give the same conclusion for convergence of the Fourier series at this point.

(c) *Dirichlet's test:* Here we note that at times $t_0 = n\frac{1}{2}$, $n \in \mathbb{Z}$, the limits $f(t_0+)$, $f(t_0-)$, $f'(t_0+)$, and $f'(t_0-)$ exist. Moreover, $\frac{1}{2}(f(t_0+)+f(t_0-)) = 0$, and so Dirichlet's test gives convergence of the Fourier series to 0 at these points.

2. We next consider the triangular wave signal g first considered in Slide 189. This signal is differentiable except at times $n\frac{1}{2}$, $n \in \mathbb{Z}$. Therefore, at these times of differentiability we have $\lim_{N\to\infty} D^{\text{per}}_{T,N} g(t_0) = g(t_0)$. For the nondifferentiable times we do the following, using $t_0 = 0$ as typical.

Pointwise convergence of Fourier series

Examples (Pointwise convergence of Fourier series (cont'd))

(a) *Dini's test:* For $\epsilon \in \mathbb{R}_{>0}$ we must have $s \in \mathbb{C}$ such that

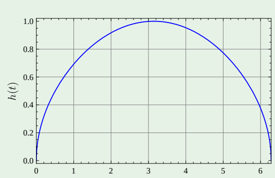
$$\int_{-\epsilon}^{\epsilon} \left|\frac{g(-t)-s}{t}\right| dt = \int_{-\epsilon}^{0} \left|\frac{t-s}{t}\right| dt + \int_0^{\epsilon} \left|\frac{-t-s}{t}\right| dt < \infty.$$

This will hold for $s = 0$, and so gives convergence of the Fourier series to 0 at $t_0 = 0$. This similarly holds for all times $n\frac{1}{2}$, $n \in \mathbb{Z}$.

(b) *Alternate Dini's test:* This amounts to Dini's test since g is even.

(c) *Dirichlet's test:* At $t_0 = n\frac{1}{2}$, $n \in \mathbb{Z}$, the limits $g(t_0+)$, $g(t_0-)$, $g'(t_0+)$, and $g'(t_0-)$ exist, and $\frac{1}{2}(g(t_0+)+g(t_0-)) = 0$. This agrees with Dini.

3. We introduce a new signal, this being the 2π-periodic extension, denoted h, of the signal $t \mapsto (\sin \frac{t}{2})^{1/2}$. Here's the signal:



Pointwise convergence of Fourier series

Examples (Pointwise convergence of Fourier series (cont'd))

This signal is differentiable at all times except those of the form $t_0 = 2n\pi$, $n \in \mathbb{Z}$. At the points of differentiability we have, as usual $\lim_{N\to\infty} D^{\mathrm{per}}_{T,N} h(t) = h(t)$. For the nondifferentiable points, e.g., $t_0 = 0$, we have the following.

(a) *Dini's test:* Because $(\sin \frac{t}{2})^{1/2} \approx \frac{\sqrt{t}}{\sqrt{2}}$ for $t \approx 0$, we must have, for $\epsilon \in \mathbb{R}_{>0}$, an $s \in \mathbb{C}$ such that

$$\int_{-\epsilon}^{0} \left| \frac{\frac{\sqrt{-t}}{\sqrt{2}} - s}{t} \right| \mathrm{d}t + \int_{0}^{\epsilon} \left| \frac{\frac{\sqrt{t}}{\sqrt{2}} - s}{t} \right| \mathrm{d}t < \infty.$$

This holds for $s = 0$, and so Dini allows us to conclude that at times $2\pi n$ the Fourier series converges to 0.

(b) *Alternate Dini's test:* This amounts to Dini's test since h is even.

(c) *Dirichlet's test:* The limits $h'(2\pi n+)$ and $h'(2\pi n)$ do not exist, so Dirichlet's test does not apply.

Pointwise convergence of Fourier series

- In the notes, as Theorem IV-5.2.31, you will find stated a very powerful pointwise convergence test, called Jordan's test. This theorem involves signals with "locally bounded variation." You are not allowed to use this theorem in this course since all signals we encounter will have "locally bounded variation," although you do not know what this means. Thus, the answer, "The Fourier series at t_0 converges to $\frac{1}{2}(f(t_0+) + f(t_0-))$ by Jordan's test," will be a correct answer to most pointwise convergence problems in this course. But you are not allowed to state this answer, unless your answer is accompanied by an illustration that you understand the importance of bounded variation. For example, you could give an explanation of how functions of bounded variation are (1) in the dual space of the set of continuous functions, and (2) how Borel measures are the Radon–Nikodym derivative of functions of bounded variation with respect to the Lebesgue measure.
- So. . . just stick to the Dini tests and Dirichlet's test.

Reading for Lecture 26

Material related to this lecture can be found in the following sections of the course notes:

1. Section IV-5.2.4.

Lecture 27
Uniform convergence of Fourier series

- Next we consider the matter of uniform convergence of the sequence $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ of partial sums for the Fourier series.
- A fundamental theorem is the following.

Theorem (Uniform convergence of Fourier series)

Let $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$. If $\mathscr{F}_{\mathrm{CD}}(f) \in \ell^1(\mathbb{Z}(T^{-1});\mathbb{C})$ then the following statements hold:

(i) $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges uniformly to a (necessarily continuous) T-periodic signal g;

(ii) $f(t) = g(t)$ for almost every $t \in \mathbb{R}$.

Proof.

The nth term in the CDFT for f satisfies

$$|\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}| = |\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})| \triangleq M_n,$$

Uniform convergence of Fourier series

Proof (cont'd).

and furthermore the series $\sum_{n\in\mathbb{Z}} M_n$ converges. This shows that $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges uniformly by the Weierstrass M-test, and we denote the limit signal by g. This signal is continuous as a consequence of the theorem on Slide 121. To see that f and g are equal almost everywhere we first note that, swapping the sum and the integral (this is allowed since the sum converges uniformly),

$$\begin{aligned}\mathscr{F}_{\mathrm{CD}}(g)(nT^{-1}) &= \int_0^T g(t)\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\\ &= \int_0^T \sum_{m\in\mathbb{Z}} \frac{1}{T}\mathscr{F}_{\mathrm{CD}}(f)(mT^{-1})\mathrm{e}^{2\pi\mathrm{i}m\frac{t}{T}}\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t\\ &= \frac{1}{T}\sum_{m\in\mathbb{Z}}\mathscr{F}_{\mathrm{CD}}(f)(mT^{-1})\int_0^T \mathrm{e}^{2\pi\mathrm{i}m\frac{t}{T}}\mathrm{e}^{-2\pi\mathrm{i}n\frac{t}{T}}\,\mathrm{d}t = \mathscr{F}_{\mathrm{CD}}(f)(nT^{-1}).\end{aligned}$$

The theorem now follows directly from the injectivity of the CDFT. □

Uniform convergence of Fourier series

Corollary (A test for uniform convergence)

Let $f \in \mathrm{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and suppose that there exists a piecewise continuous signal $f' : [0,T] \to \mathbb{C}$ with the property that

$$f(t) = f(0) + \int_0^t f'(\tau)\,\mathrm{d}\tau.$$

Then $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges uniformly to f.

Proof.

We shall show that the hypotheses of previous theorem hold. Since $(D^{\mathrm{per}}_{N,T}f)_{N\in\mathbb{Z}_{>0}}$ converges pointwise to f by Dirichlet's test and since f is continuous, we may write

$$f(t) = \frac{1}{T}\sum_{n\in\mathbb{Z}}\mathscr{F}_{\mathrm{CD}}(f)(nT^{-1})\mathrm{e}^{2\pi\mathrm{i}n\frac{t}{T}}.$$

Uniform convergence of Fourier series

Proof (cont'd).

By our CDFT differentiation theorem on Slide 196, the CDFT of f' is given by

$$\mathscr{F}_{\mathsf{CD}}(f')(nT^{-1}) = \frac{2\pi \mathrm{i} n}{T}\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1}).$$

By Bessel's inequality (here we use some ideas from the L^2-CDFT that we have not yet discussed) we then have

$$\frac{1}{T}\sum_{n\in\mathbb{Z}}|\mathscr{F}_{\mathsf{CD}}(f')(nT^{-1})|^2 \le \|f'\|_2^2 < \infty,$$

so that the sum

$$\sum_{n\in\mathbb{Z}}|\mathscr{F}_{\mathsf{CD}}(f')(nT^{-1})|^2$$

converges. Now let

Uniform convergence of Fourier series

Proof (cont'd).

$$s_N = \sum_{|n|\le N}|\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1})|,$$

and note that

$$\begin{aligned} s_N &= |\mathscr{F}_{\mathsf{CD}}(f)(0)| + \sum_{\substack{|n|\le N\\ n\neq 0}}|\mathscr{F}_{\mathsf{CD}}(f)(nT^{-1})| = |\mathscr{F}_{\mathsf{CD}}(f)(0)| + \sum_{\substack{|n|\le N\\ n\neq 0}}\frac{|T\mathscr{F}_{\mathsf{CD}}(f')(nT^{-1})|}{|2\pi \mathrm{i} n|} \\ &= |\mathscr{F}_{\mathsf{CD}}(f)(0)| + \frac{T}{2\pi}\sum_{\substack{|n|\le N\\ n\neq 0}}|\mathscr{F}_{\mathsf{CD}}(f')(nT^{-1})|\left|\frac{1}{n}\right| \\ &\le |\mathscr{F}_{\mathsf{CD}}(f)(0)| + \frac{T}{2\pi}\Big(\sum_{\substack{|n|\le N\\ n\neq 0}}|\mathscr{F}_{\mathsf{CD}}(f')(nT^{-1})|^2\Big)^{1/2}\Big(\sum_{\substack{|n|\le N\\ n\neq 0}}\frac{1}{n^2}\Big)^{1/2}, \end{aligned}$$

using the Cauchy–Bunyakovsky–Schwarz inequality. Now note that both sums

Uniform convergence of Fourier series

Proof (cont'd).

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |\mathscr{F}_{\mathsf{CD}}(f')(nT^{-1})|^2, \quad \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^2}$$

converge. This shows that

$$\lim_{N\to\infty} s_N < \infty. \tag{4}$$

Moreover, since the sequence $(s_N)_{N\in\mathbb{Z}_{>0}}$ is increasing, (4) implies that the sequence converges, and this is what we set out to prove. $\square$

Examples (Uniform convergence of Fourier series)

We use the signals f, g, and h used in our discussion of pointwise convergence tests.

