Csiszár's Hypothesis Testing Reverse Cutoff Rate for General Sources with Memory[1]

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Abstract — **We investigate Csiszár's hypothesis testing reverse β-cutoff rate for arbitrary sources with memory. Under some conditions, we show that the reverse β-cutoff rate is given by the Rényi α-divergence rate for $\alpha = \frac{1}{1-\beta}$ and $0 < \beta < \beta_{\max}$, where $\beta_{\max}$ is the largest $\beta < 1$ for which the Rényi divergence rate is finite. For $\beta_{\max} \le \beta < 1$, an upper bound for the reverse cutoff rate is established.**

I. Introduction

In [2], Csiszár established the concept of reverse and forward β-cutoff rates for the hypothesis testing between two finite-alphabet memoryless sources X and $\bar{X}$. He defined the reverse (respectively forward) β-cutoff rate as the number $R_0 \ge 0$ that provides the best possible lower bound of form $\beta(E - R_0)$ to the type 1 correct (respectively error) exponent where $E > 0$ is the rate of exponential decay of the type 2 error probability. He then showed that both β-cutoff rates are given by Rényi's α-divergence $D_\alpha(X\|\bar{X})$ with $\alpha = \frac{1}{1-\beta}$, $\alpha > 0$, $\alpha \ne 1$. In [1], we extended one of Csiszár's results by proving that the liminf Rényi divergence rate provides an expression of the forward cutoff rate for arbitrary sources with memory and general alphabet. In this work, we investigate the reverse cutoff rate for such sources.

II. Reverse β-Cutoff Rate

Given two arbitrary sources $\mathbf{X}$ and $\bar{\mathbf{X}}$ with common alphabet $\{\mathcal{X}^n\}_{n=1}^{\infty}$ [3], we consider the general hypothesis testing problem with $\mathbf{X}$ as the null hypothesis and $\bar{\mathbf{X}}$ as the alternative hypothesis. Let $\mathcal{A}_n \subseteq \mathcal{X}^n$ be an acceptance region and define $\mu_n \triangleq Pr\{X^n \notin \mathcal{A}_n\}$ and $\lambda_n \triangleq Pr\{\bar{X}^n \in \mathcal{A}_n\}$ where μ_n and λ_n are the type 1 and type 2 error probabilities, respectively.

Definition 1 Fix $E > 0$. A rate r is called E-unachievable if there exists a sequence of acceptance regions $\mathcal{A}_n$ such that

$$\limsup_{n\to\infty} -\frac{1}{n}\log(1-\mu_n) \le r \quad \text{and} \quad \liminf_{n\to\infty} -\frac{1}{n}\log \lambda_n \ge E.$$

We then set $D_e^*(E|\mathbf{X}\|\bar{\mathbf{X}}) \triangleq \inf\{r > 0 : r \text{ is } E\text{-unachievable}\}$.

By interchanging the role of $\mathbf{X}$ and $\bar{\mathbf{X}}$ and the role of r and E, Han's result in [3][Theorem 4.1] can be used to provide an expression for $D_e^*(E|\mathbf{X}\|\bar{\mathbf{X}})$ as follows.

Proposition 1 For any $E > 0$,

$$D_e^*(E|\mathbf{X}\|\bar{\mathbf{X}}) = \inf_{R\in\mathbb{R}} \left\{R + \rho(R) + [E - \rho(R)]^+\right\},$$

where

$$\rho(R) = \lim_{n\to\infty} -\frac{1}{n}\log P_{\bar{X}^n}\left\{x^n \in \mathcal{X}^n : \frac{1}{n}\log\frac{P_{\bar{X}^n}(x^n)}{P_{X^n}(x^n)} \le R\right\},$$

provided the limit of $\rho(R)$ exists, and for any $M > 0$, there exists $K > 0$ such that

$$\liminf_{n\to\infty} -\frac{1}{n}\log P_{X^n}\left\{x^n \in \mathcal{X}^n : \frac{1}{n}\log\frac{P_{X^n}(x^n)}{P_{\bar{X}^n}(x^n)} \ge K\right\} \ge M.$$

Definition 2 Fix $\beta > 0$. $R_0 \ge 0$ is a reverse β-achievable rate if

$$D_e^*(E|\mathbf{X}\|\bar{\mathbf{X}}) \ge \beta(E - R_0)$$

for all $E > 0$. The reverse β-cutoff rate $R_0^{(r)}(\beta|\mathbf{X}\|\bar{\mathbf{X}})$ is defined as the infimum of all reverse β-achievable rates.

Theorem 1 (Reverse β-cutoff rate formula) Assume that the conditions of Proposition 1 hold, that $\rho(R)$ is convex, and that there exists an R such that $R + \rho(R) = 0$. Then,

$$R_0^{(r)}(\beta|\mathbf{X}\|\bar{\mathbf{X}}) \le \liminf_{n\to\infty} \frac{1}{n} D_{\frac{1}{1-\beta}}(X^n\|\bar{X}^n) \quad \text{for } 0 < \beta < 1,$$

and

$$R_0^{(r)}(\beta|\mathbf{X}\|\bar{\mathbf{X}}) \ge \limsup_{n\to\infty} \frac{1}{n} D_{\frac{1}{1-\beta}}(X^n\|\bar{X}^n) \quad \text{for } 0 < \beta < \beta_{\max},$$

where $D_\alpha(X^n\|\bar{X}^n)$ is Rényi's α-divergence between X^n and $\bar{X}^n$ and $\beta_{\max}$ is the largest $\beta < 1$ for which the limsup Rényi divergence rate is finite. The above two inequalities directly imply that $R_0^{(r)}(\beta|\mathbf{X}\|\bar{\mathbf{X}}) = \lim_{n\to\infty} \frac{1}{n} D_{1/(1-\beta)}(X^n\|\bar{X}^n)$ for $0 < \beta < \beta_{\max}$.

Nagaoka and Hayashi [4] recently generalized Han's result in Proposition 1 by establishing a formula for $D_e^*(E|\mathbf{X}\|\bar{\mathbf{X}})$ without requiring any assumption on $\rho(R)$. We are currently investigating the extension of our reverse cutoff rate result for this more general setting.

References

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