Appendix F
Multilinear algebra

Many of the algebraic constructions we shall carry out rely on fairly elementary multilinear algebra. While the techniques are indeed fairly elementary, it is well-advised to be careful with which of the various possible conventions one is using. Therefore, in this section we give a careful review of multilinear algebra with the primary intention of establishing and understanding our conventions. We suppose that the reader is well acquainted with the tensor product and its properties [e.g., Hungerford 1980, §IV.5].

F.1 The tensor algebra

We first provide some of the basic constructions for the general tensor algebra. This is a specific application of tensor products.

We let F be a field and let V be an F-vector space. We denote by

$$\mathrm{T}^k(\mathsf{V}) = \mathsf{V} \otimes \cdots \otimes \mathsf{V}$$

the k-fold tensor product of V and by $\mathrm{T}(\mathsf{V}) = \oplus_{k=0}^{\infty} \mathrm{T}^k(\mathsf{V})$ the direct sum of all of these products, with the understanding that $\mathrm{T}^0(\mathsf{V}) = \mathsf{F}$. Note that $\mathrm{T}(\mathsf{V})$ is naturally an F-vector space. We wish to show that it is an F-algebra by defining a suitable product. To do so we use the following result.

F.1.1 Lemma (Definition of product of tensors) *Let F be a field and let V be an F-vector space. For each $\mathrm{k}, \mathrm{l} \in \mathbb{Z}_{\geq 0}$ there exists a map $\mathrm{m}_{\mathrm{k},\mathrm{l}} \colon \mathrm{T}^{\mathrm{k}}(\mathsf{V}) \times \mathrm{T}^{\mathrm{l}}(\mathsf{V}) \to \mathrm{T}^{\mathrm{k}+\mathrm{l}}(\mathsf{V})$ such that*

$$\mathrm{m}_{\mathrm{k},\mathrm{l}}(\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_{\mathrm{k}}, \mathrm{v}_{\mathrm{k}+1} \otimes \cdots \otimes \mathrm{v}_{\mathrm{k}+\mathrm{l}}) = \mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_{\mathrm{k}} \otimes \mathrm{v}_{\mathrm{k}+1} \otimes \cdots \otimes \mathrm{v}_{\mathrm{k}+\mathrm{l}}.$$

Proof Since the tensor product is associative [Hungerford 1980, Theorem IV.5.8], there is a isomorphism of the F-vector spaces $\mathrm{T}^k(\mathsf{V}) \otimes \mathrm{T}^l(\mathsf{V})$ and $\mathrm{T}^{k+l}(\mathsf{V})$ defined by

$$(v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_{k+l}) \mapsto v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l}.$$

The map of the lemma is clearly just the bilinear map associated to this isomorphism via the universal property of the tensor product. ■

Clearly the maps $m_{k,l}$, $k, l \in \mathbb{Z}_{\geq 0}$, define a product that makes $\mathrm{T}(\mathsf{V})$ into an associative algebra over F.

We may now formally make the various definitions we use.

F.1.2 Definition (k-tensor, tensor algebra) Let F be a commutative unit and let V be an F-vector space.

(i) For $k \in \mathbb{Z}_{\geq 0}$ a ***k-tensor*** is an element of $\mathrm{T}^k(\mathsf{V})$. We adopt the convention that a ***0-tensor*** is an element of F.

(ii) The ***tensor algebra*** of V is the F-algebra $\mathrm{T}(\mathsf{V})$ with the natural F-vector space structure and the product structure as defined in Lemma F.1.1. •

One readily verifies that $\mathrm{T}(\mathsf{V})$ is an associative F-algebra. We leave the unilluminating verification of this to the reader.

The essential characterisation of the tensor product is its universality with respect to multilinear maps. For the tensor algebra, there is a similarly styled result.

F.1.3 Proposition (Characterisation of the tensor algebra) *Let F be a field, let V be an F-vector space, let A be a unitary F-algebra, and let $\mathrm{f} \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{V}; \mathsf{A})$. Then there exists a unique homomorphism $\phi_{\mathrm{f}} \colon \mathrm{T}(\mathsf{V}) \to \mathsf{A}$ of F-algebras such that the diagram*

$$\begin{tikzcd} \mathrm{T}(\mathsf{V}) \arrow[r, "\phi_{\mathrm{f}}"] & \mathsf{A} \\ \mathsf{V} \arrow[u] \arrow[ur, "\mathrm{f}"'] & \end{tikzcd}$$

commutes, where the vertical arrow is the inclusion of $\mathsf{V} = \mathrm{T}^1(\mathsf{V})$ in $\mathrm{T}(\mathsf{V})$.

Proof For existence, define a k-multilinear map $g_k \in \mathrm{L}^k(\mathsf{V}; \mathsf{A})$ by

$$g_k(v_1, \dots, v_k) = f(v_1) \cdots f(v_k);$$

it is easy to check that this map is indeed k-multilinear. Also define $g_0 \colon \mathsf{F} \to \mathsf{A}$ by $g_0(r) = re$ with $e \in \mathsf{A}$ the identity element. By the universal property of the tensor product, for each $k \in \mathbb{Z}_{\geq 0}$ there then exists a unique $g'_k \in \mathrm{Hom}_{\mathsf{F}}(\mathrm{T}^k(\mathsf{V}); \mathsf{A})$ such that

$$g'_k(v_1 \otimes \cdots \otimes v_k) = g_k(v_1, \dots, v_k), \qquad v_1, \dots, v_k \in \mathsf{V}.$$

To get the existence part of the proposition, let ϕ_f be defined so that its restriction to $\mathrm{T}^k(\mathsf{V})$ is g'_k.

For uniqueness, if ψ_f is an F-algebra homomorphism such that the diagram in the statement of the proposition commutes, then we necessarily have

$$\psi_f(v_1 \otimes \cdots \otimes v_k) = f(v_1) \cdots f(v_k), \qquad v_1, \dots, v_k \in \mathsf{V}.$$

Thus $\psi_f = \phi_f$. ■

The point is that, to determine how the tensor algebra can map into another algebra, it suffices to determine how $\mathsf{V} = \mathrm{T}(\mathsf{V})$ is mapped into the algebra. This is not surprising, and this is the essence of the proof, since elements of the form $v_1 \otimes \cdots \otimes v_k$, $v_1, \dots, v_k \in \mathsf{V}$, $k \in \mathbb{Z}_{>0}$, generate $\mathrm{T}(\mathsf{V})$.

As a useful application of the preceding characterisation of the tensor algebra, we have the following result. The result may be seen as providing a functorial characterisation of the tensor algebra.

F.1.4 Proposition (Induced homomorphisms of tensor algebras) *Let* F *be a field and let* V *and* U *be* F*-vector spaces. If* $f \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{V};\mathsf{U})$ *then there exists a unique homomorphism* f_* *of the* F*-algebras* $\mathrm{T}(\mathsf{V})$ *and* $\mathrm{T}(\mathsf{U})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{V} \arrow[r, "f"] \arrow[d] & \mathsf{U} \arrow[d] \\ \mathrm{T}(\mathsf{V}) \arrow[r, "f_*"'] & \mathrm{T}(\mathsf{U}) \end{tikzcd}$$

commutes, where the vertical arrows are the canonical inclusions.

Proof Apply Proposition F.1.3 to the map $\iota_{\mathsf{U}} \circ f$, where $\iota_{\mathsf{U}} \colon \mathsf{U} \to \mathrm{T}(\mathsf{U})$ is the inclusion. ■

In tensor analysis, one is often interested in tensors of various "types." In order to establish the connection between this and what we define above, we make the following definition.

F.1.5 Definition (Covariant/contravariant tensors) Let F be a field and let V be an F-vector space. For $r, s \in \mathbb{Z}_{\geq 0}$, a ***tensor of contravariant type* r *and covariant type* s** is an element of the F-vector space

$$\mathrm{T}^r_s(\mathsf{V}) = \mathrm{T}^r(\mathsf{V}) \otimes \mathrm{T}^s(\mathsf{V}^*). \qquad \bullet$$

In particular, a k-tensor is of covariant type 0 and contravariant type k. We shall not have much occasion in this book to use tensors of various mixed type.

We shall principally be interested in finite-dimensional vector spaces. In this case the tensor algebra inherits a natural basis from any basis for V. The following result records this, and follows immediately from standard results concerning bases for tensor products, [cf. Hungerford 1980, Corollary IV.5.12].

F.1.6 Proposition (Bases for tensor algebras) *Let* F *be a field and let* V *be an* F*-vector space. If* $\{e_a\}_{a \in A}$ *is a basis for* V *then the set*

$$\{e_{a_1} \otimes \cdots \otimes e_{a_k} \mid a_1, \dots, a_k \in A\}$$

is a basis for $\mathrm{T}^k(\mathsf{V})$*.*

F.1.7 Remark (Tensor algebras and free associative algebras) Let A be a set and let F be a field. The ***free associative*** F***-algebra*** generated by A is defined as follows. Let $M(A)$ be the free monoid generated by A. Thus an element of $M(A)$ is a finite ordered sequence of elements from A (called a ***word***), and the product between two such sequences is juxtaposition. The identity element is the empty sequence. Now let $\mathsf{A}(A)$ be the set of maps $\phi \colon M(A) \to \mathsf{F}$ with finite support. Note that $M(A)$ is naturally regarded as a subset of $\mathsf{A}(A)$. Thus $\mathsf{A}(A)$ is the F-vector space generated by $M(A)$. We make $\mathsf{A}(A)$ into an F-algebra by taking the monoid product on $M(A) \subseteq \mathsf{A}(A)$ and extending this to $\mathsf{A}(A)$ by linearity.

The point is this. If $\{e_a\}_{a \in A}$ is a basis for V then $\mathrm{T}(\mathsf{V})$ is isomorphic as an F-algebra to $\mathsf{A}(A)$, and there is a unique isomorphism which maps $1_{\mathsf{F}} \in \mathrm{T}^0(\mathsf{V})$ to the empty word and the basis vector e_a to the word $\{a\}$ of length 1. •

Now we turn to the interior product for tensors. To make the definition, we note that for $k \in \mathbb{Z}_{\geq 0}$ we may define an isomorphism from $\mathrm{T}^k(\mathsf{V}^*)$ to $\mathrm{L}^k(\mathsf{V};\mathsf{F})$ by assigning to $\alpha_1 \otimes \cdots \otimes \alpha_k$ the multilinear map

$$(v_1, \dots, v_k) \mapsto \alpha_1(v_1)\cdots\alpha_k(v_k).$$

With this in mind, we make the following definition.

F.1.8 Definition (Interior product) Let F be a field and let V be a finite-dimensional F-vector space. For $\omega \in \mathrm{T}^k(\mathsf{V}^*)$ let $\phi_\omega \in \mathrm{L}^k(\mathsf{V};\mathsf{F})$ be the associated k-multilinear map. The ***interior product*** of ω with $v \in \mathsf{V}$ is the element of $\mathrm{T}^{k-1}(\mathsf{V}^*)$ associated with the map $\psi \in \mathrm{L}^{k-1}(\mathsf{V};\mathsf{F})$ given by

$$\psi(v_1, \dots, v_{k-1}) = \phi_\omega(v, v_1, \dots, v_{k-1}).$$

We denote the interior product of ω with v by $v \lrcorner\, \omega$. •

Note that one can in particular apply the definition of the interior product to $\mathrm{TS}^k(\mathsf{V}^*)$ and $\mathrm{T}\bigwedge^k(\mathsf{V}^*)$. Moreover, it is clear that the interior product with v leaves $\mathrm{TS}(\mathsf{V}^*)$ and $\mathrm{T}\bigwedge(\mathsf{V}^*)$ invariant.

Let us give the representation of $v \lrcorner\, \omega$ in a basis. To do so we introduce some multi-index manipulations. If $i \in \{1, \dots, n\}$ and if $J = (j_1, \dots, j_n)$ is an n-multi-index of degree k and with $j_i > 0$, then we define an n-multi-index $J - 1_i$ of degree $k-1$ by subtracting 1 from j_i. With this notation we have the following result whose proof follows from the definitions.