1. First we consider the square wave f. We showed that it converges pointwise to a discontinuous limit signal. Thus convergence cannot be uniform: We showed in the theorem on Slide 121 that the pointwise limit of a sequence of bounded continuous signals (e.g., the partial sums for a Fourier series) is continuous.

Uniform convergence of Fourier series

Examples (Uniform convergence of Fourier series (cont'd))

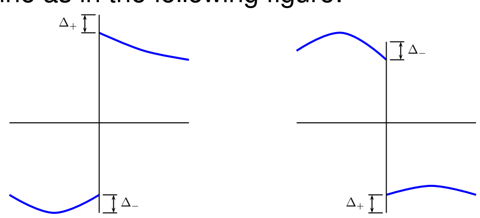
3. For the triangular wave signal g, the corollary above implies uniform convergence.
4. For the signal h we cannot apply any of the tests above. However, using a test associated with signals of bounded variation (which, remember is verboten in this course), one can show that the Fourier series for h converges uniformly to h.

Gibbs' phenomenon

- A Fourier series cannot converge uniformly to a discontinuous limit signal; in particular, the Fourier series of a discontinuous signal cannot converge to the signal.
- The exact manner of this lack of uniform convergence can be described. The precise statement of this is in the notes, but is a little notationally tedious. However, the idea is simple, so we provide this.
- Let $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{R})$, let $t_0 \in \mathbb{R}$, and suppose that the limits $f(t_0-)$, $f(t_0+)$, $f'(t_0-)$, and $f'(t_0+)$ exist. Suppose that $f(t_0-) \neq f(t_0+)$.
- Dirichlet's test tells us that

$$\lim_{N\to\infty} D^{\mathrm{per}}_{T,N}f(t_0) = \tfrac{1}{2}(f(t_0+) + f(t_0-)).$$

- Around t_0 consider the subset defined by taking the union of the graph of f with a vertical line as in the following figure:



Gibbs' phenomenon

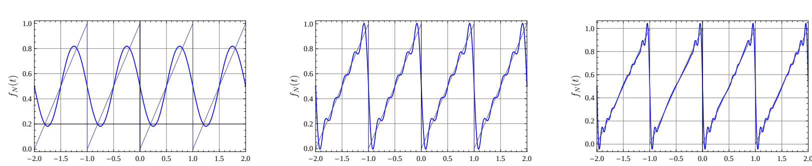
- Here

$$\Delta_+ = \Delta_- = |j(t_0)|\left(\frac{I}{\pi} - \frac{1}{2}\right) \approx 0.0895|j(t_0)|$$

where

$$I = \int_0^{\pi} \frac{\sin t}{t}\,\mathrm{d}t, \quad j(t_0) = |f(t_0+) - f(t_0-)|.$$

- Then a theorem in Bôcher in 1906 states that the graph of the Nth partial sum for the Fourier series approaches this subset of the plane as $N \to \infty$.
- This is best illustrated by a picture:



We can see that as $N \to \infty$ the graphs approach something like the figure on the preceding slide.

Reading for Lecture 27

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-5.2.5 and IV-5.2.6.

Lecture 28
Cesàro summability

- We just spent a lot of time talking about convergence of Fourier series, even though we began our discussion of Fourier series by indicating that this does not provide a left-inverse for the CDFT (recall: Kolmogorov showed that there exists $f \in \mathsf{L}^1_{\mathsf{per},T}(\mathbb{R};\mathbb{C})$ whose Fourier series diverges everywhere).
- Let us now instead provide an actual left-inverse. We do this by averaging the partial sums for the Fourier series to get the ***Cesàro sums***:

$$\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{T}\sum_{|j|\leq n}\mathscr{F}_{\mathsf{CD}}(f)(jT^{-1})\mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}}$$

Lemma

For $f \in \mathsf{L}^1_{\mathsf{per},T}(\mathbb{R};\mathbb{C})$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{T}\sum_{j=-n}^{n}\mathscr{F}_{\mathsf{CD}}(f)(jT^{-1})\mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-\tau)F^{\mathsf{per}}_{T,N}(\tau)\,\mathrm{d}\tau.$$

Cesàro summability

- As with the partial sums for the Fourier series, we let

$$F^{\mathrm{per}}_{T,N}f(t)=\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{T}\sum_{|j|\le n}\mathscr{F}_{\mathrm{CD}}(f)(jT^{-1})\mathrm{e}^{2\pi\mathrm{i}j\frac{t}{T}}.$$

Theorem

For $f\in\mathsf{L}^{(0)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ the following statements hold:

(i) if $f\in\mathsf{L}^{(p)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then $(F^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges to f in $\mathsf{L}^{p}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$;

(ii) if $f\in\mathsf{C}^{0}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then $(F^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ converges uniformly to f;

(iii) if $f\in\mathsf{L}^{(\infty)}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ and if, for $t_0\in\mathbb{R}$, the limits $f(t_0-)$ and $f(t_0+)$ exist then $(F^{\mathrm{per}}_{T,N}f(t_0))_{N\in\mathbb{Z}_{>0}}$ converges to $\frac{1}{2}(f(t_0-)+f(t_0+))$.

Cesàro summability

- The first statement in the preceding theorem indicates that taking Cesàro sums *does* provide a left-inverse for the CDFT in that the sequence of these sums converges to the original signal in $\mathsf{L}^{1}_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$.
- The other statements in the theorem indicate that the convergence properties for the Cesàro sums are generally far more robust than the partial sums for Fourier series.
- So, why not *always* use Cesàro sums? Well, there are good reasons for using Fourier partial sums over Cesàro sums sometimes. For example, in cases when the Fourier series converges, it will converge faster that the Cesàro sums. This is something of great interest in signal processing.

The L^2-CDFT

- Note that $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C}) \subset \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ (you should know this). It turns out that the restriction of $\mathscr{F}_{\mathrm{CD}}$ to $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ has some surprisingly rich properties. This is related to the fact that the norm on $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is associated to the inner product

$$\langle f, g\rangle_2 = \int_0^T f(t)\overline{g(t)}\,\mathrm{d}t.$$

This is not something we will explore here, but will be explored more deeply in Math 335. Here we simply record the most important consequence of this for our present programme.

Theorem

The CDFT restricted to $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ is a Hilbert space isomorphism from $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ to $\ell^2(\mathbb{Z}(T^{-1});\mathbb{C})$.

Theorem

If $f \in \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then the sequence $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ of partial sums for the Fourier series of f converges to f in $\mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$.

The L^2-CDFT

- Thus the Fourier series acts as an inverse for the L^2-CDFT. Note, however, that this does *not* mean that the Fourier series converges pointwise in any nice way. Convergence in L^2 is very different from pointwise convergence.
- Let us say a few words about this.
  1. By abstract arguments, if $f \in \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then there is a subsequence of $(D^{\mathrm{per}}_{T,N}f)_{N\in\mathbb{Z}_{>0}}$ which converges pointwise almost everywhere to f.
  2. In 1913 Luzin conjectured that if $f \in \mathsf{L}^2_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ then the Fourier series for f converges pointwise almost everywhere to f.
  3. Following Kolmogorov's statement that there exists $f \in \mathsf{L}^1_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ whose Fourier series diverges everywhere, it is believed that Luzin's conjecture is false.
  4. While trying to prove that Luzin's conjecture was false, Carleson in 1966 proved that it was actually correct.
  5. Kahane and Katznelson in 1966 proved that, given a subset $Z \subseteq [0, T]$ of measure zero, there exists $f \in \mathsf{C}^0_{\mathrm{per},T}(\mathbb{R};\mathbb{C})$ such that the Fourier series for f diverges at all points in Z. Thus Carleson's theorem cannot be improved.
- *Fun project:* Go to the library and pick out a book with a title like "Signals and Systems" or "Fourier Analysis for Scientists and Engineers." Look for the false statement, "The Fourier series for a continuous function (uniformly) converges to the function." Most books like this contain this statement for some reason.

The L^1-CCFT (definitions)

- We now turn to the Fourier transform for aperiodic continuous-time signals. The central ideas here follow those for the CDFT to a surprising degree. Thus our treatment will be quicker that for the CDFT.

Definition (CCFT)

The ***continuous-continuous Fourier transform*** or ***CCFT*** assigns to $f \in L^1(\mathbb{R};\mathbb{C})$ the signal $\mathscr{F}_{CC}(f)\colon \mathbb{R} \to \mathbb{C}$ by

$$\mathscr{F}_{CC}(f)(\nu) = \int_{\mathbb{R}} f(t)e^{-2\pi i\nu t}\,dt.$$

Examples (CCFT)

1. For $a \in \mathbb{C}$ with $\operatorname{Re}(a) \in \mathbb{R}_{>0}$, note that $f(t) = 1_{\geq 0}(t)e^{-at}$ is a signal in $L^1(\mathbb{R};\mathbb{C})$. We then compute

$$\mathscr{F}_{CC}(f)(\nu) = \int_{\mathbb{R}} f(t)e^{-2\pi i\nu t}\,dt = \int_0^\infty e^{-(a+2\pi i\nu)t}\,dt = \frac{1}{a+2\pi i\nu}.$$

The L^1-CCFT (definitions)

Examples (CCFT (cont'd))

2. Let $a \in \mathbb{R}_{>0}$ and consider the signal $f = \chi_{[-a,a]}$ given by the characteristic function of $[-a,a]$. We then compute

$$\mathscr{F}_{CC}(\sigma)(\nu) = \int_{-a}^{a} e^{-2\pi i\nu t}\,dt = -\frac{e^{-2\pi i\nu t}}{2\pi i\nu}\Big|_{-a}^{a} = \frac{\sin(2\pi a\nu)}{\pi\nu}.$$

