IMAGE MODELING AND RESTORATION THROUGH CONTAGION URN SCHEMES

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ABSTRACT

We introduce a novel class of nonlinear stochastic filters based on contagion urn schemes. These filters which rely on biologically inspired sampling processes, offer good restoration results on heavily corrupted binary images.

1. INTRODUCTION

We present a new approach to binary image filtering using contagion urn schemes. Techniques modeling images as Markov random fields (MRF) have been extensively investigated in the past [1, 2]. MRF's appropriately represent spatial dependencies and the MRF-Gibbs equivalence allows for the computation of the maximum a posteriori (MAP) estimate of the original image [1]. This approach suffers nevertheless from various drawbacks among which are the difficulty in estimating the model parameters and the necessity to solve non-convex optimization problems.

In this work, we turn to a nonlinear filtering procedure that preserves key features of the Gibbs sampling process but is driven instead by urn schemes that models the development of an infectious disease. Urn sampling schemes are of interest since they may be used to model Markov chains and also MRF's [3]. Furthermore, contagion urn schemes can be employed to generate what could be described as "'autocatalytic" processes [4]: they constitute positive-feedback systems that yield limiting patterns of the self-reinforcing type.

2. CONTAGION URN SCHEMES

In 1923, Polya and Eggenberger [5] introduced the following urn scheme as a model for the spread of a contagious disease through a population. An urn originally contains T balls, of which W are white and B are black ($T = W+B$). We make successive draws from the urn; after each draw, we return to the urn $1+\Delta$ balls of the same color as was just drawn, where $\Delta > 0$. Let $\rho = W/T$ and $\delta = \Delta/T$. Define the binary process $\{Z_n\}_{n=1}^{\infty}$ as follows:

$$Z_n = \begin{cases} 1, & \text{if the } n^{th} \text{ ball drawn is white;} \\ 0, & \text{if the } n^{th} \text{ ball drawn is black.} \end{cases}$$

It can be shown that the process $\{Z_n\}$ is stationary and non-ergodic [6, 7]. The urn scheme has infinite memory, in the sense that the very *first* ball drawn from the urn and the 999,999th ball drawn from the urn have an identical effect on the outcome of the millionth draw.

Alternatively, in [8], a finite-memory version of the Polya urn scheme is proposed in which the effects of the "disease" fade in time. As before, consider an urn with T balls, of which W are white and B are black. At the j'th draw, $j = 1, 2, \ldots$, we select a ball from the urn and replace it with $1+\Delta$ balls of the same color ($\Delta > 0$); then, M draws later ($M > 0$) - after the $(j+M)$'th draw - we retrieve from the urn Δ balls of the color picked at time j. With this modification of the original Polya urn scheme, the total number of balls in the urn is constant ($T + M\Delta$ balls) after an initialization period of M draws. It also limits the effect of any draw to M draws in the future. In this case, it can be shown that the process $\{Z_n\}$ is a stationary ergodic Markov process of order M [8]. We will refer to the above two schemes as the Polya scheme of memory M: letting $M \to \infty$ yields the original Polya model; if $0 < M < \infty$, we get the finite-memory urn model.

3. FILTERING BY CONTAGION

We propose an iterative filtering scheme inspired by the above urn sampling processes. Consider a binary image $I_n = [p_n^{(i,j)}]$ of size $K \times L$, where $p_n^{(i,j)} \in \{0, 1\}$ is the intensity of pixel (i, j) at iteration n, $n = 0, 1, \ldots$, $(i,j) \in \mathcal{I}$ where $\mathcal{I} \triangleq \{(i,j) : i = 0, \ldots, K-1; j = 0, \ldots, L-1\}$. We associate an urn $u_n^{(i,j)} \triangleq (B_n^{(i,j)}, W_n^{(i,j)})$ with each pixel (i, j) at time n, where $B_n^{(i,j)}$ and $W_n^{(i,j)}$ are respectively the number of black and white balls in the urn. We allow for spatial interactions at each time instant by *associating* the urns of the neighboring pixels in the determination of the new sampled intensity value. Let I_0 be an image degraded by noise (at time $n=0$) that we wish to restore. We use the following procedure.

1. *Urn Initialization:* For each pixel (i,j), the initial urn composition $u_0^{(i,j)} = (B_0^{(i,j)}, W_0^{(i,j)})$ is obtained by computing the relative frequencies of white and black pixels in a spatial neighborhood centered on (i,j).

2. *Edge Detection:* Horizontal $(E_{0,h})$ and vertical $(E_{0,v})$ edge maps are derived from I_0, of size $K\times(L-1)$ and $(K-1)\times L$, respectively. The horizontal edge map $E_{0,h}$ consists of an array of $e_h^{(r,s)}$ such that $e_h^{(r,s)} = 1$ if an edge is detected between neighboring elements $p_0^{(r,s)}$ and $p_0^{(r,s+1)}$, and zero otherwise, $r = 0,\ldots,K-1$, $s = 0,\ldots,L-2$. The vertical edge map $E_{0,v}$ is defined similarly; i.e., $E_{0,v} = [e_v^{(l,m)}]$ such that $e_v^{(l,m)} = 1$ if an edge has been found between $p_0^{(l,m)}$ and $p_0^{(l+1,m)}$, and zero otherwise, $l = 0,\ldots,K-2$, $m = 0,\ldots,L-1$.