F.1.9 Proposition (Component formula for interior product) *Let* F *be a field and let* V *be a finite-dimensional* F*-vector space with basis* $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ *and dual basis* $\{\mathrm{e}^1, \dots, \mathrm{e}^n\}$*. Accepting an abuse of notation denote*

$$\mathrm{e}^{\mathrm{J}}_{\gamma,\odot} = \gamma_{\mathrm{j}_1}(\mathrm{e}^{\mathrm{j}_1}) \odot \cdots \odot \gamma_{\mathrm{j}_n}(\mathrm{e}^{\mathrm{j}_n})$$

for $\mathrm{J} = (\mathrm{j}_1, \dots, \mathrm{j}_n) \in \mathbb{Z}^n_{\geq 0}$*. Suppose that* $\omega \in \mathrm{T}^k(\mathsf{V}^*)$ *is given by*

$$\omega = \sum_{\mathrm{i}_1,\dots,\mathrm{i}_k \in \{1,\dots,n\}} \omega_{\mathrm{i}_1\cdots\mathrm{i}_k} \mathrm{e}^{\mathrm{i}_1} \otimes \cdots \otimes \mathrm{e}^{\mathrm{i}_k}$$

and that $\mathrm{v} \in \mathsf{V}$ *is given by* $\mathrm{v} = \sum_{\mathrm{i}=1}^n \mathrm{v}^\mathrm{i} \mathrm{e}_\mathrm{i}$*. Then the following statements hold:*

(i) $\mathrm{v} \lrcorner\, \omega = \sum_{\mathrm{i}=1}^{n} \sum_{\mathrm{i}_1,\dots,\mathrm{i}_{k-1} \in \{1,\dots,n\}} \omega_{\mathrm{i}_1\cdots\mathrm{i}_{k-1}} \mathrm{v}^\mathrm{i} \mathrm{e}^{\mathrm{i}_1} \otimes \cdots \otimes \mathrm{e}^{\mathrm{i}_{k-1}}$;

(ii) *if* $\omega \in \mathrm{TS}^k(\mathsf{V}^*)$ *then* $\mathrm{v} \lrcorner\, \omega = \sum_{\mathrm{i}=1}^{n} \sum_{\substack{\mathrm{J}=(\mathrm{j}_1,\dots,\mathrm{j}_n)\\ |\mathrm{J}|=k,\mathrm{j}_\mathrm{i}>0}} \omega_\mathrm{J} \mathrm{v}^\mathrm{i} \mathrm{e}^{\mathrm{J}-1_\mathrm{i}}_{\gamma,\odot}$;

(iii) *if* $\omega \in \mathrm{T}\bigwedge^k(\mathsf{V})$ *then* $\mathrm{v} \lrcorner\, \omega = \sum_{\mathrm{i}=1}^{n} \sum_{\substack{\mathrm{i}_1,\dots,\mathrm{i}_{k-1} \in \{1,\dots,n\}\\ \mathrm{i}_1 < \cdots < \mathrm{i}_k}} \omega_{\mathrm{i}\mathrm{i}_1\cdots\mathrm{i}_{k-1}} \mathrm{v}^\mathrm{i} \mathrm{e}^{\mathrm{i}_1} \,\overline{\wedge}\, \cdots \,\overline{\wedge}\, \mathrm{e}^{\mathrm{i}_{k-1}}$.

F.2 The symmetric algebra and symmetric tensor algebra

Symmetric and alternating tensors will be rather important for us. These can be a little confusing to deal with because they arise naturally in two ways. While, for the cases we are interested in these two ways are isomorphic, there are some annoying constants to deal with. We shall need to get these constants straight, so we delve a little systematically into the necessary background so as to indicate the origin of the various choices one can make.

F.2.1 Tensor products of algebras

When one takes the tensor product of F-algebras A and B, the tensor product naturally inherits the structure of an F-algebra. In order to describe the product, let us first prove the following sensibility result.

F.2.1 Proposition (Sensibility of the product for the tensor product of algebras) *Let F be a field and let $\mathsf{A}_1, \dots, \mathsf{A}_k$ be unitary F-algebras. Define a map $\mathrm{m}\colon (\otimes_{j=1}^k \mathsf{A}_j) \times (\otimes_{j=1}^k \mathsf{A}_j) \to \otimes_{j=1}^k \mathsf{A}_j$ by first defining*

$$\mathrm{m}(v_1 \otimes \cdots \otimes v_k, u_1 \otimes \cdots \otimes u_k) = v_1 u_1 \otimes \cdots \otimes v_k u_k,$$

and then by extension using linearity. Then m defines a product on the F-vector space $\otimes_{j=1}^k \mathsf{A}_j$ which renders it an F-algebra.

Proof This is a matter of checking the definition, and we leave this to the reader. ■

F.2.2 Definition (Tensor product of algebras) Let F be a field and let $\mathsf{A}_1, \dots, \mathsf{A}_k$ be unitary F-algebras. The ***tensor product*** of $\mathsf{A}_1, \dots, \mathsf{A}_k$ is the F-algebra $\otimes_{j=1}^k \mathsf{A}_j$ with the product as defined in Proposition F.2.1. •

The following obvious result will be useful.

F.2.3 Proposition (The tensor product of commutative algebras is commutative) *Let F be a field and let $\mathsf{A}_1, \dots, \mathsf{A}_k$ be commutative unitary F-algebras. Then the F-algebra $\otimes_{j=1}^k \mathsf{A}_j$ is commutative.*

Note that for each $l \in \{1, \dots, k\}$ there is a canonical homomorphism $f_l\colon \mathsf{A}_l \to \otimes_{j=1}^k \mathsf{A}_j$ given by

$$f_l(v_l) = 1_{\mathsf{A}_1} \otimes \cdots \otimes v_l \otimes \cdots \otimes 1_{\mathsf{A}_k}, \tag{F.1}$$

where 1_{A_j} is the unit elements of the algebra A_j. Let us now indicate how one can extend algebra homomorphisms from factors in a tensor product to the tensor product.

F.2.4 Proposition (Homomorphisms from tensor products of algebras) *Let* F *be a field and let* $\mathsf{A}_1, \dots, \mathsf{A}_k, \mathsf{B}$ *be unitary* F*-algebras. For* $j \in \{1, \dots, k\}$ *let* $g_j \colon \mathsf{A}_j \to \mathsf{B}$ *be a unital homomorphism of* F*-algebras, supposing that*

$$g_j(v_j)g_l(v_l) = g_l(v_l)g_j(v_j), \qquad v_j \in \mathsf{A}_j,\ v_l \in \mathsf{A}_l,\ j, l \in \{1, \dots, k\}.$$

Then there exists a unique homomorphism $g \colon \otimes_{j=1}^k \mathsf{A}_j \to \mathsf{B}$ *of* F*-algebras such that the diagram*

$$\begin{tikzcd}
\mathsf{A}_j \arrow[r, "g_j"] \arrow[d, "f_j"'] & \mathsf{B} \\
\otimes_{j=1}^k \mathsf{A}_j \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes for each $j \in \{1, \dots, k\}$.

Proof One can define a map $g' \colon \prod_{j=1}^k \mathsf{A}_j \to \mathsf{B}$ by

$$g'(v_1, \dots, v_k) = g_1(v_1) \cdots g_k(v_k),$$

and verify that this map is k-multilinear. Therefore, by universality of the tensor product, there exists a homomorphism $g \colon \otimes_{j=1}^k \mathsf{A}_j \to \mathsf{B}$ of F-vector spaces such that

$$g(v_1 \otimes \cdots \otimes v_k) = g_1(v_1) \cdots g_k(v_k), \qquad v_j \in \mathsf{A}_j,\ j \in \{1, \dots, k\}.$$

Using the fact that the maps $g_1, \dots, g_k$ are homomorphisms of F-algebras, one can easily show that g is also a homomorphism of F-algebras. This establishes the existence of the map in the statement of the proposition.

For uniqueness, suppose that $h \colon \otimes_{j=1}^k \mathsf{A}_j \to \mathsf{B}$ is another map having the properties asserted in the statement of the proposition. Since

$$h(1_{\mathsf{A}_1} \otimes \cdots \otimes v_j \otimes \cdots \otimes 1_{\mathsf{A}_k}) = g_j(v_j), \qquad v_l \in \mathsf{A}_j,\ j \in \{1, \dots, k\},$$

and since h is a homomorphism of F-algebras, we immediately have

$$h(v_1 \otimes \cdots \otimes v_k) = g_1(v_1) \cdots g_k(v_k), \qquad v_j \in \mathsf{A}_j,\ j \in \{1, \dots, k\}.$$

giving $h = g$, as desired. ■

F.2.2 The symmetric algebra

In this section we define a new F-algebra associated to the tensor algebra of an F-vector space. This new vector space has defined on it a natural product, and this product is symmetric. The benefit of this construction is that the product structure is naturally inherited from that on $\mathsf{T}(\mathsf{V})$. One of the less attractive aspects of the construction is that it is done using quotients, and so is not as concrete as the other construction we detail involving symmetric tensors. Nonetheless, the two constructions will turn out to be isomorphic, possibly up to some constants, in the cases which are of interest to us.

First the definition.

F.2.5 Definition (Symmetric algebra) Let F be a field and let V be an F-vector space. Let $I_S(\mathsf{V})$ be the two-sided ideal of $\mathrm{T}(\mathsf{V})$ generated by elements of the form

$$v_1 \otimes v_2 - v_2 \otimes v_1, \qquad v_1, v_2 \in \mathsf{V}.$$

The ***symmetric algebra*** of V is the F-algebra $\mathrm{S}(\mathsf{V}) = \mathrm{T}(\mathsf{V})/I_S(\mathsf{V})$. The product in $\mathrm{S}(\mathsf{V})$ is denoted by

$$(A_1 + I_S(\mathsf{V})) \cdot (A_2 + I_S(\mathsf{V})) \triangleq A_1 \otimes A_2 + I_S(\mathsf{V}).$$ •

It is clear that $\mathrm{S}(\mathsf{V})$ is an F-algebra. There is a natural degree associated with elements of $\mathrm{S}(\mathsf{V})$ with the elements of degree $k \in \mathbb{Z}_{\geq 0}$ are given by $\mathrm{S}^k(\mathsf{V}) = \mathrm{T}^k(\mathsf{V})/I_S^k(\mathsf{V})$ where $I_S^k(\mathsf{V}) = I_S(\mathsf{V}) \cap \mathrm{T}^k(\mathsf{V})$. In particular, since $I_S^0(\mathsf{V}) = I_S^1(\mathsf{V}) = \{0\}$ it follows that $\mathrm{S}^0(\mathsf{V}) \simeq \mathsf{F}$ and $\mathrm{S}^1(\mathsf{V}) \simeq \mathsf{V}$.

Let us show that the symmetric algebra is symmetric.

F.2.6 Proposition (The product in S(V) is commutative) *Let F be a field and let V be an F-vector space. Then the symmetric algebra is a commutative algebra.*

Proof Since $\{v_1 \otimes \cdots \otimes v_k \mid v_1, \dots, v_k \in \mathsf{V}\}$ generates $\mathrm{T}(\mathsf{V})$ it suffices to show that

$$\begin{aligned}(v_1 \otimes \cdots \otimes v_k + I_S(\mathsf{V})) &\cdot (u_1 \otimes \cdots \otimes u_l + I_S(\mathsf{V})) \\ &= (u_1 \otimes \cdots \otimes u_l + I_S(\mathsf{V})) \cdot (v_1 \otimes \cdots \otimes v_k + I_S(\mathsf{V})), \qquad v_1, \dots, v_k, u_1, \dots, u_l \in \mathsf{V}.\end{aligned}$$

By a trivial inductive argument using associativity of the tensor product it actually suffices to show that

$$(v + I_S(\mathsf{V})) \cdot (u + I_S(\mathsf{V})) = (u + I_S(\mathsf{V})) \cdot (v + I_S(\mathsf{V})), \qquad v, u \in \mathsf{V}.$$

However, since $v \otimes u + I_S(\mathsf{V}) = u \otimes v + I_S(\mathsf{V})$ this immediately follows. ■

We note that the set

$$\{v_1 \otimes \cdots \otimes v_k + I_S(\mathsf{V}) \mid v_1, \dots, v_k \in \mathsf{V}\}$$

generates $\mathrm{S}(\mathsf{V})$. It is, therefore, convenient to have a compact representation of these generators. We shall adopt the following notation:

$$v_1 \cdot \cdots \cdot v_k \triangleq v_1 \otimes \cdots \otimes v_k + I_S(\mathsf{V}).$$

Note that it matters not the order in which one writes the elements in the expression $v_1 \cdot \cdots \cdot v_k$.

For the tensor algebra, the characterisation of Proposition F.1.3 is extremely important and useful. For the symmetric algebra there is an analogous characterisation which we now give.

F.2.7 Proposition (Characterisation of the symmetric algebra) *Let* F *be a field, let* V *be an* F*-vector space, let* A *be a unitary* F*-algebra, and let* $\mathrm{f} \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{V};\mathsf{A})$ *have the property that* $\mathrm{f}(\mathrm{v}_1)\mathrm{f}(\mathrm{v}_2) = \mathrm{f}(\mathrm{v}_2)\mathrm{f}(\mathrm{v}_1)$ *for all* $\mathrm{v}_1, \mathrm{v}_2 \in \mathsf{V}$*. Then there exists a unique* $\phi_{\mathrm{f}} \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{S}(\mathsf{V});\mathsf{A})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{S}(\mathsf{V}) \arrow[r, "\phi_{\mathrm{f}}"] & \mathsf{A} \\ \mathsf{V} \arrow[u] \arrow[ur, "\mathrm{f}"'] & \end{tikzcd}$$

commutes, where the vertical arrow is the inclusion of V *in* $\mathsf{S}(\mathsf{V})$*.*

Proof By Proposition F.1.3 let $\phi'_f \colon \mathrm{T}(\mathsf{V}) \to \mathsf{A}$ be the unique F-algebra homomorphism such that ϕ'_f agrees with f on V. We claim that ϕ'_f vanishes on the ideal $I_S(\mathsf{V})$. Certainly ϕ'_f vanishes on the generators of the ideal $I_S(\mathsf{V})$ since f does. That ϕ'_f vanishes on all of $I_S(\mathsf{V})$ follows since ϕ'_f is an F-algebra homomorphism and since $I_S(\mathsf{V})$ is a two-sided ideal, and so generated as a subspace by elements of the form

$$v_1 \otimes \cdots \otimes v_{l-1} \otimes v_l \otimes \cdots \otimes v_k - v_1 \otimes \cdots \otimes v_l \otimes v_{l-1} \otimes \cdots \otimes v_k, \qquad v_1, \dots, v_k \in \mathsf{V},\ k \in \mathbb{Z}_{>0}.$$

It then follows that there exists a linear map ϕ_f of the F-vector spaces $\mathsf{S}(\mathsf{V})$ and A such that the diagram

$$\begin{tikzcd} \mathrm{T}(\mathsf{V}) \arrow[r, "\phi'_f"] \arrow[d] & \mathsf{A} \\ \mathsf{S}(\mathsf{V}) \arrow[ur, "\phi_f"'] & \end{tikzcd}$$

commutes, where the vertical arrow is the canonical projection. That ϕ_f is a homomorphism of F-algebras follows since $I_S(\mathsf{V})$ is an ideal. Since the canonical projection maps $\mathsf{V} \subseteq \mathrm{T}(\mathsf{V})$ to $\mathsf{V} \simeq \mathsf{S}^1(\mathsf{V})$, this gives the existence part of the proof.