3. Next we consider a Gaussian $\gamma_a(t) = e^{-at^2}$ where a is a positive real number, and compute

$$\begin{aligned}\mathscr{F}_{CC}(\gamma_a)(\nu) &= \int_{\mathbb{R}} e^{-at^2-2\pi i\nu t}\,dt \\ &= \int_{\mathbb{R}} e^{-at^2-2\pi i\nu t+\frac{\pi^2\nu^2}{a}}e^{-\frac{\pi^2\nu^2}{a}}\,dt \\ &= e^{-\frac{\pi^2\nu^2}{a}}\int_{\mathbb{R}} e^{-a(t+i\frac{\pi\nu}{a})^2}\,dt.\end{aligned}$$

The L^1-CCFT (definitions)

Examples (CCFT (cont'd))

This last integral is an integral along the line through $\mathrm{i}\frac{\pi\nu}{a} \in \mathbb{C}$ and parallel to the real axis. To perform this integral we use contour integration in $\mathbb{C}$. Let us take the case of $\nu \in \mathbb{R}_{>0}$ first. We define a contour Γ_R given by

$$\Gamma_R = \{(x,0) \mid x \in [-R,R]\} \cup \left\{(R,y) \;\middle|\; y \in [0,\tfrac{\pi\nu}{a}]\right\} \\ \cup \left\{(x,\tfrac{\pi\nu}{a}) \;\middle|\; x \in [-R,R]\right\} \cup \left\{(-R,y) \;\middle|\; y \in [0,\tfrac{\pi\nu}{a}]\right\},$$

and we take the counterclockwise sense for performing the integration. Since the function $z \mapsto \mathrm{e}^{-az^2}$ is analytic in $\mathbb{C}$ we have

$$\begin{aligned} 0 &= \int_{\Gamma_R} \mathrm{e}^{-az^2}\,\mathrm{d}z \\ &= \int_{-R}^{R} \mathrm{e}^{-ax^2}\,\mathrm{d}x + \int_0^{\frac{\pi\nu}{a}} \mathrm{e}^{-a(R+\mathrm{i}y)^2}\,\mathrm{d}y \\ &\quad + \int_R^{-R} \mathrm{e}^{-a(x+\mathrm{i}\frac{\pi\nu}{a})^2}\,\mathrm{d}x + \int_{\frac{\pi\nu}{a}}^{0} \mathrm{e}^{-a(-R+\mathrm{i}y)^2}\,\mathrm{d}y. \end{aligned}$$

The L^1-CCFT (definitions)

Examples (CCFT (cont'd))

We claim that the second and fourth integrals are zero in the limit as $R \to \infty$. To see this for the second integral, note that

$$|\mathrm{e}^{-a(R+\mathrm{i}y)^2}| = |\mathrm{e}^{-aR^2}\mathrm{e}^{-2a\mathrm{i}Ry}\mathrm{e}^{ay^2}| \le \mathrm{e}^{-aR^2}\mathrm{e}^{\frac{\pi^2\nu^2}{a}}.$$

Thus

$$\left|\int_0^{\frac{\pi\nu}{a}} \mathrm{e}^{-a(R+\mathrm{i}y)^2}\,\mathrm{d}y\right| \le \int_0^{\frac{\pi\nu}{a}} |\mathrm{e}^{-a(R+\mathrm{i}y)^2}|\,\mathrm{d}y \le \mathrm{e}^{\frac{\pi^2\nu^2}{a}} \int_0^{\frac{\pi\nu}{a}} \mathrm{e}^{-aR^2}\,\mathrm{d}y.$$

We then compute

$$\lim_{R\to\infty} \int_0^{\frac{\pi\nu}{a}} \mathrm{e}^{-aR^2}\,\mathrm{d}y = \lim_{R\to\infty} \frac{\pi\nu}{a}\mathrm{e}^{-aR^2} = 0.$$

This gives the vanishing of the second integral as $R \to \infty$. The same sort of argument gives the same conclusion as regards the fourth integral.

The L^1-CCFT (definitions)

Examples (CCFT (cont'd))

Therefore, we get

$$\int_{\mathbb{R}} e^{-a(t+i\frac{\pi\nu}{a})^2}\, dt = \lim_{R\to\infty}\int_{-R}^{R} e^{-a(t+i\frac{\pi\nu}{a})^2}\, dt = \lim_{R\to\infty}\int_{-R}^{R} e^{-at^2}\, dt.$$

Thus we have

$$\mathscr{F}_{CC}(\gamma_a)(\nu) = e^{-\frac{\pi^2\nu^2}{a}}\int_{\mathbb{R}} e^{-at^2}\, dt,$$

this being valid for $\nu \in \mathbb{R}_{>0}$. A similar analysis to the above gives the same formula for $\nu \in \mathbb{R}_{<0}$. To evaluate the integral on the right we perform a trick. We denote

$$I = \int_{\mathbb{R}} e^{-at^2}\, dt$$

The L^1-CCFT (definitions)

Examples (CCFT (cont'd))

so that

$$\begin{aligned}
I^2 &= \Big(\int_{\mathbb{R}} e^{-ax^2}\, dx\Big)\Big(\int_{\mathbb{R}} e^{-ay^2}\, dy\Big) = \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-a(x^2+y^2)}\, dy\Big) dx \\
&= \int_0^{\infty}\Big(\int_0^{2\pi} r e^{-ar^2}\, d\theta\Big) dr = 2\pi\int_0^{\infty} r e^{-ar^2}\, dr \\
&= \frac{\pi}{a}\int_0^{\infty} e^{-\rho}\, d\rho = \frac{\pi}{a},
\end{aligned}$$

where in the third step we make a change of variable for the integral over the plane from Cartesian to polar coordinates using the relation $dxdy = rdrd\theta$. In any case, $I = \sqrt{\frac{\pi}{a}}$ and so

$$\mathscr{F}_{CC}(\gamma_a)(\nu) = \sqrt{\frac{\pi}{a}}e^{-\frac{\pi^2\nu^2}{a}}.$$

Thus we see that the Gaussian has the feature that its CCFT is almost equal to itself. Indeed, if $a = \pi$ then the CCFT is *exactly* equal to the original signal.

Reading for Lecture 28

Material related to this lecture can be found in the following sections of the course notes:

1. Section IV-5.2.7.
2. Sections IV-5.3.1 and IV-5.3.2 (mainly Theorem IV-5.3.8).

Lecture 29
The L^1-CCFT (properties)

- Now let us turn to some of the properties of the CCFT.
- First some things more or less elementary. Recall from Slide 14 the reparameterisations τ_a and σ of the time-domain $\mathbb{R}$.
- Let us also define $\overline{\mathscr{F}}_{\mathrm{CC}}(f)(\nu) = \displaystyle\int_{\mathbb{R}} f(t)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}t$.

Proposition

(i) $\overline{\mathscr{F}_{\mathrm{CC}}(f)} = \overline{\mathscr{F}}_{\mathrm{CC}}(\bar{f})$;

(ii) $\mathscr{F}_{\mathrm{CC}}(\sigma^* f) = \sigma^*(\mathscr{F}_{\mathrm{CC}}(f)) = \overline{\mathscr{F}}_{\mathrm{CC}}(f)$;

(iii) *if f is even (resp. odd) then $\mathscr{F}_{\mathrm{CC}}(f)$ is even (resp. odd);*

(iv) *if f is real and even (resp. real and odd) then $\mathscr{F}_{\mathrm{CC}}(f)$ is real and even (resp. imaginary and odd);*

(v) $\mathscr{F}_{\mathrm{CC}}(\tau_a^* f)(\nu) = \mathrm{e}^{-2\pi\mathrm{i}a\nu}\mathscr{F}_{\mathrm{CC}}(f)(\nu)$.

The L^1-CCFT (properties)

Theorem (Riemann–Lebesgue lemma for CCFT)

For $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$

(i) $\mathscr{F}_{\mathsf{CC}}(f)$ *is a bounded, uniformly continuous function and*

(ii) $\lim_{|\nu|\to\infty} |\mathscr{F}_{\mathsf{CC}}(f)(\nu)| = 0.$

- An immediate consequence of this is that the CCFT is a map $\mathscr{F}_{\mathsf{CC}}\colon \mathsf{L}^1(\mathbb{R};\mathbb{C}) \to \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$.
- In fact, one can easily show that the CCFT is a continuous linear map (see Slide 94) from $(\mathsf{L}^1(\mathbb{R};\mathbb{C}), \|\cdot\|_1)$ to $(\mathsf{C}^0_0(\mathbb{R};\mathbb{C}), \|\cdot\|_\infty)$; this is Corollary IV-6.1.8 in the course notes.

Differentiation and the CCFT

- As with the CDFT, there are relations between the differentiability of a signal and the rate of decay of its CCFT. Moreover, because the CCFT is itself a continuous-frequency signal, it too can have differentiability properties, and these can be accounted for.

Proposition (The CCFT and differentiation)

Suppose that $f \in \mathsf{C}^0(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^1(\mathbb{R};\mathbb{C})$ *and that there exists a signal* $f'\colon \mathbb{R} \to \mathbb{C}$ *with the following properties:*

(i) for every $T \in \mathbb{R}_{>0}$, f' *is piecewise continuous on* $[-T, T]$*;*

(ii) f' *is discontinuous at a finite number of points;*

(iii) $f' \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$*;*

(iv) $f(t) = f(0) + \int_0^t f'(\tau)\,\mathrm{d}\tau.$

Then

$$\mathscr{F}_{\mathsf{CC}}(f')(\nu) = (2\pi\mathrm{i}\nu)\mathscr{F}_{\mathsf{CC}}(f)(\nu).$$

Differentiation and the CCFT

Corollary (The CCFT and higher-order derivatives)

If $f \in \mathsf{C}^{r-1}(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^1(\mathbb{R};\mathbb{C})$ for $r \in \mathbb{Z}_{>0}$ and suppose that there exists a signal $f^{(r)}\colon \mathbb{R} \to \mathbb{C}$ with the following properties:

(i) for every $T \in \mathbb{R}_{>0}$, $f^{(r)}$ is piecewise continuous on $[-T,T]$;

(ii) $f^{(r)}$ is discontinuous at a finite number of points;

(iii) $f^{(j)} \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ for $j \in \{1,\dots,r\}$;

(iv) $f^{(r-1)}(t) = f^{(r-1)}(0) + \int_0^t f^{(r)}(\tau)\,\mathrm{d}\tau$.