3. *Iterative Image Sampling:* For $n > 0$, find the value of each pixel (i,j) by sampling from neighboring urns $\mathcal{V}_{n-1}^{(i,j)}$ with $\mathcal{V}_{n-1}^{(i,j)} \triangleq \{u_{n-1}^{(r,s)} : (r,s) \in \mathcal{N}_d^{(i,j)}\}$, where $\mathcal{N}_d^{(i,j)}$ is the neighborhood set of pixel (i,j) [1]: $\mathcal{N}_d^{(i,j)} \triangleq \{(r,s)\in\mathcal{I} : (i-r)^2+(j-s)^2 \le d\}$. The sampling is performed according to Polya's model of memory M described earlier on each neighboring urn and the outcomes of draws from these urns are denoted by $Z_{n-1}^{(r,s)}, (r,s) \in \mathcal{N}_d^{(i,j)}$. The sampling from neighboring elements is inhibited if an edge exists (in step 2) between them. The new pixel configuration $[p_n^{(i,j)}]$ is then obtained by combining outcomes from the set of participating urns $\mathcal{Z}_{n-1}^{(i,j)}$; i.e.,

$$\begin{aligned}\mathcal{Z}_{n-1}^{(i,j)} \;=\; & \{u_{n-1}^{(r,s)}, (r,s) \in \mathcal{N}_d^{(i,j)} \wedge \\ & (((r \ne i) \Longrightarrow (e_v^{(r,s)} = 0)) \vee \\ & ((s \ne j) \Longrightarrow (e_h^{(r,s)} = 0)))\}.\end{aligned} \tag{1}$$

   Then
   $$p_n^{(i,j)} = select(\mathcal{Z}_{n-1}^{(i,j)}),$$
   where the *select* operation is a generic operation combining the various draws in $\mathcal{Z}_{n-1}^{(i,j)}$. A possible choice is to take a linear combination of these draws with weights that represent the relative importance of a given neighbor (in a fashion similar to that of the way potentials of cliques are set in MRF's). Alternatively, *select* can be chosen as the mean or median operator.

4. *Urn Updating:* If $p_n^{(i,j)} = 1$, add Δ white balls to the urn; if it is zero, add Δ black balls to the urn. In the case of the infinite memory urn scheme (original Polya scheme) , this yields a new urn composition for each pixel as
   $$u_n^{(i,j)} : \begin{cases} W_n^{(i,j)} = W_{n-1}^{(i,j)} + (p_n^{(i,j)}) * \Delta, \\ B_n^{(i,j)} = B_{n-1}^{(i,j)} + (1 - p_n^{(i,j)}) * \Delta. \end{cases}$$
   In the case of finite-memory urn scheme, we have instead:
   $$u_n^{(i,j)} : \begin{cases} W_n^{(i,j)} = W_{n-1}^{(i,j)} + (p_n^{(i,j)} - p_{n-M}^{(i,j)}) * \Delta, \\ B_n^{(i,j)} = B_{n-1}^{(i,j)} + (1 - (p_n^{(i,j)} - p_{n-M}^{(i,j)})) * \Delta. \end{cases}$$

5. Let $n = n+1$. If $n \le N$, proceed to step 3; otherwise, stop, where N is a prescribed number of iterations.

4. STATISTICAL PROPERTIES

The resulting sequence of generated images exhibits both spatial and temporal dependencies represented by a Markovian relationship in terms of the urns $u_n^{(r,s)}$, more specifically:

$$Pr\{u_n^{(i,j)} | U_{n-1}, U_{n-2}, \ldots, U_0\} = Pr\{u_n^{(i,j)} | \mathcal{Z}_{n-1}^{(i,j)}\},$$

where $U_n \triangleq [u_n^{(i,j)}]$ is the urn matrix associated with I_n, and $\mathcal{Z}_{n-1}^{(i,j)}$ is the set of participating urns defined in the previous section. Consider the infinite memory Polya scheme. The asymptotic properties of the joint distribution can be characterized in the simple case $d=0$; i.e., when all spatial interactions are shut-off at each sampling step. In this case, as the number of draws increases indefinitely, the proportion of white balls in each urn (or the sample average of Z_n) converges with probability one to Z [7]. This limiting proportion Z is a continuous random variable with support the interval $(0,1)$ and Beta probability density function with parameters $(\rho/\delta, (1-\rho)/\delta)$:

$$f_Z(z) = \frac{\Gamma(1/\delta)}{\Gamma(\rho/\delta)\Gamma((1-\rho)/\delta)} z^{\frac{\rho}{\delta}-1}(1-z)^{\frac{1-\rho}{\delta}-1},$$

for $0 < z < 1$; and zero otherwise. $\Gamma(\cdot)$ is the gamma function, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$. The behavior of this pdf is as follows: assuming $\delta = 1$ for simplicity, if the original proportion of white balls in the urn is close to one, then the limiting distribution of Z, will be skewed towards 1. A similar behavior is obtained for the case when ρ is close to zero. Therefore the limiting pattern will reflect the underlying probability $Pr(p_1^{(i,j)} = x) = \rho^x(1-\rho)^{(1-x)}$.