To show uniqueness, let $\psi_f \colon \mathsf{S}(\mathsf{V}) \to \mathsf{A}$ be an F-algebra homomorphism agreeing with f on V and such that the diagram in the statement of the proposition commutes. Thus, since ψ_f is an F-algebra homomorphism,

$$\psi_f(v_1 \cdot \cdots \cdot v_k) = f(v_1) \cdots f(v_k), \qquad v_1, \dots, v_k \in \mathsf{V}.$$

However, given the construction of ϕ'_f in the proof of Proposition F.1.3 and the related construction of ϕ_f above, it follows that

$$\psi_f(v_1 \cdot \cdots \cdot v_k) = \psi_f(v_1 \cdot \cdots \cdot v_k), \qquad v_1, \dots, v_k \in \mathsf{V}.$$

Since the elements $v_1 \cdot \cdots \cdot v_k$, $v_1, \dots, v_k \in \mathsf{V}$, $k \in \mathbb{Z}_{\geq 0}$, generate $\mathsf{S}(\mathsf{V})$, the result follows. ■

The main value of the result is that it allows for the unique extension to $\mathsf{S}(\mathsf{V})$ of a linear map from V. The idea, roughly, is that if one has commutativity of the image for elements of $\mathsf{V} \simeq \mathsf{S}^1(\mathsf{V})$, this carries over to the generators $v_1 \cdot \cdots \cdot v_k$, $v_1, \dots, v_k \in \mathsf{V}$, $k \in \mathbb{Z}_{>0}$, for $\mathsf{S}(\mathsf{V})$ in a natural way.

As with tensor algebras, linear maps between vector spaces induce homomorphisms of their symmetric algebras.

F.2.8 Proposition (Induced homomorphisms of symmetric algebras) *Let* F *be a field and let* V *and* U *be* F*-vector spaces. If* $f \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{V};\mathsf{U})$ *then there exists a unique homomorphism* f_* *of the* F*-algebras* $\mathsf{S}(\mathsf{V})$ *and* $\mathsf{S}(\mathsf{U})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{V} \arrow[r, "f"] \arrow[d] & \mathsf{U} \arrow[d] \\ \mathsf{S}(\mathsf{V}) \arrow[r, "f_*"'] & \mathsf{S}(\mathsf{U}) \end{tikzcd}$$

commutes, where the vertical arrows are the canonical inclusions.

Proof Apply Proposition F.2.7 to the map $\iota_{\mathsf{U}} \circ f$, where $\iota_{\mathsf{U}} \colon \mathsf{U} \to \mathsf{S}(\mathsf{U})$ is the inclusion. ■

Let us now consider bases for symmetric algebras of finite-dimensional vector spaces.

F.2.9 Proposition (The symmetric algebra of a finite-dimensional vector space) *Let* F *be a field and let* V *be a finite-dimensional* F*-vector space with basis* $\mathscr{E} = \{e_1, \dots, e_n\}$*. For an* n*-multi-index* $J = (j_1, \dots, j_n)$ *denote*

$$e_{\cdot}^{J} = e_1^{j_1} \cdot \cdots \cdot e_n^{j_n}.$$

Then the set

$$\mathscr{E}_{\cdot} = \{e_{\cdot}^{J} \mid J \textit{ an } n \textit{ multi-index}\}$$

is a basis for $\mathsf{S}(\mathsf{V})$*.*

Proof Let us first prove a lemma.

1 Lemma *Let* F *be a field and let* $\mathsf{V}_1, \dots, \mathsf{V}_k$ *be* F*-vector spaces. There then exists a unique isomorphism* Φ *of the* F*-algebras* $\otimes_{j=1}^{k} \mathsf{S}(\mathsf{V}_j)$ *and* $\mathsf{S}(\oplus_{j=1}^{k} \mathsf{V}_j)$ *such that the diagram*

$$\begin{tikzcd} \mathsf{S}(\mathsf{V}_j) \arrow[r, "\iota_{j*}"] \arrow[d, "f_j"'] & \mathsf{S}(\oplus_{j=1}^{k} \mathsf{V}_j) \\ \otimes_{j=1}^{k} \mathsf{S}(\mathsf{V}_j) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

commutes for each $j \in \{1, \dots, k\}$*, where* $\iota_j \colon \mathsf{V}_j \to \oplus_{j=1}^{k} \mathsf{V}_j$ *is the canonical inclusion,* ι_{j*} *is the induced homomorphism of symmetric algebras (cf. Proposition F.2.8), and* f_j *is the homomorphism defined by*

$$f_j(v_j) = 1_{\mathsf{S}(\mathsf{V}_1)} \otimes \cdots \otimes v_j \otimes \cdots \otimes 1_{\mathsf{S}(\mathsf{V}_k)}.$$

Proof For brevity denote $\mathsf{V} = \oplus_{j=1}^{k} \mathsf{V}_j$. Since we have the homomorphisms $\iota_{j*} \colon \mathsf{S}(\mathsf{V}_j) \to \mathsf{S}(\mathsf{V})$ for each $j \in \{1, \dots, k\}$ and since $\mathsf{S}(\mathsf{V})$ is commutative, by Proposition F.2.4 there exists a unique homomorphism Φ for which the diagram in the proposition commutes. Thus need only show that Φ is an isomorphism.

For $j \in \{1, \dots, k\}$ define $g_j \colon \mathsf{V}_j \to \otimes_{j=1}^{k} \mathsf{S}(\mathsf{V}_j)$ by $g_j = f_j \circ \iota_j$. Then let $g \colon \mathsf{V} \to \otimes_{j=1}^{k} \mathsf{S}(\mathsf{V}_j)$ be defined by its agreeing with g_j on V_j for each $j \in \{1, \dots, k\}$. Note that $\otimes_{j=1}^{k} \mathsf{S}(\mathsf{V}_j)$ is commutative by Proposition F.2.3. We may thus apply Proposition F.2.7 to assert the existence of a unique F-algebra homomorphism $\Psi\colon \mathsf{S}(\mathsf{V}) \to \otimes_{j=1}^{k}\mathsf{S}(\mathsf{V})$ such that $\Psi \circ \iota_{\mathsf{V}} = g$. It is now an exercise using the definitions to show that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are both the identity map. ▼

We write $\mathsf{V} = \oplus_{j=1}^{n}(\mathsf{F}e_j)$. By the lemma we then have

$$\mathsf{S}(\mathsf{V}) \simeq \otimes_{j=1}^{n}\mathsf{S}(\mathsf{F}e_j). \tag{F.2}$$

Taking components of degree k we have

$$\mathsf{S}^k(\mathsf{V}) \simeq \oplus(\mathsf{S}^{j_1}(\mathsf{F}(e_1)) \otimes \cdots \otimes \mathsf{S}^{j_n}(\mathsf{F}e_n)),$$

where the direct sum is over all multi-indices $(j_1, \dots, j_n)$ of degree k. Note that, by the properties of bases for tensor products [Hungerford 1980, Corollary IV.5.12], a basis for

$$\oplus(\mathsf{S}^{j_1}(\mathsf{F}(e_1)) \otimes \cdots \otimes \mathsf{S}^{j_n}(\mathsf{F}e_n))$$

is given by elements of the form $b_{j_1} \otimes \cdots \otimes b_{j_n}$ where $(j_1, \dots, j_n)$ is an n-multi-index of degree k and where b_{j_l} runs over a basis for $\mathsf{S}^{j_l}(\mathsf{F}e_l)$, $l \in \{1, \dots, n\}$. Since the image of $b_{j_1} \otimes \cdots \otimes b_{j_n}$ under the inverse of the isomorphism (F.2) is simply $b_{j_1} \cdot \cdots \cdot b_{j_n}$, it follows that it suffices to prove the proposition when $\mathsf{V} = \mathsf{F}e$ for some $e \in \mathsf{V}$, i.e., when V is of dimension 1. However, in this case we have $\mathsf{S}(\mathsf{F}e) = \mathsf{T}(\mathsf{F}e)$ ($I_S(\mathsf{V}) = \{0\}$ in this case), and so the result follows from Proposition F.1.6 since the product in $\mathsf{S}(\mathsf{V})$ is simply the tensor product. ■

A consequence of the preceding result is that one can determine the dimension of the degree k component of the symmetric algebra. Recall that $\binom{k}{l} = \frac{k!}{l!(k-l)!}$ for $k, l \in \mathbb{Z}_{\geq 0}$ with $l \leq k$.

F.2.10 Corollary (Dimension of degree k component of S(V)) *Let* F *be a field and let* V *be a finite-dimensional* F*-vector space of dimension* n*. Then, for* $\mathrm{k} \in \mathbb{Z}_{\geq 0}$, $\mathsf{S}^{\mathrm{k}}(\mathsf{V})$ *is of dimension* $\binom{\mathrm{n+k-1}}{\mathrm{n-1}}$.

Proof The dimension of $\mathsf{S}^k(\mathsf{V})$ is equal to the number of n-multi-indices of degree k. This is in turn equal to the number of combinations, allowing repetitions, of k elements from a set of n objects. A combination of this form, corresponding to a multi-index $J = (j_1, \dots, j_n)$ of degree k, can be written after ordering as

$$(\underbrace{1, \dots, 1}_{j_1 \text{ terms}}, \dots, \underbrace{n, \dots, n}_{j_n \text{ terms}})$$

Thus this combination is uniquely determined by the location of the $n-1$ divisions between the elements $1, 2, \dots, n$ in the list. That is to say, to each such combination is uniquely associated $n-1$ divisions chosen from a possible $n+k-1$ divisions. The number of such divisions is $\binom{n+k-1}{n-1}$, from elementary combinatorics. ■

F.2.3 Symmetric tensors

In the preceding section we constructed the symmetric algebra of a vector space as a quotient algebra of the tensor algebra by a certain ideal. In this section we consider symmetric tensors, which form a subspace of the tensor algebra.

We fix $k \in \mathbb{Z}_{\geq 0}$ and let $\sigma \in \mathfrak{S}_k$. We let F be a field with V an F-vector space. We can define a map $\sigma\colon \prod_{j=1}^{k} \mathsf{V} \to \mathrm{T}^k(\mathsf{V})$ (slight abuse of notation) by

$$\sigma(v_1, \dots, v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

It is clear that σ is k-multilinear and so defines a homomorphism $\sigma \in \mathrm{Hom}_{\mathsf{F}}(\mathrm{T}^k(\mathsf{V}); \mathrm{T}^k(\mathsf{V}))$ (another slight abuse of notation) satisfying

$$\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \qquad v_1, \dots, v_k \in \mathsf{V}.$$

We now make the following definition.

F.2.11 Definition (Symmetric tensor) Let F be a field and let V be an F-vector space. An element $A \in \mathrm{T}^k(\mathsf{V})$, $k \in \mathbb{Z}_{\geq 0}$, is a ***symmetric tensor of order* k** if $\sigma(A) = A$ for all $\sigma \in \mathfrak{S}_k$. The set of symmetric elements of $\mathrm{T}^k(\mathsf{V})$ is denoted by $\mathrm{TS}^k(\mathsf{V})$, and we denote

$$\mathrm{TS}(\mathsf{V}) = \oplus_{k \in \mathbb{Z}_{\geq 0}} \mathrm{TS}^k(\mathsf{V}).$$ •

One readily checks that $\mathrm{TS}(\mathsf{V})$ is a subspace of $\mathrm{T}(\mathsf{V})$. We can define an F-linear map $\mathrm{Sym}'_k\colon \mathrm{T}^k(\mathsf{V}) \to \mathrm{TS}^k(\mathsf{V})$ by

$$\mathrm{Sym}'_k(A) = \sum_{\sigma \in \mathfrak{S}_k} \sigma(A).$$

Note that if $A \in \mathrm{TS}^k(\mathsf{V})$ then $\mathrm{Sym}'_k(A) = k!A$. If $A \mapsto k!A$ is invertible in $\mathrm{TS}^k(\mathsf{V})$, e.g., if F is a field of characteristic zero, then we can define $\mathrm{Sym}_k\colon \mathrm{T}^k(\mathsf{V}) \to \mathrm{TS}^k(\mathsf{V})$ by

$$\mathrm{Sym}_k(A) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma(A).$$

This homomorphism has the advantage of being a projection when it is defined. One can extend Sym'_k (and Sym_k, when it is defined) to all of $\mathrm{TS}(\mathsf{V})$ by homogeneity. The resulting map will be denoted by Sym' (and Sym, when it is defined).