Then

$$\mathscr{F}_{\mathrm{CC}}(f^{(r)})(\nu) = (2\pi\mathrm{i}\nu)^r \mathscr{F}_{\mathrm{CC}}(f)(\nu).$$

- We then have the following facts, some of which we will not get to for a few lectures.
  1. If $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ then the CCFT satisfies

$$\lim_{|\nu|\to\infty} |\mathscr{F}_{\mathrm{CC}}(f)(\nu)| = 0.$$

     This is the Riemann–Lebesgue Lemma.

Differentiation and the CCFT

2. If $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ then $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$. This is nontrivial and will be discussed when we discuss the L^2-CCFT subsequently.
3. If f satisfies the conditions of differentiation proposition above, then $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$. This follows from things we have not yet stated about uniform convergence of Fourier integrals.
4. If $f \in \mathsf{C}^r(\mathbb{R};\mathbb{C})$ and if $f^{(k)} \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ for $k \in \{0,1,\dots,r\}$ then the CCFT of f has the property that

$$\lim_{|\nu|\to\infty} \nu^j \mathscr{F}_{\mathrm{CC}}(f)(\nu) = 0$$

   for $j \in \{0,1,\dots,r\}$.
5. If $f \in \mathsf{C}^\infty(\mathbb{R};\mathbb{C})$ and if $f^{(k)} \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ for $k \in \mathbb{Z}_{\geq 0}$ then the CCFT of f has the property that

$$\lim_{|\nu|\to\infty} \nu^k \mathscr{F}_{\mathrm{CC}}(f)(\nu) = 0$$

   for any $k \in \mathbb{Z}_{\geq 0}$.

Differentiation and the CCFT

Proposition (Differentiability of transformed signals)

For $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$, if the signals $t \mapsto t^j f(t)$, $j \in \{0, 1, \dots, k\}$, are in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$ then $\mathscr{F}_{\mathrm{CC}}(f)$ is k-times continuously differentiable and

$$\mathscr{F}_{\mathrm{CC}}(f)^{(k)}(\nu) = \int_{\mathbb{R}} (-2\pi\mathrm{i}t)^k f(t) \mathrm{e}^{-2\pi\mathrm{i}\nu t}\, \mathrm{d}t.$$

In particular, if $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ has compact support then $\mathscr{F}_{\mathrm{CC}}(f)$ is infinitely differentiable.

Examples (CCFT and differentiation)

1. Take
$$f(t) = \begin{cases} \mathrm{e}^{-t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$
Since f is not continuous, we cannot expect that $\mathscr{F}_{\mathrm{CC}}(f)$ will decay quickly to zero as $|\nu| \to \infty$. Since f decays to zero faster than any polynomial as $|t| \to \infty$, it follows that $\mathscr{F}_{\mathrm{CC}}(f)$ is infinitely differentiable.

Differentiation and the CCFT

Examples (CCFT and differentiation (cont'd))

2. Take $f(t) = \frac{1}{1+t^{2k}}$ for $k \geq 1$. By induction one can show that the rth derivative of f has the form
$$f^{(r)}(t) = \frac{P_r}{(1+t^{2k})^{-r-1}},$$
where P_r is a homogeneous polynomial of degree $(2k-1)r$. Thus $f^{(r)}(t)$ goes to zero as $|t| \to \infty$ like $|t|^{-2k-r}$. Therefore, $f^{(r)} \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ if and only if $-2k - r \leq -2 \iff r \geq 2(1-k)$. This holds for all $r \geq 0$ and all $k \geq 1$ and so f and all of its derivatives are in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$. Thus $\mathscr{F}_{\mathrm{CC}}(f)$ decays to zero as $|\nu| \to \infty$ faster than any polynomial. Also, $t \mapsto t^r f(t)$ is in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$ if and only if $r - 2k \leq -2 \iff r \leq 2(k-1)$. Thus $\mathscr{F}_{\mathrm{CC}}(f)$ is $2(k-1)$ times continuously differentiable.
3. Finally, we take $f(t) = \mathrm{e}^{-t^2}$. We see that f and all of its derivatives are in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$ (why?) and so $\mathscr{F}_{\mathrm{CC}}(f)$ decays to zero as $|\nu| \to \infty$ faster than any polynomial. Also, $t \mapsto t^k f(t)$ is in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$ for any $k \in \mathbb{Z}_{>0}$ (why?) and so $\mathscr{F}_{\mathrm{CC}}(f)$ is infinitely differentiable.

Reading for Lecture 29

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-6.1.2 and IV-6.1.3.

Lecture 30
CCFT inversion (warm up)

- The story we will tell for the inversion of the CCFT mirrors that for the CDFT. Thus we skip the motivation and get right to it.

Theorem (CCFT is injective)

$\mathscr{F}_{\mathrm{CC}}\colon \mathsf{L}^1(\mathbb{R};\mathbb{C}) \to \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$ *is injective.*

Proof.

The proof bears some resemblance to that for the corresponding assertion for the CDFT, but is a little more complicated. We thus skip it, and refer to the course notes. □

- The map $\mathscr{F}_{\mathrm{CC}}\colon \mathsf{L}^1(\mathbb{R};\mathbb{C}) \to \mathsf{C}^0_0(\mathbb{R};\mathbb{C})$ is *not* surjective.
- Being injective it possesses a left-inverse. As with the CDFT we shall do the following.
  1. Propose a left-inverse that actually doesn't work, but give conditions under which it does work.
  2. Give an actual left-inverse.

Fourier integrals

- Our proposed left-inverse for the CCFT mimics the Fourier series for the CDFT. Rather than summing over the discrete collection of harmonics that we have for the CDFT, we integrate over the continuous set of harmonics for the CCFT.

Definition (Fourier integral)

For $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ the ***Fourier integral*** for f is

$$\mathsf{FI}[f](t) = \int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu,$$

disregarding whether the integral converges.

- The "partial sums" for the Fourier integral are

$$f_{\Omega}(t) \triangleq \int_{-\Omega}^{\Omega} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu, \qquad \Omega \in \mathbb{R}_{>0},$$

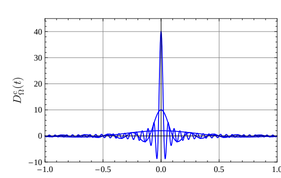
which is always well-defined since $\mathscr{F}_{\mathrm{CC}}(f)$ is continuous.

Fourier integrals

- As with the CDFT, the Fourier integral does not give a left-inverse for the CCFT. We will study conditions on signals which ensure that the Fourier integral does recover them from their CCFT.
- As with Fourier series, the key to understanding convergence of Fourier integrals is the ***continuous Dirichlet kernel***:

$$D_{\Omega}(t) = \begin{cases} \dfrac{\sin(2\pi\Omega t)}{\pi t}, & t \neq 0, \\ 2\Omega, & t = 0. \end{cases}$$

For $\Omega \in \{1, 5, 20\}$ we plot the continuous Dirichlet kernel:



Fourier integrals

Lemma

For $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ *we have* $f_\Omega(t) = \int_{\mathbb{R}} f(t-\tau)D_\Omega(\tau)\,\mathrm{d}\tau$*,* $\Omega \in \mathbb{R}_{>0}$*.*

- Following our notation for the partial sums for the Fourier series, we denote
$$D_\Omega f(t) = \int_{\mathbb{R}} f(t-\tau)D_\Omega(\tau)\,\mathrm{d}\tau.$$
- We will now study the behaviour of these integrals as $\Omega \to \infty$.
- The story bears much resemblance to that for the CDFT and convergence of Fourier series.

Pointwise convergence of Fourier integrals

Theorem (Pointwise convergence of Fourier integrals)

Let $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$*, let* $t_0 \in \mathbb{R}$*, and let* $s \in \mathbb{C}$*. The following statements are equivalent:*

(i) $\lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} \mathscr{F}_{\mathsf{CC}}(f)(\nu)e^{2\pi\mathrm{i}\nu t_0}\,\mathrm{d}\nu = s$*;*

(ii) $\lim_{\Omega\to\infty} \int_{\mathbb{R}} (f(t_0-t)-s)D_\Omega(t)\,\mathrm{d}t = 0$ *(integral understood in the conditional sense);*

(iii) for each $\epsilon \in \mathbb{R}_{>0}$ *we have*
$$\lim_{\Omega\to\infty} \int_{-\epsilon}^{\epsilon} (f(t_0-t)-s)D_\Omega(t)\,\mathrm{d}t = 0;$$

- Again we see that the localisation principle holds. Since the continuous Dirichlet kernel looks a lot like the discrete Dirichlet kernel around $t = 0$, this explains why pointwise convergence for Fourier integrals is so similar to that for Fourier series. So we will race through this a little.

Pointwise convergence of Fourier integrals

Theorem (Dini's test)

Let $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ *and let* $t_0 \in \mathbb{R}$. *If there exists* $\epsilon \in \mathbb{R}_{>0}$ *so that*

$$\int_{-\epsilon}^{\epsilon} \left| \frac{f(t_0 - t) - s}{t} \right| \mathrm{d}t < \infty,$$

then $(D_\Omega f(t_0))_{\Omega \in \mathbb{R}_{>0}}$ *converges to* s.

Corollary (An alternative version of Dini's test)

Let $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ *and let* $t_0 \in \mathbb{R}$. *If there exists* $\epsilon \in \mathbb{R}_{>0}$ *so that*

$$\int_{0}^{\epsilon} \left| \frac{\frac{1}{2}(f(t_0 + t) + f(t_0 - t)) - s}{t} \right| \mathrm{d}t < \infty,$$

then $(D_\Omega f(t_0))_{\Omega \in \mathbb{R}_{>0}}$ *converges to* s.

Pointwise convergence of Fourier integrals

Corollary (Fourier integrals converge at points of differentiability)

If $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ *is differentiable at* t_0 *then* $(D_\Omega f(t_0))_{\Omega \in \mathbb{R}_{>0}}$ *converges to* $f(t_0)$.