5. EXPERIMENTAL RESULTS

The above procedure was used to filter images corrupted by a combination of bursty and iid noise. The iid additive noise process is a good model for transmission errors that arise in wireline communication systems, while the bursty noise process models transmission errors that occur in wireless communication channels (e.g., the digital cellular channel).

In our experiments, the *select* operator is chosen to be the median filter. An infinite memory Polya urn scheme is used, and a second order neighborhood is assumed ($d = 2$). Δ is set to 20. Results of experiments conducted on binary images of Lena (512×512) and a headscan (512×512) are displayed in Figure 2. The original images are shown respectively in Figures 2.(a) and 2.(g).

Lena is first corrupted by iid noise with bit error rate of 0.20, as shown in Figure 2.(b). The edge map of Figure 2.(b) is given in Figure 2.(d). The filtered image for $N = 20$ is shown in Figures 2.(c). The resulting probability of error of the filtered image is plotted as a function of number of iterations N and is shown in Figure 1. We can clearly remark that most of the gain is achieved at $N = 10$.

Next, Lena is corrupted by a combination of iid noise and bursty noise. The iid noise has a bit error rate equal 0.1. The bursty noise process is a highly correlated Markov process with a correlation parameter of 0.91 and a 0.10 error rate. The overall bit error rate of the combined noise equals 0.18. The corrupted image is shown in Figure 2.(e). The filtered image (for $N = 20$) shown in Figure 2.(f), yields a probability of error of 0.054. For comparison, a median filter iteratively applied for $N = 10$ times to the noisy image yielded performance comparable to that of the contagion filter. This fact should be expected since the *select* operator was chosen to be the median. However, on closer inspection of the full size images, we notice that the contagion filter is more likely to preserve details. For lack of space, these results are not presented here. The reader is referred to the site *http://www.cfar.umd.edu/~burlina/polya.html.*

Finally, we show the result of applying the contagion filter to the headscan image. An additive iid noise with bit error rate 0.1 is used (Figure 2.(h)). The filtered image (probability of error of 0.035) is shown in Figure 2.(i) for $N = 5$. In sum, we conclude that the contagion filtering scheme offers good performance.

6. CONCLUSION

We have presented a nonlinear filtering procedure inspired from contagion urn schemes. This has been applied to the filtering of heavily corrupted binary images. Extensions to filtering of grey level images as well as applications to medical imaging are currently under investigation.

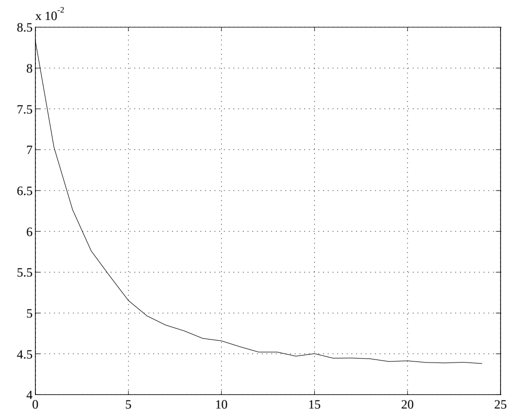

Figure 1: Pe. vs. time step N.

7. REFERENCES

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(a) Binary Lena

(b) Iid noise on Lena

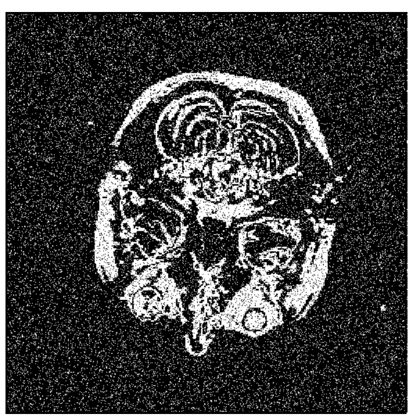
(c) Restored from iid

(d) Edge Map of 2.(b)

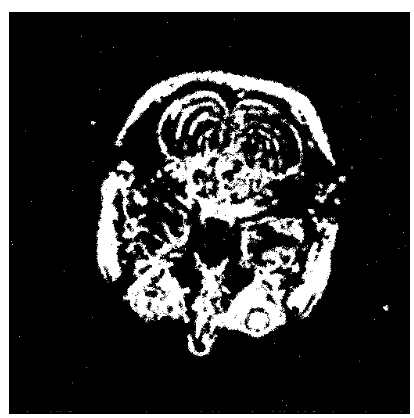
(e) Bursty + iid noise on Lena

(f) Restored from bursty + iid noise

(g) Headscan



(h) Iid noise on headscan



(i) Restored from iid

Figure 2: Restoration of images with a combination of iid and bursty noise using contagion urn schemes.