We additionally render $\mathrm{TS}(\mathsf{V})$ a subalgebra by defining on it a suitable product. Note that the tensor product itself will not typically suffice since the tensor product of two symmetric tensors is generally not symmetric (and is *never* symmetric in the cases of most interest to us). For $k, l \in \mathbb{Z}_{\geq 0}$ let us define a subset $\mathfrak{S}_{k,l}$ of $\mathfrak{S}_{k+l}$ consisting of permutations σ satisfying

$$\sigma(1) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(k+l).$$

For $A \in \mathrm{TS}^k(\mathsf{V})$ and $B \in \mathrm{TS}^l(\mathsf{V})$ we then define

$$A \odot B = \sum_{\sigma \in \mathfrak{S}_{k,l}} \sigma(A \otimes B).$$

Equipped with this product, let us record some properties of $\mathrm{TS}(\mathsf{V})$.

F.2.12 Proposition (Properties of the algebra TS(V)) *Let* F *be a field and let* V *be an* F*-vector space. For* $\mathrm{k},\mathrm{l},\mathrm{m} \in \mathbb{Z}_{\geq 0}$ *and* $\mathrm{A} \in \mathrm{TS}^{\mathrm{k}}(\mathsf{V})$*,* $\mathrm{B} \in \mathrm{TS}^{\mathrm{l}}(\mathsf{V})$*, and* $\mathrm{C} \in \mathrm{TS}^{\mathrm{m}}(\mathsf{V})$*, the following statements hold:*

(i) $\mathrm{A} \odot \mathrm{B} \in \mathrm{TS}^{\mathrm{k+l}}(\mathsf{V})$*;*

(ii) $\mathrm{A} \odot \mathrm{B} = \mathrm{B} \odot \mathrm{A}$*;*

(iii) $\mathrm{A} \odot (\mathrm{B} \odot \mathrm{C}) = (\mathrm{A} \odot \mathrm{B}) \odot \mathrm{C}$*.*

In particular, $\mathrm{TS}(\mathsf{V})$ *is a commutative subalgebra of* $\mathrm{T}(\mathsf{V})$*.*

Proof Before getting to the proof we first engage in a general discussion which reveals the character of the product in $\mathrm{TS}(\mathsf{V})$. The work here will pay off in the remainder of the proof, and also in our discussion of exterior algebra.

The first part of this discussion concerns the symmetric group. Let $k_1, \dots, k_m \in \mathbb{Z}_{\geq 0}$ be such that $\sum_{j=1}^{m} k_m = k$. Let $\mathfrak{S}_{k_1|\cdots|k_m}$ be the subgroup of $\mathfrak{S}_k$ with the property that elements σ of $\mathfrak{S}_{k_1|\cdots|k_m}$ take the form

$$\begin{pmatrix} 1 & \cdots & k_1 & \cdots & k_1 + \cdots + k_{m-1} + 1 & \cdots & k_1 + \cdots + k_m \\ \sigma_1(1) & \cdots & \sigma_1(k_1) & \cdots & k_1 + \cdots + k_{m-1} + \sigma_m(1) & \cdots & k_1 + \cdots + k_{m-1} + \sigma_m(k_m) \end{pmatrix},$$

where $\sigma_j \in \mathfrak{S}_{k_j}$, $j \in \{1, \dots, m\}$. The assignment $(\sigma_1, \dots, \sigma_m) \mapsto \sigma$ with σ as above is an isomorphism of $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_m}$ with $\mathfrak{S}_{k_1|\cdots|k_m}$. Also denote by $\mathfrak{S}_{k_1,\dots,k_m}$ the subset of $\mathfrak{S}_k$ having the property that $\sigma \in \mathfrak{S}_{k_1,\dots,k_m}$ satisfies

$$\sigma(k_1 + \cdots + k_j + 1) < \cdots < \sigma(k_1 + \cdots + k_j + k_{j+1}), \qquad j \in \{0, 1, \dots, m-1\}.$$

Now we have the following lemma.

1 Lemma *With the above notation, the map* $(\sigma_1, \sigma_2) \mapsto \sigma_1 \circ \sigma_2$ *from* $\mathfrak{S}_{\mathrm{k}_1,\dots,\mathrm{k}_\mathrm{m}} \times \mathfrak{S}_{\mathrm{k}_1|\cdots|\mathrm{k}_\mathrm{m}}$ *to* $\mathfrak{S}_\mathrm{k}$ *is a bijection.*

Proof Let P be the set of partitions $(S_1, \dots, S_m)$ of $\{1, \dots, k\}$ (i.e., $\{1, \dots, k\} = \overset{\circ}{\cup}{}_{j=1}^{m} S_j$) such that $\mathrm{card}(S_j) = k_j$, $j \in \{1, \dots, m\}$. Note that $\mathfrak{S}_k$ acts in a natural way on P. Now specifically choose $S = (S_1, \dots, S_m) \in P$ by

$$S_j = \{k_0 + \cdots + k_{j-1} + 1, \dots, k_1 + \cdots + k_j\}, \qquad j \in \{1, \dots, m\},$$

taking $k_0 = 0$. For a general $T = (T_1, \dots, T_m) \in P$ let $\mathfrak{S}_{S\to T}$ be the set of $\sigma \in \mathfrak{S}_k$ that map S to T. Clearly if $\sigma \in \mathfrak{S}_{S\to T}$ then

$$\mathfrak{S}_{S\to T} = \{\sigma \circ \sigma' \mid \sigma' \in \mathfrak{S}_{k_1|\cdots|k_m}\}.$$

That is to say, $\mathfrak{S}_{S\to T}$ is a left coset of $\mathfrak{S}_{k_1|\cdots|k_m}$ in $\mathfrak{S}_k$. Now note that there exists a unique $\sigma \in \mathfrak{S}_{k_1,\dots,k_m} \cap \mathfrak{S}_{S\to T}$. This gives the result. ▼

Now let us reinterpret the product in $\mathrm{TS}(\mathsf{V})$. The preceding lemma tells us that each left coset of $\mathfrak{S}_{k_1|\cdots|k_m}$ in $\mathfrak{S}_k$ is uniquely identified with an element of $\mathfrak{S}_{k_1,\dots,k_m}$. Thus the sum in the expression

$$\sum_{\sigma \in \mathfrak{S}_{k_1,\dots,k_m}} \sigma(A_1 \otimes \cdots \otimes A_m)$$

is a sum over representatives in $\mathfrak{S}_k/\mathfrak{S}_{k_1|\cdots|k_m}$. Let us show that this sum is, in fact, independent of the choice of representative. That is to say, if $\sigma \in \mathfrak{S}_{k_1,\dots,k_m}$ and if $\sigma' \in \mathfrak{S}_{k_1|\cdots|k_m}$ then

$$\sum_{\sigma\in\mathfrak{S}_{k_1,\dots,k_m}} \sigma\circ\sigma'(A_1\otimes\cdots\otimes A_k) = \sum_{\sigma\in\mathfrak{S}_{k_1,\dots,k_m}} \sigma(A_1\otimes\cdots\otimes A_m) \tag{F.3}$$

for all $A_j \in \mathsf{TS}^{k_j}(\mathsf{V})$, $j\in\{1,\dots,m\}$. This equality, however, follows since

$$\sigma'(A_1\otimes\cdots\otimes A_m) = A_1\otimes\cdots\otimes A_m,$$

as may be verified by first considering how σ' acts on generators for $\mathsf{T}(\mathsf{V})$ then using the fact that $A_1,\dots,A_m$ are symmetric.

Using this interpretation of the product in $\mathsf{TS}(\mathsf{V})$ let us give a useful lemma.

2 Lemma *With the above notation,*

$$\mathrm{A}_1\odot\cdots\odot\mathrm{A}_\mathrm{m} = \sum_{\sigma\in\mathfrak{S}_{\mathrm{k}_1,\dots,\mathrm{k}_\mathrm{m}}} \sigma(\mathrm{A}_1\otimes\cdots\otimes\mathrm{A}_\mathrm{m}).$$

Proof This is vacuously true for $m = 1$, so suppose it true for $m\in\{1,\dots,r-1\}$ for $r\in\mathbb{Z}_{>0}$. Thus

$$A_2\odot\cdots\odot A_r = \sum_{\sigma\in\mathfrak{S}_{k_2,\dots,k_m}} \sigma(A_2\otimes\cdots\otimes A_r).$$

We now have the natural identifications

$$\mathfrak{S}_{k_1|k_2+\cdots+k_r}/\mathfrak{S}_{k_1|\cdots|k_r} \simeq \mathfrak{S}_{k_2,\dots,k_r}$$

and

$$\mathfrak{S}_k/\mathfrak{S}_{k_1|k_2+\cdots+k_r} \simeq \mathfrak{S}_{k_1,k_2+\cdots+k_r}.$$

Thus $\mathfrak{S}_k/\mathfrak{S}_{k_1|\cdots|k_r}$ is identified with the set

$$\{\sigma_1\circ\sigma_2 \mid \sigma_1\in\mathfrak{S}_{k_1,k_2+\cdots+k_r},\ \sigma_2\in\mathfrak{S}_{k_2,\dots,k_r}\}$$

Thus both this set and $\mathfrak{S}_{k_1,\dots,k_r}$ are representatives for $\mathfrak{S}_k/\mathfrak{S}_{k_1|\cdots|k_r}$. Now, using (F.3) we have

$$\begin{aligned} A_1\odot\cdots\odot A_r &= A_1\odot(A_2\odot\cdots\odot A_r) \\ &= \sum_{\sigma\in\mathfrak{S}_{k_1,k_2+\cdots+k_r}} \sigma(A_1\otimes(A_2\odot\cdots\odot A_r)) \\ &= \sum_{\sigma\in\mathfrak{S}_{k_1,k_2+\cdots+k_r}} \sum_{\sigma'\in\mathfrak{S}_{k_2,\dots,k_r}} \sigma\circ\sigma'(A_1\otimes\cdots\otimes A_r) \\ &= \sum_{\sigma''\in\mathfrak{S}_{k_1,\dots,k_r}} \sigma''(A_1\otimes\cdots\otimes A_r), \end{aligned}$$

as desired. ▼

(i) Let $\sigma \in \mathfrak{S}_{k+l}$ and by the lemma write $\sigma = \sigma_1 \circ \sigma_2$ for $\sigma_1 \in \mathfrak{S}_{k,l}$ and $\sigma_2 \in \mathfrak{S}_{k|l}$. We now have

$$\begin{aligned}\sigma(A \odot B) &= \sum_{\sigma' \in \mathfrak{S}_{k,l}} \sigma \circ \sigma'(A \otimes B) \\ &= \sum_{\sigma' \in \mathfrak{S}_{k,l}} \sigma_2 \circ \sigma_1 \circ \sigma'(A \otimes B) \\ &= \sum_{\sigma'' \in \mathfrak{S}_{k,l}} \sigma_2 \circ \sigma''(A \otimes B) \\ &= \sigma_2(A \odot B) = A \odot B,\end{aligned}$$

using (F.3) in the third step and using the fact that $\mathfrak{S}_{k|l}$ fixes $\mathrm{TS}^k(\mathsf{V}) \otimes \mathrm{TS}^l(\mathsf{V})$.

(ii) Let $\sigma_0 \in \mathfrak{S}_{k+l}$ be defined by

$$\sigma_0(1, \dots, k, k+1, \dots, k+l) = (k+1, \dots, k+l, 1, \dots, k),$$

and note that the map $\sigma \mapsto \sigma \circ \sigma_0$ is an isomorphism of the subgroups $\mathfrak{S}_{k,l}$ and $\mathfrak{S}_{l,k}$. Also, for $v_1, \dots, v_k, u_1, \dots, u_l \in \mathsf{V}$ we have

$$\sigma_0((v_1 \otimes \cdots \otimes v_k) \otimes (u_1 \otimes \cdots \otimes u_l)) = (u_1 \otimes \cdots \otimes u_l) \otimes (v_1 \otimes \cdots \otimes v_k).$$

Thus, for $A' \in \mathrm{T}^k(\mathsf{V})$ and $B' \in \mathrm{T}^l(\mathsf{V})$, we have

$$\sigma_0(A' \otimes B') = B' \otimes A'.$$

We then have

$$\begin{aligned}B \odot A = \sum_{\sigma \in \mathfrak{S}_{k,l}} \sigma(B \otimes A) &= \sum_{\sigma \in \mathfrak{S}_{k,l}} \sigma \circ \sigma_0(A \otimes B) \\ &= \sum_{\sigma' \in \mathfrak{S}_{l,k}} \sigma'(A \otimes B) = A \odot B.\end{aligned}$$

(iii) This follows immediately from Lemma 2. ■

A consequence of Lemma 2 that was given in the proof is the following useful relation.

F.2.13 Corollary (Product of elements of degree one in TS(V)) *Let* F *be a field, let* V *be an* F*-vector space, and let* $\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{k} \in \mathsf{V}$*. Then*

$$\mathrm{v}_1 \odot \cdots \odot \mathrm{v}_\mathrm{k} = \mathrm{Sym}'_\mathrm{k}(\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_\mathrm{k})$$

Proof This follows since $\mathfrak{S}_{1,\cdots,1} = \mathfrak{S}_k$. ■

For nice fields there is an alternative formula for the product in $\mathrm{TS}(\mathsf{V})$. Indeed, this is the formula one most often sees, although it does not make sense for vector spaces over general fields.