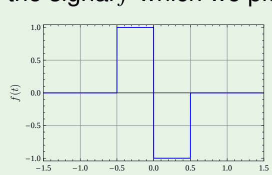
Theorem (Dirichlet's test)

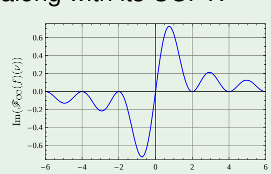
Let $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ *and suppose that the limits* $f(t_0-)$, $f(t_0+)$, $f'(t_0-)$, *and* $f'(t_0+)$ *exist for* $t_0 \in \mathbb{R}$. *Then*

$$\lim_{\Omega \to \infty} \int_{-\Omega}^{\Omega} \mathscr{F}_{\mathrm{CC}}(f)(\nu) e^{2\pi \mathrm{i} \nu t_0} \, \mathrm{d}\nu = \frac{1}{2}\big(f(t_0+) + f(t_0-)\big).$$

Examples (Pointwise convergence of Fourier integrals)

1. We consider the signal f which we plot along with its CCFT:





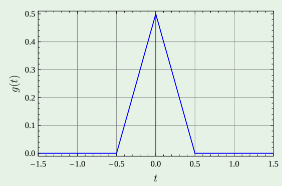
Pointwise convergence of Fourier integrals

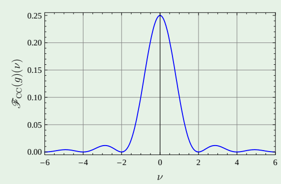
Examples (Pointwise convergence of Fourier integrals (cont'd))

The analysis here is very much like that for the Fourier series for the square wave, so we omit the details. As always, at points of differentiability the Fourier integral converges to the signal. The points remaining are $\{-\frac{1}{2}, 0, \frac{1}{2}\}$. The results of applying the convergence tests are:

(a) *Dini's test:* Fails to predict convergence.
(b) *Alternate Dini's test:* Gives convergence to $\frac{1}{2}$ at $t_0 = -\frac{1}{2}$, $-\frac{1}{2}$ at $t_0 = \frac{1}{2}$, and 0 at $t_0 = 0$.
(c) *Dirichlet's test:* Gives the same convergence as Alternate Dini.

2. Here we consider g with its CCFT as plotted below:





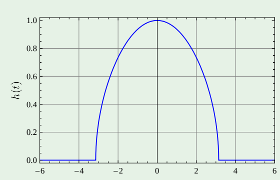
Pointwise convergence of Fourier integrals

Examples (Pointwise convergence of Fourier integrals (cont'd))

In this case, all three convergence tests may be applied to give the conclusion that the Fourier integral for g converges to g for all $t_0 \in \mathbb{R}$.

3. The last signal we consider is

$$h(t) = \begin{cases} \sqrt{\sin \frac{t+\pi}{2}}, & |t| \le \pi, \\ 0, & \text{otherwise} \end{cases}$$



The tests can be applied as follows.

(a) *Dini's test:* Gives convergence to h at all points.
(b) *Alternate Dini's test:* Gives convergence to h at all points.
(c) *Dirichlet's test:* Fails to apply.

Uniform convergence of Fourier integrals

- We wish to study uniform convergence of the family of signals $(D_\Omega f)_{\Omega\in\mathbb{R}_{\geq 0}}$. We should be clear about what this means since this is not a sequence.

Definition

Let $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$. The family $(D_\Omega f)_{\Omega\in\mathbb{R}_{\geq 0}}$ ***converges uniformly*** to $g\colon \mathbb{R}\to\mathbb{C}$ if, for every $\epsilon\in\mathbb{R}_{>0}$, there exists $\Omega_0\in\mathbb{R}_{\geq 0}$ such that

$$|g(t) - D_\Omega f(t)| < \epsilon$$

for every $t\in\mathbb{R}$ and $\Omega\geq\Omega_0$.

Theorem (Uniform convergence of Fourier integrals)

If $f\in\mathsf{L}^1(\mathbb{R};\mathbb{C})$ and if $\mathscr{F}_{\mathrm{CC}}(f)\in\mathsf{L}^1(\mathbb{R};\mathbb{C})$, then the following statements hold:

(i) $(D_\Omega f)_{\Omega\in\mathbb{R}_{>0}}$ converges uniformly to a continuous signal g as $\Omega\to\infty$;

(ii) $f(t) = g(t)$ for almost every $t\in\mathbb{R}$.

Uniform convergence of Fourier integrals

Corollary (A test for uniform convergence)

Let $f\in\mathsf{C}^0(\mathbb{R};\mathbb{C})$ and suppose that there exists a signal $f'\colon\mathbb{R}\to\mathbb{C}$ such that

(i) for every $T\in\mathbb{R}_{>0}$, f' is piecewise continuous on $[-T,T]$,

(ii) f' is discontinuous at a finite number of points,

(iii) $f'\in\mathsf{L}^1(\mathbb{R};\mathbb{C})\cap L^2(\mathbb{R};\mathbb{C})$, and

(iv) $f(t) = \int_{-\infty}^{t} f'(\tau)\,\mathrm{d}\tau$.

Then $(D_\Omega f)_{\Omega\in\mathbb{R}_{>0}}$ converges uniformly to f. In particular, if $f, f^{(1)}, f^{(2)}\in\mathsf{C}^0(\mathbb{R};\mathbb{C})\cap\mathsf{L}^1(\mathbb{R};\mathbb{C})$ then $(D_\Omega f)_{N\in\mathbb{Z}_{>0}}$ converges uniformly to f.

- Note that the hypotheses in the above corollary are those of the differentiation rule for the CCFT on Slide 257, but we additionally require that $f'\in\mathsf{L}^2(\mathbb{R};\mathbb{C})$. This condition is needed because in the proof of the result we use the Cauchy–Bunyakovsky–Schwarz inequality, cf. the proof of the corresponding result for the CDFT on Slide 235.

Reading for Lecture 30

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-6.2.1, IV-6.2.2, IV-6.2.4, and IV-6.2.5.

Lecture 31
Uniform convergence of Fourier integrals

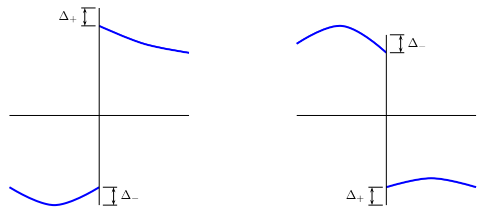
Examples (Uniform convergence of Fourier integrals)

We consider the three signals f, g, and h introduced starting on Slide 269.

1. Since the Fourier integral converges pointwise to a discontinuous limit, convergence cannot be uniform.
2. Note that
$$g(t) = \int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau$$
and since $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$, the Fourier integral converges uniformly to g by the corollary above.
3. The corollary does not apply since h' is not piecewise continuous. And the theorem may apply, but it requires us to compute the CCFT of h. So, we are unable to conclude whether the Fourier integral converges uniformly with what we know. There is, however, a bounded variation test that works. But you do not know this. . .

Gibbs' phenomenon redux

- Suppose that $f \in \mathsf{L}^1(\mathbb{R};\mathbb{R})$ and that the limits $f(t_0-), f(t_0+), f'(t_0-)$, and $f'(t_0+)$ exist.
- Near t_0 define a subset of the plane:



  Here Δ_+ and Δ_- are just as they are for the Gibbs phenomenon for the CDFT on Slide 240.
- Then, in a neighbourhood of t_0 the graph of $D_\Omega f(t_0)$ approaches the subset of the plane above as $\Omega \to \infty$.

Cesàro convergence

- Again we introduce an averaging process to try to rectify the less than perfect convergence properties of Fourier integrals.
- Thus we define

$$\frac{1}{\Omega}\int_0^\Omega \Big(\int_{-\omega}^{\omega} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu\Big)\,\mathrm{d}\omega,$$

  and consider the limit of the resulting signals as $\Omega \to \infty$.

Lemma

For $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C})$ we have

$$\frac{1}{\Omega}\int_0^\Omega \Big(\int_{-\omega}^{\omega} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu\Big)\,\mathrm{d}\omega = \int_{\mathbb{R}} f(t-\tau)F_\Omega(\tau)\,\mathrm{d}\tau,$$

where

$$F_\Omega(t) = \begin{cases} \frac{\sin^2(\pi\Omega t)}{\pi^2\Omega t^2}, & t \neq 0, \\ \Omega, & t = 0 \end{cases}$$

*is the **Fejér kernel**.*

Cesàro convergence

Theorem

For $f \in \mathsf{L}^{(0)}(\mathbb{R};\mathbb{C})$ the following statements hold:

(i) if $f \in \mathsf{L}^{(p)}(\mathbb{R};\mathbb{C})$ then $(F_\Omega f)_{\Omega\in\mathbb{R}}$ converges to f in $\mathsf{L}^p(\mathbb{R};\mathbb{C})$;

(ii) if $f \in \mathsf{C}^0_{\mathrm{bdd}}(\mathbb{R};\mathbb{C})$ then $(F_\Omega f|K)_{\Omega\in\mathbb{R}_{>0}}$ converges uniformly to $f|K$ for every compact subset $K \subseteq \mathbb{R}$;

(iii) if $f \in \mathsf{C}^0_{\mathrm{unif,bdd}}(\mathbb{R};\mathbb{C})$ is then $(F_\Omega f)_{\Omega\in\mathbb{R}_{>0}}$ converges uniformly to f;

(iv) if $f \in \mathsf{L}^{(\infty)}(\mathbb{R};\mathbb{C})$ and if, for $t_0 \in \mathbb{R}$, the limits $f(t_0-)$ and $f(t_0+)$ exist then $(F_\Omega f(t_0))_{\Omega\in\mathbb{R}_{>0}}$ converges to $\frac{1}{2}(f(t_0-)+f(t_0+))$.

- Thus, just as with Cesàro means for Fourier series, the Cesàro means for Fourier integrals provide a means of recovering a signal from its CCFT. That is, it provides us with an explicit left-inverse for the CCFT.
- Despite their desirable convergence properties, there are some trade-offs involved in using Cesàro means; we refer to Slide 245 for discussion.

The L^2-CCFT

- Note that $\mathsf{L}^2(\mathbb{R};\mathbb{C}) \not\subseteq \mathsf{L}^1(\mathbb{R};\mathbb{C})$ and so, if the CCFT can even be defined for signals in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$, the approach cannot be direct as it was for the CDFT.
- Here is the procedure.
  1. We state the following fact without proof.

Lemma

If $f \in \mathsf{L}^1(\mathbb{R};\mathbb{C}) \cap \mathsf{L}^2(\mathbb{R};\mathbb{C})$ then $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$.