F.2.14 Proposition (Alternative formula for product in TS(V)) *Let* F *be a field of characteristic zero and let* V *be an* F*-vector space. If* $\mathrm{k}, \mathrm{l} \in \mathbb{Z}_{\geq 0}$, *and if* $\mathrm{A} \in \mathrm{TS}^{\mathrm{k}}(\mathsf{V})$ *and* $\mathrm{B} \in \mathrm{TS}^{\mathrm{l}}(\mathsf{V})$. *Then*

$$\mathrm{A} \odot \mathrm{B} = \frac{(\mathrm{k}+\mathrm{l})!}{\mathrm{k}!\mathrm{l}!} \mathrm{Sym}_{\mathrm{k}+\mathrm{l}}(\mathrm{A} \otimes \mathrm{B}).$$

Proof Since F is a field of characteristic zero the map $A \mapsto k!A$ is an isomorphism of $\mathrm{TS}^k(\mathsf{V})$ for every $k \in \mathbb{Z}_{\geq 0}$. Using Lemma 1 of Proposition F.2.12, for $\sigma \in \mathfrak{S}_{k+1}$ we write $\sigma = \sigma_1 \circ \sigma_2$ for $\sigma_1 \in \mathfrak{S}_{k,l}$ and $\sigma_2 \in \mathfrak{S}_{k|l}$. We have

$$\begin{aligned} \mathrm{Sym}_{k+l}(A \otimes B) &= \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \sigma(A \otimes B) \\ &= \frac{1}{(k+l)!} \sum_{\sigma_1 \in \mathfrak{S}_{k,l}} \sum_{\sigma_2 \in \mathfrak{S}_{k|l}} \sigma_1 \circ \sigma_2(A \otimes B) \\ &= \frac{k!l!}{(k+l)!} \sum_{\sigma_1 \in \mathfrak{S}_{k,l}} \sigma_1(A \otimes B) = \frac{k!l!}{(k+l)!} A \odot B, \end{aligned}$$

using the fact that $\sigma_2(A \otimes B) = A \otimes B$ for all $\sigma_2 \in \mathfrak{S}_{k|l}$. ■

This alternative formula for the product in TS(V) explains why other formulae can be posed for a product in TS(V). For example, one could define a product by $(A, B) \mapsto \mathrm{Sym}_{k+l}(A \otimes B)$, this making sense at least for vector spaces. Our definition for the product has multiple advantages. For one, it is naturally defined in all cases. Also, it serves to eliminate certain annoying constants that can arise is the tensor analysis of symmetric tensors. Finally, the product we use allows the identification of TS(V) with S(V) in those cases of most interest to us. The matter of alternative definitions for products is typically more problematic in exterior algebra, where these competing conventions become quite annoying in dealing with differential forms in differential geometry.

Now let us consider bases for TS(V). It is convenient to introduce, for $k \in \mathbb{Z}_{\geq 0}$, a map $\gamma_k \colon \mathsf{V} \to \mathrm{TS}^k(\mathsf{V})$ defined by

$$\gamma_k(v) = \underbrace{v \otimes \cdots \otimes v}_{k \text{ times}}.$$

We adopt the convention that $\gamma_0(v) = 1_{\mathsf{F}}$. Let us prove some properties of this map γ_k.

F.2.15 Lemma (Properties of γ_{k}) *Let* F *be a field and let* V *be an* F*-vector space. For* $\mathrm{k}, \mathrm{m} \in \mathbb{Z}_{>0}$ *let* $\mathrm{k}_1, \dots, \mathrm{k}_{\mathrm{m}} \in \mathbb{Z}_{\geq 0}$ *be such that* $\mathrm{k}_1 + \cdots + \mathrm{k}_{\mathrm{m}} = \mathrm{k}$ *and let* $\mathrm{S}_{\mathrm{k}_1,\dots,\mathrm{k}_{\mathrm{m}}}$ *be the set of maps* $\phi \colon \{1, \dots, \mathrm{k}\} \to \{1, \dots, \mathrm{m}\}$ *such that*

$$\mathrm{card}(\phi^{-1}(1)) = \mathrm{k}_1, \dots, \mathrm{card}(\phi^{-1}(\mathrm{m})) = \mathrm{k}_{\mathrm{m}}.$$

Then the following statements hold:

(i) $\underbrace{\mathrm{v}\odot\cdots\odot\mathrm{v}}_{k\ \text{times}} = \mathrm{k}!\gamma_{\mathrm{k}}(\mathrm{v})$;

(ii) *for* $\mathrm{v}_1,\dots,\mathrm{v}_\mathrm{m}\in\mathsf{V}$ *we have*

$$\gamma_{\mathrm{k}}(\mathrm{v}_1+\cdots+\mathrm{v}_\mathrm{m}) = \sum_{\substack{\mathrm{k}_1,\dots,\mathrm{k}_\mathrm{m}\\ \mathrm{k}_1+\cdots+\mathrm{k}_\mathrm{m}=\mathrm{k}}} \gamma_{\mathrm{k}_1}(\mathrm{v}_1)\odot\cdots\odot\gamma_{\mathrm{k}_\mathrm{m}}(\mathrm{v}_\mathrm{m});$$

(iii) *for* $\mathrm{v}_1,\dots,\mathrm{v}_\mathrm{m}\in\mathsf{V}$, *for* $\mathrm{k},\mathrm{m}\in\mathbb{Z}_{>0}$, *and for* $\mathrm{k}_1,\dots,\mathrm{k}_\mathrm{m}\in\mathbb{Z}_{\geq 0}$ *such that* $\mathrm{k}_1+\cdots+\mathrm{k}_\mathrm{m}=\mathrm{k}$,

$$\gamma_{\mathrm{k}_1}(\mathrm{v}_1)\odot\cdots\odot\gamma_{\mathrm{k}_\mathrm{m}}(\mathrm{v}_\mathrm{m}) = \sum_{\sigma\in\mathfrak{S}_{\mathrm{k}_1,\dots,\mathrm{k}_\mathrm{m}}} \mathrm{v}_{\sigma(1)}\otimes\cdots\otimes\mathrm{v}_{\sigma(\mathrm{k})};$$

(iv) *for* $\mathrm{k},\mathrm{l}\in\mathbb{Z}_{\geq 0}$ *and for* $\mathrm{v}\in\mathsf{V}$ *we have*

$$\gamma_{\mathrm{k}}(\mathrm{v})\odot\gamma_{\mathrm{l}}(\mathrm{v}) = \frac{(\mathrm{k}+\mathrm{l})!}{\mathrm{k}!\mathrm{l}!}\gamma_{\mathrm{k}+\mathrm{l}}(\mathrm{v});$$

(v) *for* $\mathrm{v}_1,\dots,\mathrm{v}_\mathrm{m}\in\mathsf{V}$ *we have*

$$(-1)^{\mathrm{m}}\mathrm{v}_1\odot\cdots\odot\mathrm{v}_\mathrm{m} = \sum_{\mathrm{k}=1}^{\mathrm{m}}\ \sum_{\{\mathrm{j}_1,\dots,\mathrm{j}_\mathrm{k}\}\subseteq\{1,\dots,\mathrm{m}\}} (-1)^{\mathrm{k}}\gamma_{\mathrm{m}}(\mathrm{v}_{\mathrm{j}_1}+\cdots+\mathrm{v}_{\mathrm{j}_\mathrm{k}}).$$

Proof (i) This follows from Corollary F.2.13.

(ii) We prove the result for $m=2$, the general case following from an induction from this case. We first claim that

$$(v_1+v_2)\otimes\cdots\otimes(v_1+v_2) = \sum_{\substack{k_1,k_2\\ k_1+k_2=k}}\ \sum_{\sigma\in\mathfrak{S}_{k_1,k_2}} \sigma(\underbrace{v_1\otimes\cdots\otimes v_1}_{k_1\ \text{times}}\otimes\underbrace{v_2\otimes\cdots\otimes v_2}_{k_2\ \text{times}}).$$

This can be proved using an induction on k, and we leave the details to the reader. We now have

$$\begin{aligned}\gamma_k(v_1+v_2) &= \sum_{\substack{k_1,k_2\\ k_1+k_2=k}}\ \sum_{\sigma\in\mathfrak{S}_{k_1,k_2}} \sigma(v_1\otimes\cdots\otimes v_1\otimes v_2\otimes\cdots\otimes v_2)\\ &= \sum_{\substack{k_1,k_2\\ k_1+k_2=k}}\ \sum_{\sigma\in\mathfrak{S}_{k_1,k_2}} \sigma(\gamma_{k_1}(v_1)\otimes\gamma_{k_2}(v_2))\\ &= \sum_{\substack{k_1,k_2\\ k_1+k_2=k}} \gamma_{k_1}(v_1)\odot\gamma_{k_2}(v_2).\end{aligned}$$

(iii) By Lemma 2 of Proposition F.2.12 we have

$$\gamma_{k_1}(v_1)\odot\cdots\odot\gamma_{k_m}(v_m) = \sum_{\sigma\in\mathfrak{S}_{k_1,\dots,k_m}} \sigma(\underbrace{v_1\otimes\cdots\otimes v_1}_{k_1\ \text{times}}\otimes\cdots\otimes\underbrace{v_m\otimes\cdots\otimes v_m}_{k_m\ \text{times}}).$$

By definition of $S_{k_1,\dots,k_m}$ (and a moment's thought) the result follows.

(iv) This is a special case of (iii).

(v) Here we prove a lemma. Let

$$\mathrm{Pol}^k(\mathsf{V};\mathsf{U}) = \{f\colon \mathsf{V}\to\mathsf{U} \mid \text{ there exists } \phi\in \mathrm{L}^k(\mathsf{V};\mathsf{U}) \text{ such that } f(v)=\phi(v,\dots,v)\}$$

be the set of polynomial mappings from V to U that are homogeneous of degree k.

1 **Sublemma** *Let* F *be a field and let* V *and* U *be* F*-vector spaces. Then, for* $\mathrm{f}\in\mathrm{Pol}^{\mathrm{k}}(\mathsf{V};\mathsf{U})$ *there exists a unique* $\phi\in\mathrm{L}^{\mathrm{k}}_{\mathrm{sym}}(\mathsf{V};\mathsf{U})$ *such that*

$$\mathrm{f}(\mathrm{v}) = \phi(\mathrm{v},\dots,\mathrm{v}).$$

Moreover, for $\mathrm{v}_1,\dots,\mathrm{v}_\mathrm{k}\in\mathsf{V}$ *we have*

$$\phi(\mathrm{v}_1,\dots,\mathrm{v}_\mathrm{k}) = \frac{1}{\mathrm{k}!}\sum_{\mathrm{l}=1}^{\mathrm{k}}\sum_{\{\mathrm{j}_1,\dots,\mathrm{j}_\mathrm{l}\}\subseteq\{1,\dots,\mathrm{k}\}} (-1)^{\mathrm{k}-\mathrm{l}}\phi(\mathrm{v}_{\mathrm{j}_1}+\dots+\mathrm{v}_{\mathrm{j}_\mathrm{l}},\dots,\mathrm{v}_{\mathrm{j}_1}+\dots+\mathrm{v}_{\mathrm{j}_\mathrm{l}}). \tag{F.4}$$

Proof For the evistence part of the result, let $\phi'\in\mathrm{L}^k(\mathsf{V};\mathsf{U})$ be such that

$$f(v) = \phi'(v,\dots,v)$$

and define $\phi\in\mathrm{L}^k_{\mathrm{sym}}(\mathsf{V};\mathsf{U})$ by

$$\phi(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\phi'(v_{\sigma(1)},\dots,v_{\sigma(k)}).$$

Clearly we have $f(v)=\phi(v,\dots,v)$.

For the uniqueness part of the result we will show that, if $\phi\in\mathrm{L}^k_{\mathrm{sym}}(\mathsf{V};\mathsf{U})$ satisfies $f(v)=\phi(v,\dots,v)$, then ϕ must be given by (F.4). We prove this by examining the terms in the sum. For $l\in\{1,\dots,k\}$ and for $\{j_1,\dots,j_l\}\subseteq\{1,\dots,k\}$, if we expand the expression

$$\phi(v_{j_1}+\dots+v_{j_l},\dots,v_{j_1}+\dots+v_{j_l})$$

using multilinearity of ϕ, we obtain the sum of all terms of the form $\phi(v_{r_1},\dots,v_{r_k})$ where $r_1,\dots,r_k\in\{j_1,\dots,j_l\}$. Thus this is a sum with l^k terms. Therefore, the right-hand side of the expression (F.4) will itself be a linear combination of terms of the form $\phi(v_{r_1},\dots,v_{r_k})$ where $r_1,\dots,r_k\in\{1,\dots,k\}$. To prove the proposition we shall show that the coefficient in the linear combination is 0 unless $r_1,\dots,r_k$ are distinct. When $r_1,\dots,r_k$ are distinct, we shall show that the coefficient in the linear combination is 1. This will prove the proposition since the terms on the right-hand side corresponding to the case when $r_1,\dots,r_k$ are distinct correspond exactly to the terms on the left-hand side of the expression (F.4).