2. Let $f \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$. For $j \in \mathbb{Z}_{>0}$ define

$$f_j(t) = \begin{cases} f(t), & t \in [-j,j], \\ 0, & \text{otherwise}. \end{cases}$$

Note that $f_j \in \mathsf{L}^2([-j,j];\mathbb{C}) \subseteq \mathsf{L}^1([-j,j];\mathbb{C}) \subseteq \mathsf{L}^1(\mathbb{R};\mathbb{C})$. We state without proof the following fact.

Lemma

The sequence $(f_j)_{j\in\mathbb{Z}_{>0}}$ converges to f in the Banach space $\mathsf{L}^2(\mathbb{R};\mathbb{C})$.

The L^2-CCFT

3. We state without proof the following fact.

Lemma

The sequence $(\mathscr{F}_{\mathrm{CC}}(f_j))_{j\in\mathbb{Z}_{>0}}$ *is a Cauchy sequence in* $\mathsf{L}^2(\mathbb{R};\mathbb{C})$.

4. Completeness of $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ ensures that there exists $\mathscr{F}_{\mathrm{CC}}(f) \in \mathsf{L}^2(\mathbb{R};\mathbb{C})$ such that

$$\lim_{j\to\infty}\left(\int_{\mathbb{R}}\left|\mathscr{F}_{\mathrm{CC}}(f)(\nu) - \chi_{[-j,j]}f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t\right|^2\right)^{1/2} = 0.$$

We call $\mathscr{F}_{\mathrm{CC}}(f)$ the ***L^2-CCFT*** of f.

- Note that the L^2-CCFT does not define a function of frequency, but only an equivalence class of signals agreeing almost everywhere. This is rather different than the CCFT for signals in $\mathsf{L}^1(\mathbb{R};\mathbb{C})$.
- Note that since the Fourier integral so closely resembles the CCFT itself, differing only by a sign in the exponent of the exponential function, the above construction applies verbatim to the inverse via the Fourier integral.

Reading for Lecture 31

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-6.2.5, IV-6.2.6, IV-6.2.7, IV-6.3.1, and IV-6.3.3.

Lecture 32
The L^2-CCFT

Theorem

The L^2*-CCFT is a Hilbert space isomorphism from* $L^2(\mathbb{R};\mathbb{C})$ *to* $L^2(\mathbb{R};\mathbb{C})$*. Moreover, the inverse is defined by*

$$\int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu,$$

where the integral is defined as in the procedure above.

Examples (The L^2-CCFT)

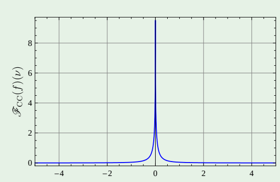
1. Take $f \in L^2(\mathbb{R};\mathbb{C})$ defined by $f(t) = \frac{1}{1+|t|}$. Note that $f \notin L^1(\mathbb{R};\mathbb{C})$. We define $f_j = \chi_{[-j,j]}f$, as in the procedure above. Since $f_j \in L^1(\mathbb{R};\mathbb{C})$ we can define $\mathscr{F}_{\mathrm{CC}}(f_j)$. Then, for almost every $\nu \in \mathbb{R}$, the limit

$$\lim_{j\to\infty} \int_{-j}^{j} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t$$

The L^2-CCFT

Examples (The L^2-CCFT (cont'd))

exists in $L^2(\mathbb{R};\mathbb{C})$ and defines $\mathscr{F}_{\mathrm{CC}}(f)(\nu)$. A sort of closed form expression for this CCFT is possible. We plot this:



Whatever we know about this function, we at least know it is in $L^2(\mathbb{R};\mathbb{C})$.

2. Let us consider $f = \chi_{[-1,1]}$ which is in $L^1(\mathbb{R};\mathbb{C}) \cap L^2(\mathbb{R};\mathbb{C})$. We had previously computed

$$\mathscr{F}_{\mathrm{CC}}(f)(\nu) = \frac{\sin(2\pi\nu)}{\pi\nu}.$$

This function is in $L^2(\mathbb{R};\mathbb{C})$, but not in $L^1(\mathbb{R};\mathbb{C})$. Thus the integral

$$\int_{\mathbb{R}} \mathscr{F}_{\mathrm{CC}}(f)(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}\nu$$

The L^2-CCFT

Examples (The L^2-CCFT (cont'd))

to determine f from its CCFT cannot be applied directly. However, the procedure above is applicable, and so, for almost every $t \in \mathbb{R}$ we have

$$\lim_{j\to\infty}\int_{-j}^{j} \mathscr{F}_{\mathsf{CC}}(f)(\nu)\mathsf{e}^{2\pi\mathsf{i}\nu t}\,\mathrm{d}\nu$$

converges to $f(t)$. Moreover, the limit defines a signal in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$.

- Note: We indicate in the examples that the limits exist for almost every ν and t, respectively. This is not obvious and does not follow from convergence in $\mathsf{L}^2(\mathbb{R};\mathbb{C})$ in a direct way. It does, however, follow from Carleson's deep theorem on almost everywhere convergence of Fourier series mentioned on Slide 247.

The ℓ^1-DCFT (definition)

- Next we consider the Fourier transform for discrete-time aperiodic signals.

Definition (DCFT)

The ***discrete-continuous Fourier transform*** or ***DCFT*** assigns to $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$ the signal $\mathscr{F}_{\mathsf{DC}}(f)\colon \mathbb{R} \to \mathbb{C}$ by

$$\mathscr{F}_{\mathsf{DC}}(f)(\nu) = \Delta\sum_{n\in\mathbb{Z}} f(n\Delta)\mathsf{e}^{-2\pi\mathsf{i}n\Delta\nu}, \qquad \nu \in \mathbb{R}.$$

- This transform is often called the "discrete-time Fourier transform."
- Note that the DCFT looks a lot like a Fourier series. We shall use this fact to relate much of what we say about the DCFT to what we have already said about the CDFT.

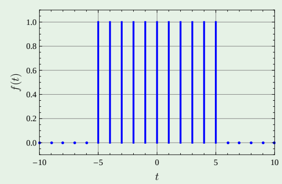
The ℓ^1-DCFT (definition)

Examples (DCFT)

1. Let us define $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ by

$$f(t) = \begin{cases} 1, & t \in \{-N\Delta, -\Delta, 0, \Delta, \dots, N\Delta\}, \\ 0, & \text{otherwise}. \end{cases}$$

Here's the graph:



The sum defining the DCFT of f is finite and we can show that

$$\mathscr{F}_{\mathrm{DC}}(f)(\nu) = \Delta \sum_{n=-N}^{N} \mathrm{e}^{-2\pi i n\Delta\nu} = \Delta D^{\mathrm{per}}_{\Delta^{-1},N}(\nu).$$

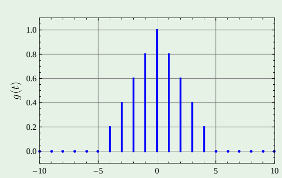
The ℓ^1-DCFT (definition)

Examples (DCFT (cont'd))

2. We consider the signal $g\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ given by

$$g(t) = \begin{cases} -\frac{t}{N\Delta} + 1, & t \in \{0, \Delta, \dots, (N-1)\Delta\}, \\ \frac{t}{N\Delta} + 1, & t \in \{-(N-1)\Delta, \dots, -\Delta\}, \\ 0, & \text{otherwise}. \end{cases}$$

Here's the graph:



The sum defining the DCFT of f is finite and we can show that

$$\mathscr{F}_{\mathrm{DC}}(g)(\nu) = \Delta F^{\mathrm{per}}_{\Delta^{-1},N}(\nu).$$

The ℓ^1-DCFT (properties)

- Recall from Slide 14 the reparameterisations τ_a and σ of the time-domain $\mathbb{R}$.
- Let us also define $\overline{\mathscr{F}}_{\mathrm{DC}}(f)(\nu) = \Delta \sum_{n\in\mathbb{Z}_{>0}} f(n\Delta)\mathrm{e}^{2\pi\mathrm{i}n\Delta\nu}$.

Proposition

(i) $\overline{\mathscr{F}_{\mathrm{DC}}(f)} = \overline{\mathscr{F}}_{\mathrm{DC}}(\bar{f})$;

(ii) $\mathscr{F}_{\mathrm{DC}}(\sigma^* f) = \sigma^*(\mathscr{F}_{\mathrm{DC}}(f)) = \overline{\mathscr{F}}_{\mathrm{DC}}(f)$;

(iii) *if* f *is even (resp. odd) then* $\mathscr{F}_{\mathrm{DC}}(f)$ *is even (resp. odd);*

(iv) *if* f *is real and even (resp. real and odd) then* $\mathscr{F}_{\mathrm{DC}}(f)$ *is real and even (resp. imaginary and odd);*

(v) *if* $a \in \mathbb{Z}(\Delta)$ *then* $\mathscr{F}_{\mathrm{DC}}(\tau_a^* f)(\nu) = \mathrm{e}^{-2\pi\mathrm{i}a\nu}\mathscr{F}_{\mathrm{DC}}(f)(\nu)$.

The ℓ^1-DCFT (properties)

Theorem

For $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$, $\mathscr{F}_{\mathrm{DC}}(f) \in \mathrm{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$.

Proof.

This follows from the Weierstrass M-test, cf. the proof of the Theorem on Slide 233. □

- As with the other Fourier transforms we have seen, it is the case that the DCFT is a continuous linear map, in this case between the Banach spaces $(\ell^1(\mathbb{Z}(\Delta);\mathbb{C}), \|\cdot\|_1)$ and $(\mathrm{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C}), \|\cdot\|_\infty)$. This is Theorem IV-7.1.7 in the notes.

Coefficient decay and the DCFT

- Since $\mathscr{F}_{\mathrm{DC}}(f)$ is defined on the continuous time-domain $\mathbb{R}$, we can ask about its smoothness properties; we already know that it is continuous if $f \in \ell^1(\mathbb{Z}(\Delta); \mathbb{C})$.