Let us fix $r_1,\dots,r_k\in\{1,\dots,k\}$ (not necessarily distinct) and examine how many terms of the form $\phi(v_{r_1},\dots,v_{r_k})$ appear in the sum on the right in (F.4). This will depend on how many distinct elements of $\{1,\dots,k\}$ appear in the set $\{r_1,\dots,r_k\}$. Let us suppose that there are s distinct elements. For $l\geq s$, in the set of subsets $\{j_1,\dots,j_l\}\subseteq\{1,\dots,k\}$ there will be $D(k,l,s)$ members which contain $\{r_1,\dots,r_k\}$ as a subset, where

$$D(k,l,s) = \frac{(k-s)!}{(l-s)!(k-l)!}.$$

To see this, note that to each subset $\{j_1,\dots,j_l\} \subseteq \{1,\dots,k\}$ that contains $\{r_1,\dots,r_k\}$ as a subset, there corresponds a unique subset of $l-s$ elements of a set of $k-s$ elements (the complement to $\{r_1,\dots,r_k\}$ in $\{j_1,\dots,j_l\}$). There are $D(k,l,s)$ such subsets after we note that $D(k,l,s) = \binom{k-s}{l-s}$. This means that there will be $D(k,l,s)$ terms of the form $\phi(v_{r_1},\dots,v_{r_k})$ which appear in the sum

$$\sum_{\{j_1,\dots,j_l\}\subseteq\{1,\dots,k\}} (-1)^{k-l}\phi(v_{j_1}+\cdots+v_{j_l},\dots,v_{j_1}+\cdots+v_{j_l}).$$

Therefore, there will be $\sum_{l=s}^{k}(-1)^{k-l}D(k,l,s)$ terms of the form $\phi(v_{r_1},\dots,v_{r_k})$ in the right-hand side of the expression (F.4). We claim that

$$\sum_{l=s}^{k}(-1)^{k-l}D(k,l,s) = \begin{cases} 1, & s=k,\\ 0, & s<k. \end{cases}$$

For $s=k$ the equality is checked directly. For $s<k$ we note that, for $r_1,r_2 \in \mathsf{F}$ and for $k-s>0$, we have

$$(r_1+r_2)^{k-s} = \sum_{j=0}^{k-s}\binom{k-s}{j} r_1^j r_2^{k-s-j} = \sum_{l=s}^{k} D(k,l,s) r_1^{l-s} r_2^{k-l}.$$

Letting $r_1 = 1_{\mathsf{F}}$ and $r_2 = -1_{\mathsf{F}}$ we obtain

$$\sum_{l=s}^{k}(-1)^{k-l}D(k,l,s) = 0,$$

as desired. ▼

This part of the proof follows from part (i) and the lemma by replacing f and ϕ in the lemma with γ_k and

$$(v_1,\dots,v_m) \mapsto m! v_1 \odot \cdots \odot v_m,$$

respectively. ■

With this notation we have the following result.

F.2.16 Proposition (The symmetric tensor algebra of a finite-dimensional vector space) *Let* F *be a field and let* V *be a finite-dimensional* F*-vector space with basis* $\mathscr{E} = \{\mathrm{e}_1,\dots,\mathrm{e}_n\}$*. For an* n*-multi-index* $\mathrm{J} = (\mathrm{j}_1,\dots,\mathrm{j}_n)$ *denote*

$$\mathrm{e}^{\mathrm{J}}_{\gamma,\odot} = \gamma_{\mathrm{j}_1}(\mathrm{e}_1)\odot\cdots\odot\gamma_{\mathrm{j}_n}(\mathrm{e}_n).$$

Then the set

$$\{\mathrm{e}^{\mathrm{J}}_{\gamma,\odot} \mid \mathrm{J} \text{ is an } n\text{-multi-index}\}$$

is a basis for $\mathsf{TS}(\mathsf{V})$*.*

Proof Recall from Proposition F.1.6 that the set

$$\mathscr{T}_k = \{e_{j_1} \otimes \cdots \otimes e_{j_k} \mid j_1, \dots, j_k \in \{1, \dots, n\}\}$$

is a basis for $\mathrm{T}^k(\mathsf{V})$. We will show that

$$\mathscr{S}_k = \{e^J_{\gamma,\odot} \mid J \text{ is an } n\text{-multi-index of degree } k\}$$

is a basis for $\mathrm{TS}^k(\mathsf{V})$. Note that $\mathrm{TS}^k(\mathsf{V})$ is the subspace of fixed points for the action of $\mathfrak{S}_k$ on $\mathrm{T}^k(\mathsf{V})$ given by $(\sigma, A) \mapsto \sigma(A)$. Note that the set $\mathscr{T}_k$ is invariant under this action. From part (iii) of Lemma F.2.15,

$$\begin{aligned}\gamma_{j_1}(e_1) \odot \cdots \odot \gamma_{j_n}(e_n) &= \sum_{\sigma \in \mathfrak{S}_{j_1,\dots,j_n}} \sigma(\underbrace{e_1 \otimes \cdots \otimes e_1}_{j_1 \text{ times}} \otimes \cdots \otimes \underbrace{e_n \otimes \cdots \otimes e_n}_{j_n \text{ times}}) \\ &= \sum_{\phi \in S_{j_1,\dots,j_n}} e_{\phi(1)} \otimes \cdots \otimes e_{\phi(k)}. \end{aligned} \tag{F.5}$$

In particular, for $J = (j_1, \dots, j_n)$ with $|J| = k$, $e^J_{\gamma,\odot}$ is the sum over all basis elements for $\mathrm{T}^k(\mathsf{V})$ in which e_l appears j_l times for each $l \in \{1, \dots, n\}$. Now note that $\sigma(e^J_{\gamma,\odot}) = e^J_{\gamma,\odot}$ for each $\sigma \in \mathfrak{S}_k$ and so

$$\operatorname{span}_{\mathsf{F}}(\mathscr{S}_k) \subseteq \mathrm{TS}^k(\mathsf{V}).$$

Next suppose that $A \in \mathrm{TS}^k(\mathsf{V})$ and write

$$A = \sum_{i_1,\dots,i_k \in \{1,\dots,n\}} A^{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}.$$

We then have $\sigma(A) = A$ for each $\sigma \in \mathfrak{S}_k$ and so

$$\sum_{i_1,\dots,i_k \in \{1,\dots,n\}} A^{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} = \sum_{i_1,\dots,i_k \in \{1,\dots,n\}} A^{i_1 \cdots i_k} e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(k)}}$$

for each $\sigma \in \mathfrak{S}_k$. Therefore, it follows that

$$A^{i_1 \cdots i_k} = A^{i_{\sigma(1)} \cdots i_{\sigma(k)}}$$

for each $\sigma \in \mathfrak{S}_k$. For $i_1, \dots, i_k \in \{1, \dots, n\}$ define an n-multi-index $(j_1, \dots, j_n)$ of degree k by asking that j_l be the number of times l appears in the list $i_1, \dots, i_k$. Then, with J so defined, define $A^J = A^{i_1 \cdots i_k}$ and note that

$$A^J = A^{i_{\sigma(1)} \cdots i_{\sigma(k)}}, \qquad \sigma \in \mathfrak{S}_k.$$

Therefore, given (F.5) and the interpretation following that formula,

$$A = \sum_{J \in \mathbb{Z}^n_{\geq 0}, |J| = k} A^J \sum_{\phi \in S_{j_1,\dots,j_n}} e_{\phi(1)} \otimes \cdots \otimes e_{\phi(k)} = \sum_{J \in \mathbb{Z}^n_{\geq 0}, |J| = k} A^J e^J_{\gamma,\odot}.$$

Thus

$$\mathrm{TS}^k(\mathsf{V}) \subseteq \operatorname{span}_{\mathsf{F}}(\mathscr{S}_k).$$

Thus $\mathscr{S}_k$ generates $\mathrm{TS}^k(\mathsf{V})$.

Finally, we need to show that $\mathscr{S}_k$ is linearly independent. Suppose that

$$\sum_{J\in\mathbb{Z}^n_{\geq 0},|J|=k} A^J e^J_{\gamma,\odot} = 0.$$

Then, following the computations above and switching from multi-indices to indices, we have

$$\sum_{i_1,\dots,i_k\in\{1,\dots,n\}} A^{i_1\cdots i_k} e_{i_1}\otimes\cdots\otimes e_{i_k} = 0.$$

This gives $A^{i_1\cdots i_k} = 0$ for every $i_1,\dots,i_k \in \{1,\dots,n\}$ and so gives the result. ■

It is easy to see from the proof that the condition that V be finitely generated can be relaxed with the penalty of a little more notation. However, since the finitely generated case is of most interest to us, we stick to this.

The proof of the proposition contains some notation that is useful to record. Note that if $A \in \mathrm{TS}^k(\mathsf{V})$ then, using the proposition, we can write

$$A = \sum_{J\in\mathbb{Z}^n_{\geq 0},|J|=k} A^J e^J_{\gamma,\odot},$$

for $A^J \in \mathsf{F}$ with J an n-multi-index of degree k. However, since $\mathrm{TS}^k(\mathsf{V}) \subseteq \mathrm{T}^k(\mathsf{V})$, we can also write

$$A = \sum_{i_1,\dots,i_k\in\{1,\dots,n\}} A^{i_1\cdots i_k} e_{i_1}\otimes\cdots\otimes e_{i_k},$$

for $A^{i_1\cdots i_k} \in \mathsf{F}$ with $i_1,\dots,i_k \in \{1,\dots,n\}$. As is shown in the proof, symmetry of A implies that

$$A^{i_1\cdots i_k} = A^{i_{\sigma(1)}\cdots i_{\sigma(k)}}, \qquad \sigma \in \mathfrak{S}_k.$$

The matter at hand is, "What is the relationship between the A^J's and the $A^{i_1\cdots i_k}$'s?" This relationship is given in the proof and it is as follows.

1. From A^J to $A^{i_1\cdots i_k}$: Let $J = (j_1,\dots,j_n)$. For an index $i_1\cdots i_k$ having the property that, for each $l \in \{1,\dots,n\}$, the number l appears j_l times in the list $i_1\cdots i_k$, define $A^{i_1\cdots i_k} = A^J$.
2. From $A^{i_1\cdots i_k}$ to A^J: Define the n-multi-index $J = (j_1,\dots,j_n)$ by asking that, for each $l \in \{1,\dots,n\}$, j_l is the number of times l appears in the list $i_1\cdots i_k$. Define $A^J = A^{i_1\cdots i_k}$.

This business is entirely analogous to the discussion concerning indices and multi-indices in Section 1.1.2.

F.2.4 Homomorphisms involving symmetric algebras and symmetric tensors

In the preceding two sections we introduced two commutative algebras $\mathrm{S}(\mathsf{V})$ and $\mathrm{TS}(\mathsf{V})$. In this section we explore some natural correspondences between these algebras and other sorts of algebraic objects. These alternative ways of thinking about the algebras S(V) and TS(V) will be of direct benefit for us. We also show that in the case of interest to us—that when V is a finite-dimensional vector space—all objects are canonically isomorphic.

First let us describe the natural homomorphisms that exist between S(V) and TS(V). By Proposition F.1.3 there exists a unique F-algebra homomorphism from T(V) to TS(V) which extends the canonical injection of V into TS(V). Since $\mathrm{Sym}' \colon \mathrm{T(V)} \to \mathrm{TS(V)}$ is such an F-algebra homomorphism, it must be the only one. By Proposition F.2.7 there exists a unique homomorphism ϕ_V from S(V) to TS(V) which extends the inclusion of V in S(V). Moreover, the diagram

$$\begin{tikzcd} \mathrm{T(V)} \arrow[d] \arrow[dr, "\mathrm{Sym}'"] & \\ \mathrm{S(V)} \arrow[r, "\phi_V"'] & \mathrm{TS(V)} \end{tikzcd} \tag{F.6}$$

commutes, where the vertical arrow is the canonical projection.

We also have, just by composition of the injection of TS(V) in T(V) with the projection from T(V) to S(V) the following commutative diagram

$$\begin{tikzcd} & \mathrm{T(V)} \arrow[d] \\ \mathrm{TS(V)} \arrow[ur] \arrow[r, "\psi_V"'] & \mathrm{S(V)} \end{tikzcd}$$

which defines the F-algebra homomorphism ψ_V.

The following result indicates how the homomorphisms ϕ_V and ψ_V are related.

F.2.17 Proposition (Relationship between S(V) and TS(V)) *Let* F *be a field and let* V *be an* F*-vector space. Then the following statements hold:*

(i) $\psi_V \circ \phi_V(A) = k!A$ *for* $A \in \mathrm{S(V)}$;

(ii) $\phi_V \circ \psi_V(A) = k!A$ *for* $A \in \mathrm{TS(V)}$.

Proof (i) For $v_1, \dots, v_k \in V$ we have

$$\phi_V(v_1 \cdot \dots \cdot v_k) = v_1 \odot \dots \odot v_k = \sum_{\sigma \in \mathfrak{S}_k} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}.$$

Now, by definition of ψ_V, we have

$$\begin{aligned} \psi_V \circ \phi_V(v_1 \cdot \dots \cdot v_k) &= \sum_{\sigma \in \mathfrak{S}_k} \psi_V(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}) \\ &= k! v_1 \cdot \dots \cdot v_k, \end{aligned}$$

giving the result.

(ii) Consider an element of $\mathrm{TS}^k(\mathsf{V})$ of the form

$$\sum_{j=1}^{m} v_1^j \otimes \cdots \otimes v_k^j,$$

for $v_l^j \in \mathsf{V}$, $j \in \{1,\dots,m\}$, $l \in \{1,\dots,k\}$ (every element of $\mathrm{TS}^k(\mathsf{V})$ is a linear combination of such elements). Then

$$\psi_{\mathsf{V}}\Big(\sum_{j=1}^{m} v_1^j \otimes \cdots \otimes v_k^j\Big) = \sum_{j=1}^{m} v_1^j \cdot \cdots \cdot v_k^j.$$

By definition of ϕ_{V} (especially the commutative diagram (F.6)) we have

$$\begin{aligned}\phi_{\mathsf{V}} \circ \psi_{\mathsf{V}}\Big(\sum_{j=1}^{m} v_1^j \otimes \cdots \otimes v_k^j\Big) &= \sum_{j=1}^{m} \mathrm{Sym}_k'(v_1^j \otimes \cdots \otimes v_k^j)\\ &= \mathrm{Sym}_k'\Big(\sum_{j=1}^{m} v_1^j \otimes \cdots \otimes v_k^j\Big)\\ &= k! \sum_{j=1}^{m} v_1^j \otimes \cdots \otimes v_k^j,\end{aligned}$$

as desired. ■

In particular, if the map $A \mapsto k!A$ is an isomorphism of $\mathrm{TS}^k(\mathsf{V})$ for each $k \in \mathbb{Z}_{\geq 0}$, then $\frac{1}{k!}\psi_{\mathsf{V}}|\mathrm{TS}^k(\mathsf{V})$ is the inverse of $\phi_{\mathsf{V}}|\mathrm{TS}^k(\mathsf{V})$ for each $k \in \mathbb{Z}_{\geq 0}$. This gives the following corollary.

F.2.18 Corollary (Symmetric tensors and the symmetric algebra for vector spaces over fields of characteristic zero) *Let F be a field of characteristic zero and let V be an F-vector space. Then ϕ_{V} is an isomorphism of F-algebras $\mathrm{S}(\mathsf{V})$ and $\mathrm{TS}(\mathsf{V})$.*

Why do we use ϕ_{V} rather than ψ_{V} as the isomorphism? The reason is that ϕ_{V} preserves the bases for $\mathrm{S}(\mathsf{V})$ and $\mathrm{TS}(\mathsf{V})$ in a nice way. Let us describe this. Suppose that V is finite-dimensional with basis $\mathscr{E} = \{e_1,\dots,e_n\}$. For an n-multi-index $J = (j_1,\dots,j_n)$ define

$$e_{\cdot}^J = e_1^{j_1} \cdot \cdots \cdot e_n^{j_n}, \quad e_{\gamma,\odot}^J = \gamma_{j_1}(e_1) \odot \cdots \odot \gamma_{j_n}(e_n).$$

Recall from Propositions F.2.9 and F.2.16 that the sets

$$\begin{aligned}\mathscr{E}_{\cdot} &= \{e_{\cdot}^J \mid J \text{ an } n \text{ multi-index}\},\\ \mathscr{E}_{\gamma,\odot} &= \{e_{\gamma,\odot}^J \mid J \text{ is an } n\text{-multi-index}\}\end{aligned}$$

form bases for $\mathrm{S}(\mathsf{V})$ and $\mathrm{TS}(\mathsf{V})$, respectively. Note that

$$\underbrace{e_l \odot \cdots \odot e_l}_{j_l \text{ times}} = j_l!\gamma_{j_l}(e_l).$$

Under the assumption that $A \mapsto k!A$ is an isomorphism of $\mathrm{TS}^k(\mathsf{V})$ for each $k \in \mathbb{Z}_{\geq 0}$, it follows that we can define

$$e_{\odot}^{J} = e_1^{j_1} \odot \cdots \odot e_n^{j_n} = \frac{1}{j_1! \cdots j_n!} e_{\gamma,\odot}^{J},$$

where the powers are defined relative to the product $\odot$. Moreover, the set

$$\mathscr{E}_{\odot} = \{e_{\odot}^{J} \mid J \text{ is an } n\text{-multi-index}\}$$

is a basis for $\mathrm{TS}(\mathsf{V})$. This basis has the property that $\phi_{\mathsf{V}}(e_{\cdot}^{J}) = e_{\odot}^{J}$. Thus $\mathrm{TS}(\mathsf{V})$ has two naturally defined bases, $\mathscr{E}_{\gamma,\odot}$ and $\mathscr{E}_{\odot}$, in the case that $A \mapsto k!A$ is an isomorphism of $\mathrm{TS}^k(\mathsf{V})$ for each $k \in \mathbb{Z}_{\geq 0}$. The basis $\mathscr{E}_{\gamma,\odot}$ has the advantage of always being definable when V is finite-dimensional. The basis $\mathscr{E}_{\odot}$ has the advantage of corresponding with the natural basis for $\mathrm{S}(\mathsf{V})$ is cases when the latter is isomorphic to $\mathrm{TS}(\mathsf{V})$.

It is worthwhile understanding the consequences of this in terms of representing the elements of $\mathrm{TS}(\mathsf{V})$ using the two sorts of bases. We still suppose that V is finite-dimensional with basis $\{e_1, \dots, e_n\}$, and we suppose that $A \mapsto k!A$ is an isomorphism of $\mathrm{TS}^k(\mathsf{V})$ for each $k \in \mathbb{Z}_{\geq 0}$. As we saw in the proof of Proposition F.2.16, and as was further elucidated following the proof, there is a natural way of moving between index and multi-index notation for representations of symmetric tensors. Using this convention, if we write $A \in \mathrm{TS}^k(\mathsf{V})$ as

$$A = \sum_{i_1,\dots,i_k \in \{1,\dots,n\}} A^{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k},$$

we have

$$A = \sum_{J \in \mathbb{Z}_{\geq 0}^n, |J| = k} A^J e_{\gamma,\odot}^{J}, \tag{F.7}$$

where the A^J's are related to the $A^{i_1 \cdots i_k}$ is the manner prescribed following the proof of Proposition F.2.16. The point is that it is the basis elements $e_{\gamma,\odot}^{J}$ for which this expression is valid. If we instead use the basis elements $e_{\odot}^{J}$ which correspond to the basis of $\mathrm{S}(\mathsf{V})$ under the isomorphism ϕ_{V}, then we instead have

$$A = \sum_{J \in \mathbb{Z}_{\geq 0}^n, |J| = k} \frac{A^J}{J!} e_{\odot}^{J}, \tag{F.8}$$

where $J!$ is stands for $j_1! \cdots j_n!$ if $J = (j_1, \dots, j_n)$. Both representations (F.7) and (F.8) have their uses. For identifications of $\mathrm{TS}(\mathsf{V})$ with polynomials the basis $\mathscr{E}_{\gamma,\odot}$ is most natural. For thinking of $\mathrm{TS}(\mathsf{V})$ as the space of partial derivatives (see Section 1.1.2) the basis $\mathscr{E}_{\odot}$ is most natural.

F.3 The exterior algebra and the algebra of alternating tensors

In the preceding several sections we delved intensely into the symmetric algebra and algebra of symmetric tensors, and arrived at some interesting conclusions as concerns the interpretations of bases for these algebras. This whole operation may be repeated in its entirety for “alternating” (meaning skew-symmetric, essentially) algebras. Since many of the details here are exactly like their symmetric counterparts, we shall merely reproduce the definitions and main results without proofs. Only in the few places where there are differences with the symmetric case will we dwell for a moment.

F.3.1 The exterior algebra

First we consider the quotient construction of the exterior algebra.

F.3.1 Definition (Exterior algebra) Let F be a field and let V be an F-vector space. Let $I_A(\mathsf{V})$ be the two-sided ideal of $\mathrm{T}(\mathsf{V})$ generated by elements of the form $v \otimes v$ for $v \in \mathsf{V}$. The ***exterior algebra*** of V is the F-algebra $\bigwedge(\mathsf{V}) = \mathrm{T}(\mathsf{V})/I_A(\mathsf{V})$. The product in $\bigwedge(\mathsf{V})$ is denoted by

$$(B_1 + I_A(\mathsf{V})) \wedge (B_2 + I_A(\mathsf{V})) \triangleq B_1 \otimes B_2 + I_A(\mathsf{V}),$$

and called the ***wedge product***. •

As with $\mathrm{S}(\mathsf{V})$, one has a notion of degree for elements of $\bigwedge(\mathsf{V})$. The elements of degree $k \in \mathbb{Z}_{\geq 0}$ are given by $\bigwedge^k(\mathsf{V}) = \mathrm{T}^k(\mathsf{V})/I_A^k(\mathsf{V})$ where $I_A^k(\mathsf{V}) = I_A(\mathsf{V}) \cap \mathrm{T}^k(\mathsf{V})$. In particular, since $I_A^0(\mathsf{V}) = I_A^1(\mathsf{V}) = \{0\}$ it follows that $\bigwedge^0(\mathsf{V}) \simeq \mathsf{F}$ and $\bigwedge^1(\mathsf{V}) \simeq \mathsf{V}$.

Now let us characterise the algebra structure in $\bigwedge(\mathsf{V})$.

F.3.2 Proposition (The product in $\bigwedge(\mathsf{V})$ is alternating) *Let* F *be a field and let* V *be an* F*-vector space. If* $\mathrm{B}_1 + \mathrm{I}_\mathrm{A}(\mathsf{V}) \in \bigwedge^{\mathrm{k}_1}(\mathsf{V})$ *and if* $\mathrm{B}_2 + \mathrm{I}_\mathrm{A}(\mathsf{V}) \in \bigwedge^{\mathrm{k}_2}(\mathsf{V})$ *then*

(i) $(\mathrm{B}_1 + \mathrm{I}_\mathrm{A}(\mathsf{V})) \wedge (\mathrm{B}_2 + \mathrm{I}_\mathrm{A}(\mathsf{V})) = (-1)^{\mathrm{k}_1\mathrm{k}_2}(\mathrm{B}_2 + \mathrm{I}_\mathrm{A}(\mathsf{V})) \wedge (\mathrm{B}_1 + \mathrm{I}_\mathrm{A}(\mathsf{V}))$ *and*

(ii) $(\mathrm{B}_1 + \mathrm{I}_\mathrm{A}(\mathsf{V})) \wedge (\mathrm{B}_1 + \mathrm{I}_\mathrm{A}(\mathsf{V})) = 0$ *if* k_1 *is odd.*

The exterior algebra has a characterisation in terms of extending maps into a general algebra, just as does the symmetric algebra.

F.3.3 Proposition (Characterisation of the exterior algebra) *Let* F *be a field, let* V *be an* F*-vector space, let* A *be a unitary* F*-algebra, and let* $\mathrm{f} \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{V}; \mathsf{A})$ *have the property that* $\mathrm{f(v)f(v)} = 0$ *for all* $\mathrm{v} \in \mathsf{V}$*. Then there exists a unique* $\phi_\mathrm{f} \in \mathrm{Hom}_{\mathsf{F}}(\bigwedge(\mathsf{V}); \mathsf{A})$ *such that the diagram*

$$\begin{tikzcd} \bigwedge(\mathsf{V}) \arrow[r, "\phi_\mathrm{f}"] & \mathsf{A} \\ \mathsf{V} \arrow[u] \arrow[ur, "\mathrm{f}"'] & \end{tikzcd}$$

commutes, where the vertical arrow is the inclusion of V *in* $\bigwedge(\mathsf{V})$*.*

Linear maps induce homomorphisms of their exterior algebras.

F.3.4 Proposition (Induced homomorphisms of exterior algebras) *Let* F *be a field and let* V *and* U *be* F*-vector spaces. If* $f \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{V};\mathsf{U})$ *then there exists a unique homomorphism* f_* *of the* F*-algebras* $\mathsf{S}(\mathsf{V})$ *and* $\mathsf{S}(\mathsf{U})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{V} \arrow[r, "f"] \arrow[d] & \mathsf{U} \arrow[d] \\ \bigwedge(\mathsf{V}) \arrow[r, "f_*"'] & \bigwedge(\mathsf{U}) \end{tikzcd}$$

commutes, where the vertical arrows are the canonical inclusions.

Using the characterisation of the exterior algebra of a direct sum we can prove the following description of a basis for the exterior algebra of a finite-dimensional vector space.

F.3.5 Proposition (The exterior algebra of a finite-dimensional vector space) *Let* F *be a field and let* V *be a finite-dimensional* F*-vector space with basis* $\mathscr{E} = \{e_1, \dots, e_n\}$*. Then the set*

$$\mathscr{E}_\wedge = \{e_{i_1} \wedge \dots \wedge e_{i_k} \mid i_1, \dots, i_k \in \{1, \dots, n\},\ i_1 < \dots < i_k,\ k \in \mathbb{Z}_{\geq 0}\}$$

is a basis for $\bigwedge(\mathsf{V})$*.*

Proof This is proved using the following lemma.

1 Lemma *Let* F *be a field and let* $\mathsf{V}_1, \dots, \mathsf{V}_k$ *be* F*-vector spaces. There then exists a unique isomorphism* Φ *of the* F*-algebras* $\otimes_{j=1}^k \bigwedge(\mathsf{V}_j)$ *and* $\bigwedge(\oplus_{j=1}^k \mathsf{V}_j)$ *such that the diagram* right?

$$\begin{tikzcd} \bigwedge(\mathsf{V}_j) \arrow[r, "\iota_{j*}"] \arrow[d, "f_j"'] & \bigwedge(\oplus_{j=1}^k \mathsf{V}_j) \\ \otimes_{j=1}^k \bigwedge(\mathsf{V}_j) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

commutes for each $j \in \{1, \dots, k\}$*, where* $\iota_j \colon \mathsf{V}_j \to \oplus_{j=1}^k \mathsf{V}_j$ *is the canonical inclusion,* ι_{j*} *is the induced homomorphism of exterior algebras (cf. Proposition F.3.4), and* f_j *is the homomorphism defined in (F.1).*

Proof The proof here follows that for the lemma from the proof of Proposition F.2.9. ▼

The proposition follows from the lemma like Proposition F.2.9 follows from the lemma in its proof. ■

For the symmetric algebra and symmetric tensors we expended some effort understanding the connection between index notation and multi-index notation. This connection really relied on the fact that the objects under consideration were symmetric. Thus we will not see this come up in our discussion of the exterior algebra and alternating tensors.