Proposition (Differentiability of the DCFT of a signal)

For $k \in \mathbb{Z}_{>0}$, suppose that $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ has the property that the signal $t \mapsto t^k f(t)$ is in $\ell^1(\mathbb{Z}(\Delta); \mathbb{C})$. Then $\mathscr{F}_{\mathrm{DC}}(f) \in \mathrm{C}^k(\mathbb{R}; \mathbb{C})$ and

$$\mathscr{F}_{\mathrm{DC}}(f)^{(k)}(\nu) = \Delta \sum_{n \in \mathbb{Z}} (-2\pi \mathrm{i} n\Delta)^k f(n\Delta) \mathrm{e}^{-2\pi \mathrm{i} n \Delta \nu}.$$

Proof.

This follows from an induction using the Weierstrass M-test, along with the fact that differentiation and summation can be swapped for a series which converges and for which the series of derivatives also converges uniformly. $\square$

Reading for Lecture 32

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-6.3.1 and IV-6.3.3.
2. Sections IV-7.1.1, IV-7.1.2, and IV-7.1.3.

Lecture 33
DCFT inversion

- As with the CDFT and the CCFT, the following result is key.

Theorem (DCFT is injective)

$\mathscr{F}_{\mathrm{DC}} \colon \ell^1(\mathbb{Z}(\Delta); \mathbb{C}) \to \mathrm{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R}; \mathbb{C})$ *is injective.*

Proof.

Because $\mathscr{F}_{\mathrm{DC}}$ is linear, to show that it is injective it suffices to show that only the zero signal in $\ell^1(\mathbb{Z}(\Delta); \mathbb{C})$ maps to the zero signal in $\mathrm{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R}; \mathbb{C})$. Thus suppose that $f \in \ell^1(\mathbb{Z}(\Delta); \mathbb{C})$ has the property that $\mathscr{F}_{\mathrm{DC}}(f)(\nu) = 0$ for every $\nu \in \mathbb{R}$. Thus

$$\sum_{n\in\mathbb{Z}} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu} = 0, \qquad \nu \in \mathbb{R}.$$

The sum on the left converges uniformly by the Weierstrass M-test. For $m \in \mathbb{Z}$

$$0 = \sum_{n\in\mathbb{Z}} f(n\Delta) \int_0^{\Delta^{-1}} \mathrm{e}^{2\pi\mathrm{i}m\Delta\nu}\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}\,\mathrm{d}\nu = \frac{f(m\Delta)}{\Delta},$$

DCFT inversion

Proof (cont'd).

swapping the sum and the integral since the sum converges uniformly, and also the fact that

$$\int_0^{\Delta^{-1}} \mathrm{e}^{2\pi\mathrm{i}m\Delta\nu}\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}\,\mathrm{d}\nu = \begin{cases} \Delta^{-1}, & n = m, \\ 0, & n \neq m. \end{cases}$$

This gives the result. □

Proposition

$\mathscr{F}_{\mathrm{DC}} \colon \ell^1(\mathbb{Z}(\Delta); \mathbb{C}) \to \mathrm{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R}; \mathbb{C})$ *is not surjective.*

Proof.

When we discussed Fourier series, we claimed that there exists $F \in \mathrm{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R}; \mathbb{C})$ whose Fourier series diverges at $t = 0$. Such a signal, therefore, cannot be the uniform limit of a series of the form

DCFT inversion

Proof (cont'd).

$$\sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{-2\pi \mathrm{i} n\Delta\nu}.$$

Thus $F \notin \mathrm{image}(\mathscr{F}_{\mathrm{DC}})$. □

- We seek a left-inverse for the DCFT. The story here is much easier than for the CDFT and the CCFT since the natural guess actually works.

Theorem

The map $\mathscr{F}_{\mathrm{DC}}^{-1}\colon \mathrm{C}^0_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C}) \to \mathrm{c}_0(\mathbb{Z}(\Delta);\mathbb{C})$ *defined by*

$$\mathscr{F}_{\mathrm{DC}}^{-1}(F)(n\Delta) = \int_0^{\Delta^{-1}} F(\nu)\mathrm{e}^{2\pi \mathrm{i} n\Delta\nu}\,\mathrm{d}\nu$$

has the property that $\mathscr{F}_{\mathrm{DC}}^{-1} \circ \mathscr{F}_{\mathrm{DC}}(f) = f$ *for every* $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$*. That is to say,* $\mathscr{F}_{\mathrm{DC}}^{-1}$ *is a left-inverse for* $\mathscr{F}_{\mathrm{DC}}$*.*

DCFT inversion

Proof.

Let $f \in \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$, let $m \in \mathbb{Z}$, and compute

$$\begin{aligned}(\mathscr{F}_{\mathrm{DC}}^{-1} \circ \mathscr{F}_{\mathrm{DC}}(f))(m\Delta) &= \Delta \int_0^{\Delta^{-1}} \sum_{n\in\mathbb{Z}} f(n\Delta)\mathrm{e}^{-2\pi \mathrm{i} n\Delta\nu}\mathrm{e}^{2\pi \mathrm{i} m\Delta\nu}\,\mathrm{d}\nu \\ &= \Delta \sum_{n\in\mathbb{Z}} f(n\Delta) \int_0^{\Delta^{-1}} \mathrm{e}^{-2\pi \mathrm{i} n\Delta\nu}\mathrm{e}^{2\pi \mathrm{i} m\Delta\nu}\,\mathrm{d}\nu = f(m\Delta),\end{aligned}$$

as desired. The integral and sum can be swapped because the sum converges uniformly. □

The ℓ^2-DCFT

- Note that $\ell^1(\mathbb{Z}(\Delta);\mathbb{C}) \subset \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ and so the sum

$$\Delta \sum_{n\in\mathbb{Z}} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}$$

  does not obviously converge for $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$.
- However, our work with the CDFT helps us out here.

Theorem

If $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ and if $f_N \in \mathrm{C}^\infty_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C}) \subseteq \mathrm{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ is defined by $f_N(\nu) = \Delta\sum_{n=-N}^{N} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}$, then the sequence $(f_N)_{N\in\mathbb{Z}_{>0}}$ converges in $\mathrm{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ to a signal which we denote $\mathscr{F}_{\mathrm{DC}}(f)$. Moreover, the resulting map $\mathscr{F}_{\mathrm{DC}}\colon \ell^2(\mathbb{Z}(\Delta);\mathbb{C}) \to \mathrm{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ is a vector space isomorphism with inverse

$$\mathscr{F}_{\mathrm{DC}}^{-1}(F)(n\Delta) = \int_0^{\Delta^{-1}} F(\nu)\mathrm{e}^{2\pi\mathrm{i}n\Delta\nu}\,\mathrm{d}\nu.$$

Finally, $\mathscr{F}_{\mathrm{DC}}$ is a Hilbert space isomorphism from $(\ell^2(\mathbb{Z}(\Delta),\mathbb{C}),\langle\cdot,\cdot\rangle_2)$ to $(\mathrm{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C}),\langle\cdot,\cdot\rangle_2)$.

The ℓ^2-DCFT

- *Compare:*
  1. For the CCFT, if $f \in \mathrm{L}^2(\mathbb{R};\mathbb{C})$, then the limit

$$\lim_{T\to\infty}\int_{-T}^{T} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\,\mathrm{d}t$$

     exists in $\mathrm{L}^2(\mathbb{R};\mathbb{C})$, so defining the L^2-CCFT of f.
  2. For the DCFT, if $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$, then the limit

$$\lim_{N\to\infty}\Delta\sum_{n=-N}^{N} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}$$

     exists in $\mathrm{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$, so defining the ℓ^2-DCFT of f.

Examples (ℓ^2-DCFT)

1. Consider $f\colon \mathbb{Z} \to \mathbb{C}$ defined by

$$f(n) = \begin{cases} 0, & n = 0, \\ \mathrm{i}\frac{1-(-1)^n}{n\pi}, & \text{otherwise.}\end{cases}$$

The ℓ^2-DCFT

Examples (ℓ^2-DCFT (cont'd))

Note that $f \in \ell^2(\mathbb{Z};\mathbb{C})$ but that $f \notin \ell^1(\mathbb{Z};\mathbb{C})$. Using our Fourier series computations from the example on Slide 227 we have

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \mathrm{i}\frac{1-(-1)^n}{n\pi}\mathrm{e}^{-2\pi\mathrm{i}n\nu} = \begin{cases} 0, & \nu \in \{0, \frac{1}{2}, 1\}, \\ 1, & \nu \in (0, \frac{1}{2}), \\ -1, & \nu \in (\frac{1}{2}, 1), \end{cases}$$

with the value of $\mathscr{F}_{\mathrm{DC}}(f)$ being defined for all frequencies by periodic extension. Thus, in this case, the series defining $\mathscr{F}_{\mathrm{DC}}(f) \in \mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$ converges for every $\nu \in \mathbb{R}$. Nonetheless, one should be careful to understand that the Theorem above gives convergence in $\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$, not pointwise convergence.

2. Next consider the signal $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ defined by

$$f(n\Delta) = \begin{cases} 0, & n = 0, \\ \frac{1}{n}, & \text{otherwise}. \end{cases}$$

The ℓ^2-DCFT

Examples (ℓ^2-DCFT (cont'd))

Note that $f \in \ell^2(\mathbb{Z}(\Delta);\mathbb{C})$ but that $f \notin \ell^1(\mathbb{Z}(\Delta);\mathbb{C})$. Note that at $\nu = 0$ the limit

$$\lim_{N\to\infty} \Delta \sum_{n=-N}^{N} \frac{1}{n}\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}$$

does not exist. Nonetheless, Theorem IV-7.1.19 ensures that the limit

$$\lim_{N\to\infty} \Delta \sum_{n=-N}^{N} \frac{1}{N}\mathrm{e}^{-2\pi\mathrm{i}n\Delta\nu}$$

exists in $\mathsf{L}^2_{\mathrm{per},\Delta^{-1}}(\mathbb{R};\mathbb{C})$.

The DDFT (definition)

- We now consider the Fourier transform for periodic discrete-time signals.
- Note that if $f\colon \mathbb{Z}(\Delta) \to \mathbb{C}$ is T-periodic then $T = N\Delta$ for some $N \in \mathbb{Z}_{>0}$.
- Denote by $\ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ the set of T-periodic signals.
- Since the signals are periodic with period $T = N\Delta$, the transformed signals in the frequency domain will be discrete with sampling interval $T^{-1} = (N\Delta)^{-1}$. (Think of the CDFT.)
- Because the signals are discrete with sampling interval Δ, the transformed signals in the frequency domain will be periodic with period Δ^{-1}. (Think of the DCFT.)
- Thus the codomain of the DDFT will be $\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1};\mathbb{C})$.