We can give the dimension of the homogeneous components of the exterior algebra.

F.3.6 Corollary (Dimension of degree k component of $\bigwedge(\mathsf{V})$) *Let* F *be a field and let* V *be a finite-dimensional* F*-vector space of dimension* n*. Then, for* $k \in \mathbb{Z}_{\geq 0}$*,* $\bigwedge^k(\mathsf{V})$ *is of dimension* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$*. In particular,* $\dim(\bigwedge^n(\mathsf{V})) = 1$ *and* $\bigwedge^k(\mathsf{V}) = \{0\}$ *for* $k > n$*.*

F.3.2 Alternating tensors

Now we switch gears, as we did with the symmetric algebra and symmetric tensors, and talk about a subalgebra of $\mathrm{T}(\mathsf{V})$ that consists of tensors that are "skew-symmetric." To do this we recall the notion of the sign of a permutation. Any permutation $\sigma \in \mathfrak{S}_k$ is the composition of a finite number of transpositions:[1]

$$\sigma = \sigma_1 \circ \cdots \circ \sigma_l.$$

The number l of transpositions is not unique. However, the evenness or oddness of the number of these transpositions only depends on σ. Thus $(-1)^l$ is well-defined, and we denote this number by $\mathrm{sign}(\sigma)$. With this notation we have the following definition.

F.3.7 Definition (Alternating tensor) Let F be a field and let V be an F-vector space. An element $A \in \mathrm{T}^k(\mathsf{V})$, $k \in \mathbb{Z}_{\geq 0}$, is an ***alternating tensor of order*** $\mathbf{k}$ if $\sigma(A) = \mathrm{sign}(\sigma)A$ for all $\sigma \in \mathfrak{S}_k$. The set of alternating elements of $\mathrm{T}^k(\mathsf{V})$ is denoted by $\mathrm{T}\bigwedge^k(\mathsf{V})$, and we denote

$$\mathrm{T}\bigwedge(\mathsf{V}) = \oplus_{k \in \mathbb{Z}_{\geq 0}} \mathrm{T}\bigwedge^k(\mathsf{V}). \qquad \bullet$$

One readily checks that $\mathrm{T}\bigwedge(\mathsf{V})$ is a subspace of $\mathrm{T}(\mathsf{V})$. We can define an F-linear map $\mathrm{Alt}'_k \colon \mathrm{T}^k(\mathsf{V}) \to \mathrm{T}\bigwedge^k(\mathsf{V})$ by

$$\mathrm{Alt}'_k(A) = \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma)\sigma(A).$$

Note that if $A \in \mathrm{T}\bigwedge^k(\mathsf{V})$ then $\mathrm{Alt}'_k(A) = k!A$. If $A \mapsto k!A$ is invertible in $\mathrm{T}\bigwedge^k(\mathsf{V})$, e.g., if F is a field of characteristic zero, then we can define $\mathrm{Alt}_k \colon \mathrm{T}^k(\mathsf{V}) \to \mathrm{T}\bigwedge^k(\mathsf{V})$ by

$$\mathrm{Alt}_k(A) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma)\sigma(A).$$

This linear map has the advantage of being a projection when it is defined. One can extend Alt'_k (and Alt_k, when it is defined) to all of $\mathrm{T}\bigwedge(\mathsf{V})$ by homogeneity. The resulting map will be denoted by Alt' (and Alt, when it is defined).

We additionally render $\mathrm{T}\bigwedge(\mathsf{V})$ a subalgebra by defining on it a suitable product. Note that the tensor product itself will not typically suffice since the tensor product of two alternating tensors is generally not alternating (and is *never* alternating in the cases of most interest to us). For $A \in \mathrm{T}\bigwedge^k(\mathsf{V})$ and $B \in \mathrm{T}\bigwedge^l(\mathsf{V})$ we then define

$$A \mathbin{\bar{\wedge}} B = \sum_{\sigma \in \mathfrak{S}_{k,l}} \mathrm{sign}(\sigma)\sigma(A \otimes B).$$

Equipped with this product, let us record some properties of $\mathrm{T}\bigwedge(\mathsf{V})$.

[1] A transposition is a swapping of two elements.

F.3.8 Proposition (Properties of the algebra $T\bigwedge(V)$) *Let F be a field and let V be an F-vector space. For $k, l, m \in \mathbb{Z}_{\geq 0}$ and $A \in T\bigwedge^k(V)$, $B \in T\bigwedge^l(V)$, and $C \in T\bigwedge^m(V)$, the following statements hold:*

(i) $A \mathbin{\bar{\wedge}} B \in T\bigwedge^{k+l}(V)$;

(ii) $A \mathbin{\bar{\wedge}} B = (-1)^{kl} B \mathbin{\bar{\wedge}} A$;

(iii) $A \mathbin{\bar{\wedge}} (B \mathbin{\bar{\wedge}} C) = (A \mathbin{\bar{\wedge}} B) \mathbin{\bar{\wedge}} C$.

In particular, $T\bigwedge(V)$ is an alternating subalgebra of $T(V)$.

This also gives the following convenient formula for the product of elements of degree one.

F.3.9 Corollary (Product of elements of degree one in $T\bigwedge(V)$) *Let F be a field, let V be an F-vector space, and let $v_1, \dots, v_k \in V$. Then*

$$v_1 \mathbin{\bar{\wedge}} \cdots \mathbin{\bar{\wedge}} v_k = \mathrm{Alt}'_k(v_1 \otimes \cdots \otimes v_k)$$

This also gives the following commonly encountered formula for the product in $T\bigwedge(V)$.

F.3.10 Proposition (Alternative formula for product in $T\bigwedge(V)$) *Let F be a field of characteristic zero and let V be an F-vector space. If $k, l \in \mathbb{Z}_{\geq 0}$, and if $A \in T\bigwedge^k(V)$ and $B \in T\bigwedge^l(V)$. Then*

$$A \mathbin{\bar{\wedge}} B = \frac{(k+l)!}{k!l!} \mathrm{Alt}_{k+l}(A \otimes B).$$

F.3.11 Proposition (The alternating tensor algebra of a finite-dimensional vector space) *Let F be a field and let V be a finite-dimensional F-vector space with basis $\mathscr{E} = \{e_1, \dots, e_n\}$. Then the set*

$$\mathscr{E}_{\bar{\wedge}} = \{e_{i_1} \mathbin{\bar{\wedge}} \cdots \mathbin{\bar{\wedge}} e_{i_k} \mid i_1, \dots, i_k \in \{1, \dots, n\},\ i_1 < \cdots < i_k,\ k \in \mathbb{Z}_{\geq 0}\}$$

is a basis for $T\bigwedge(V)$.

Let us now clearly give the relationship between the components of a tensor $A \in T\bigwedge^k(V)$ thought of first as a general tensor and second as an alternating tensor. Thus let V be finite-dimensional with basis $\{e_1, \dots, e_n\}$ and let $A \in T\bigwedge^k(V)$. Since $A \in T^k(V)$ we can write

$$A = \sum_{i_1,\dots,i_k \in \{1,\dots,n\}} A^{i_1 \dots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}. \tag{F.9}$$

That A is alternating means that, for every $\sigma \in \mathfrak{S}_k$ we have

$$\sum_{i_1,\dots,i_k \in \{1,\dots,n\}} A^{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} = \sum_{i_1,\dots,i_k \in \{1,\dots,n\}} \mathrm{sign}(\sigma) A^{i_1 \cdots i_k} e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(k)}}.$$

Thus

$$A^{i_1 \cdots i_k} = \mathrm{sign}(\sigma) A^{i_{\sigma(1)} \cdots i_{\sigma(k)}}$$

for each $\sigma \in \mathfrak{S}_k$. In particular, this implies that $A^{i_1\cdots i_k} = 0_{\mathsf{F}}$ if $i_j = i_l$ for any distinct $j, l \in \{1, \dots, k\}$. Thus the sum in (F.9) is in actuality over distinct sets of indices. Now one can use this fact, along with an argument just like that for symmetric tensors, to show that

$$A = \sum_{\substack{i_1,\dots,i_k \in \{1,\dots,n\} \\ i_1 < \cdots < i_k}} A^{i_1\cdots i_k} e_{i_1} \barwedge \cdots \barwedge e_{i_k}.$$

The punchline is that the component of $A \in \mathrm{T}\bigwedge^k(\mathsf{V})$ corresponding to the basis element $e_{i_1} \barwedge \cdots \barwedge e_{i_k}$ is simply $A^{i_1\cdots i_k}$. Other bases following from other definitions of the product in $\mathrm{T}\bigwedge(\mathsf{V})$ will (annoyingly) not have this property.

Now we connect the exterior algebra to certain multilinear maps.

F.3.12 Definition (Alternating multilinear map) Let F be a field and let V and U be F-vector spaces. A k-multilinear map $\phi \in \mathrm{L}^k(\mathsf{V};\mathsf{U})$ is ***alternating*** if, for every $\sigma \in \mathfrak{S}_k$ and every $v_1, \dots, v_k \in \mathsf{V}$, it holds that

$$\phi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sign}(\sigma)\phi(v_1, \dots, v_k).$$

The set of such alternating k-multilinear maps is denoted by $\mathrm{L}^k_{\mathrm{alt}}(\mathsf{V};\mathsf{U})$. •

With this notion we have the following result.

F.3.13 Proposition (Homomorphisms of exterior algebras and multilinear maps) *Let F be a field and let V and U be F-vector spaces. The map Φ from $\mathrm{L}^{\mathrm{k}}_{\mathrm{alt}}(\mathsf{V};\mathsf{U})$ to $\mathrm{Hom}_{\mathsf{F}}(\bigwedge^{\mathrm{k}}(\mathsf{V});\mathsf{U})$ defined by*

$$\Phi(\phi)(\mathrm{v}_1 \cdot \cdots \cdot \mathrm{v}_{\mathrm{k}}) = \phi(\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{k}})$$

is an isomorphism of F-vector spaces.

Now we can relate $\bigwedge(\mathsf{V})$ and $\mathrm{T}\bigwedge(\mathsf{V})$. By arguments just like those for symmetric algebras, but replacing the use of Proposition F.2.7 with a use of Proposition F.3.3, we arrive at F-algebra homomorphisms $\phi_{\mathsf{V}} \colon \bigwedge(\mathsf{V}) \to \mathrm{T}\bigwedge(\mathsf{V})$ and $\psi_{\mathsf{V}} \colon \mathrm{T}\bigwedge(\mathsf{V}) \to \bigwedge(\mathsf{V})$ such that the diagrams

$$\begin{tikzcd} \mathrm{T}(\mathsf{V}) \arrow[d] \arrow[dr, "\mathrm{Alt}'"] & \\ \bigwedge(\mathsf{V}) \arrow[r, "\phi_{\mathsf{V}}"'] & \mathrm{T}\bigwedge(\mathsf{V}) \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathrm{T}(\mathsf{V}) \arrow[d] \\ \mathrm{T}\bigwedge(\mathsf{V}) \arrow[ur] \arrow[r, "\psi_{\mathsf{V}}"'] & \bigwedge(\mathsf{V}) \end{tikzcd}$$

commute. These homomorphisms satisfy the following relationships.

F.3.14 Proposition (Relationship between $\bigwedge(\mathsf{V})$ and $\mathrm{T}\bigwedge(\mathsf{V})$) *Let* F *be a field and let* V *be an* F*-vector space. Then the following statements hold:*

(i) $\psi_{\mathsf{V}} \circ \phi_{\mathsf{V}}(\mathrm{A}) = \mathrm{k!A}$ *for* $\mathrm{A} \in \bigwedge(\mathsf{V})$*;*

(ii) $\phi_{\mathsf{V}} \circ \psi_{\mathsf{V}}(\mathrm{A}) = \mathrm{k!A}$ *for* $\mathrm{A} \in \mathrm{T}\bigwedge(\mathsf{V})$.

The case when $A \mapsto k!A$ is an isomorphism of $\mathrm{T}\bigwedge^k(\mathsf{V})$ for each $k \in \mathbb{Z}_{\geq 0}$ is again of importance. A special case of this is the following.

F.3.15 Corollary (Alternating tensors and the exterior algebra for vector spaces over fields of characteristic zero) *Let* F *be a field of characteristic zero and let* V *be an* F*-vector space. Then* ϕ_{V} *is an isomorphism of the* F*-algebras* $\bigwedge(\mathsf{V})$ *and* $\mathrm{T}\bigwedge(\mathsf{V})$*.*

Suppose that V is finite-dimensional with basis $\mathscr{E} = \{e_1, \dots, e_n\}$ and if $A \mapsto k!A$ is an isomorphism of $\mathrm{T}\bigwedge^k(\mathsf{V})$ for each $k \in \mathbb{Z}_{\geq 0}$. Then the isomorphism ϕ_{V} has the nice feature that

$$\phi_{\mathsf{V}}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{i_1} \,\overline{\wedge}\, \cdots \,\overline{\wedge}\, e_{i_k}$$

for every $i_1, \dots, i_k \in \{1, \dots, n\}$. Thus the bases $\mathscr{E}_{\wedge}$ and $\mathscr{E}_{\overline{\wedge}}$ are in natural correspondence in this case. Note that, unlike for symmetric tensors, we do not have the ambiguity of two natural bases for $\mathrm{T}\bigwedge(\mathsf{V})$.

Bibliography

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