Definition (DDFT)

The ***discrete-discrete Fourier transform*** or ***DDFT*** assigns to $f \in \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ the signal $\mathscr{F}_{\mathrm{DD}}(f) \in \ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$ by

$$\mathscr{F}_{\mathrm{DD}}(f)(\tfrac{k}{N\Delta}) = \Delta \sum_{n=0}^{N-1} f(n\Delta)\mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}n}, \qquad k \in \mathbb{Z}.$$

Reading for Lecture 33

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-7.1.5 and IV-7.1.6.
2. Sections IV-7.2.1 and IV-7.2.2.

Lecture 34
The DDFT (definition)

Examples (DDFT)

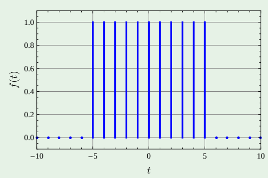
1. We consider a $N\Delta$-periodic signal defined by asking that it be the periodic extension of the signal

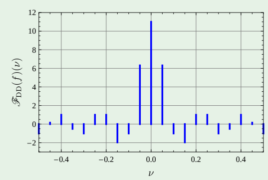
$$f(t) = \begin{cases} 1, & t \in \{-M\Delta, \dots, -\Delta, 0, \Delta, \dots, M\Delta\}, \\ 0, & \text{otherwise}, \end{cases}$$

where M is a positive integer less than $\frac{N}{2}$. One then computes

$$\mathscr{F}_{\mathrm{DD}}(f)(k(N\Delta)^{-1}) = \Delta D^{\mathrm{per}}_{\Delta^{-1},M}(k(N\Delta)^{-1}).$$

Here are the plots with $\Delta = 1$, $N = 20$, and $M = 5$:





The DDFT (definition)

Examples (DDFT (cont'd))

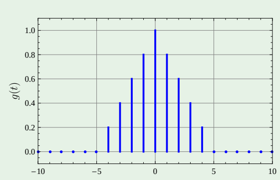
2. Here we define an $N\Delta$-periodic signal using periodic extension by

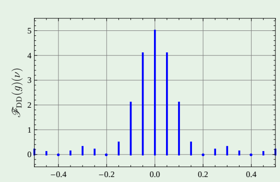
$$g(t) = \begin{cases} -\frac{t}{M\Delta} + 1, & t \in \{0, \Delta, \dots, (M-1)\Delta\}, \\ \frac{t}{M\Delta} + 1, & t \in \{-(M-1)\Delta, \dots, -\Delta\}, \\ 0, & \text{otherwise}, \end{cases}$$

for some positive integer M less than $\frac{N}{2}$. We compute

$$\mathscr{F}_{\mathrm{DD}}(g)(k(N\Delta)^{-1}) = \Delta F^{\mathrm{per}}_{\Delta^{-1},M}(k(N\Delta)^{-1})$$

which we plot for $\Delta = 1$, $N = 20$, and $M = 5$:





The DDFT (properties)

- First of all, one should verify that $\mathscr{F}_{\mathrm{DD}}(f) \in \ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$ if $f \in \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$. This is a direct computation.
- Recall from Slide 14 the reparameterisations τ_a and σ of the time-domain $\mathbb{R}$.
- Let us also define $\overline{\mathscr{F}}_{\mathrm{DD}}(f)(\frac{k}{N\Delta}) = \Delta \sum_{n=0}^{N-1} f(n\Delta)\mathrm{e}^{2\pi\mathrm{i}\frac{k}{N}n}$.

Proposition

(i) $\overline{\mathscr{F}_{\mathrm{DD}}(f)} = \overline{\mathscr{F}}_{\mathrm{DD}}(\bar{f})$;

(ii) $\mathscr{F}_{\mathrm{DD}}(\sigma^* f) = \sigma^*(\mathscr{F}_{\mathrm{DD}}(f)) = \overline{\mathscr{F}}_{\mathrm{DD}}(f)$;

(iii) *if f is even (resp. odd) then $\mathscr{F}_{\mathrm{DD}}(f)$ is even (resp. odd);*

(iv) *if f is real and even (resp. real and odd) then $\mathscr{F}_{\mathrm{DD}}(f)$ is real and even (resp. imaginary and odd);*

(v) *if $m \in \mathbb{Z}$ then $\mathscr{F}_{\mathrm{DD}}(\tau^*_{m\Delta} f)(\frac{k}{N\Delta}) = \mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}m}\mathscr{F}_{\mathrm{DC}}(f)(\frac{k}{N\Delta})$.*

The DDFT (properties)

- Since $\mathscr{F}_{\mathrm{DD}}$ is a map between finite-dimensional $\mathbb{C}$-vector spaces, the determining its deeper properties is much easier than for the other Fourier transforms.
- But we need to give some notation to describe these properties. We define an inner product on $\ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C})$ by

$$\langle f, g\rangle_{\mathrm{time}} = \Delta \sum_{n=0}^{N-1} f(n\Delta)\overline{g(n\Delta)}$$

and an inner product on $\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$ by

$$\langle F, G\rangle_{\mathrm{freq}} = (N\Delta)^{-1} \sum_{n=0}^{N-1} F(n(N\Delta)^{-1})\overline{G(n(N\Delta)^{-1})}.$$

Theorem

$\mathscr{F}_{\mathrm{DD}}$ is a Hilbert space isomorphism of the finite-dimensional Hilbert spaces $(\ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C}), \langle\cdot,\cdot\rangle_{\mathrm{time}})$ and $(\ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C}), \langle\cdot,\cdot\rangle_{\mathrm{freq}})$.

DDFT inversion

- The preceding result already says that $\mathscr{F}_{\mathrm{DD}}$ is invertible. The following result records its inverse.

Theorem

The map $\mathscr{F}_{\mathrm{DD}}\colon \ell_{\mathrm{per},N\Delta}(\mathbb{Z}(\Delta);\mathbb{C}) \to \ell_{\mathrm{per},\Delta^{-1}}(\mathbb{Z}((N\Delta)^{-1});\mathbb{C})$ *is an isomorphism with inverse defined by*

$$\mathscr{F}_{\mathrm{DD}}^{-1}(F)(k\Delta) = \frac{1}{N\Delta}\sum_{n=0}^{N-1} F(n(N\Delta)^{-1})\mathrm{e}^{2\pi\mathrm{i}\frac{k}{N}n}.$$

Proof.

This is a direct computation using the relation

$$\sum_{n=0}^{N-1} \mathrm{e}^{2\pi\mathrm{i}n\frac{j}{N}}\mathrm{e}^{-2\pi\mathrm{i}n\frac{k}{N}} = \begin{cases} N, & k=j, \\ 0, & \text{otherwise.} \end{cases}$$

This is rather analogous to the integral relations for harmonics we have used many times previously for continuous-time signals. □

The FFT

- While many signals in principle are continuous-time signals, they are approximated in practise by discretised versions which can be processed easily using a digital computer.
- For this reason, the DDFT is the most important of the Fourier transforms in practise, although it is the simplest.
- Therefore, it is of some importance to be able to efficiently compute the DDFT.
- A common measure of computational effort required for an algorithm is the number of multiplications it must perform; additions are comparatively cheap.
- A direct application of the DDFT to an $N\Delta$-periodic signal shows that it requires N multiplications for each of its N values, i.e., N^2 multiplications.
- An important algorithm by Cooley and Tukey reduces this to a computation requiring fewer than $\frac{1}{2}N\log_2 N + 2N$ multiplications in cases where N is a power of 2.
- We shall describe the main idea behind the algorithm.

The FFT

- To simplify notation we work with the map from $\mathbb{C}^N$ to $\mathbb{C}^N$ defined by

$$F_{\mathsf{DD}}(z)(k) = \sum_{n=0}^{N-1} z(n)\mathrm{e}^{2\pi\mathrm{i}\frac{k}{N}n}, \qquad k \in \{0, 1, \dots, N-1\}.$$

The relationship between this and the DDFT is clear.

Lemma

Let $N \in \mathbb{Z}_{>0}$ be even and, for $z_1, z_2 \in \mathbb{C}^{N/2}$, define

$$z = (z_1(0), z_2(0), z_1(1), z_2(1), \dots, z_1(\tfrac{N}{2}-1), z_2(\tfrac{N}{2}-1)).$$

Then

$$\begin{aligned} F_{\mathsf{DD}}(z)(k) &= \tfrac{1}{2}(F_{\mathsf{DD}}(z_1)(k) + \mathrm{e}^{-2\pi\mathrm{i}\frac{k}{N}} F_{\mathsf{DD}}(z_2)(k)), \\ F_{\mathsf{DD}}(z)(k+\tfrac{N}{2}) &= \tfrac{1}{2}(F_{\mathsf{DD}}(z_1)(k) - \mathrm{e}^{2\pi\mathrm{i}\frac{k}{N}} F_{\mathsf{DD}}(z_2)(k)) \end{aligned}$$

for $k \in \{0, 1, \dots, \frac{N}{2}-1\}$.

The FFT

Proof.

The proof is a direct computation. □

- The point is that the computation can be broken into two parts, each involving roughly half the number of complex multiplications as the original computation. This is a significant saving.
- Now, if $N = 2^r$ for some $r \in \mathbb{Z}_{>0}$ then:
  1. $\frac{N}{2}$ is even and so the lemma can be applied to reduce the computations for $F_{\mathsf{DD}}(z)$ to shorter computations for $F_{\mathsf{DD}}(z_1)$ and $F_{\mathsf{DD}}(z_2)$;
  2. $\frac{N/2}{2}$ is even and the lemma can be applied to reduce the computations for $F_{\mathsf{DD}}(z_1)$ and $F_{\mathsf{DD}}(z_2)$ to shorter computations for $F_{\mathsf{DD}}(z_{11})$, $F_{\mathsf{DD}}(z_{12})$, $F_{\mathsf{DD}}(z_{21})$, and $F_{\mathsf{DD}}(z_{22})$;
  3. this process can be continued r times.

Reading for Lecture 34

Material related to this lecture can be found in the following sections of the course notes:

1. Sections IV-7.2.2, IV-7.2.3, IV-7.2.5, and IV-7.2.6.