Csiszár's Forward Cutoff Rate for Testing Between two Arbitrary Sources[1]

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Abstract — **The Csiszár forward β-cutoff rate ($\beta < 0$) for hypothesis testing is defined as the largest rate $R_0 \geq 0$ such that for all rates $0 < E < R_0$, the smallest probability of type 1 error of sample size-n tests with probability of type 2 error $\leq e^{-nE}$ is asymptotically vanishing as $e^{-n\beta(E-R_0)}$. It was shown by Csiszár that the forward β-cutoff rate for testing between a null hypothesis X against an alternative hypothesis $\bar{\mathbf{X}}$ based on independent and identically distributed samples, is given by Rényi's α-divergence $D_\alpha(X\|\bar{X})$, where $\alpha = 1/(1-\beta)$.**

In this work, we show that the forward β-cutoff rate for the general hypothesis testing problem is given by the lim inf α-divergence rate. The result holds for an arbitrary abstract alphabet (not necessarily countable).

I. Introduction

In [2], Csiszár establishes the concept of forward β-cutoff rate for the hypothesis testing problem based on independent and identically (i.i.d.) observations. He then demonstrates that the forward β-cutoff rate is given by $D_{1/(1-\beta)}(X\|\bar{X})$, where $D_\alpha(X\|\bar{X})$ denotes the Rényi [4] α-divergence, $\alpha > 0$, $\alpha \neq 1$. This result provides a new operational significance for the α-divergence.

In this work, we extend Csiszár's result [2] by investigating the forward β-cutoff rate for the hypothesis testing between two arbitrary (not necessarily stationary, ergodic, etc.) sources with a general alphabet. We demonstrate that the lim-inf α-divergence rate provides the expression for the forward β-cutoff rate.

II. Preliminaries

Given two arbitrary sources $\mathbf{X}$ and $\bar{\mathbf{X}}$ taking values in the same source alphabet $\{\mathcal{X}^n\}_{n=1}^\infty$ [3], we may define the general hypothesis testing problem with $\mathbf{X}$ as the null hypothesis and $\bar{\mathbf{X}}$ as the alternative hypothesis. Let $\mathcal{A}_n$ be any subset of $\mathcal{X}^n$, $n = 1, 2, \ldots$ that we call the acceptance region of the hypothesis testing, and define $\mu_n \triangleq Pr\{X^n \notin \mathcal{A}_n\}$ and $\lambda_n \triangleq Pr\{\bar{X}^n \in \mathcal{A}_n\}$ where μ_n, λ_n are called type 1 error probability and type 2 error probability, respectively.

Definition 1 Fix $r > 0$. A rate E is called r-achievable if there exists an acceptance region $\mathcal{A}_n$ such that

$$\liminf_{n\to\infty} -\frac{1}{n}\log\mu_n \geq r \quad \text{and} \quad \liminf_{n\to\infty} -\frac{1}{n}\log\lambda_n \geq E.$$

Definition 2 The supremum of all r-achievable rates is denoted by $B_e(r|\mathbf{X}\|\bar{\mathbf{X}})$:

$$B_e(r|\mathbf{X}\|\bar{\mathbf{X}}) \triangleq \sup\{E \geq 0 : E \text{ is } r\textit{-achievable}\}.$$

The dual of this function is defined as:

$$D_e(E|\mathbf{X}\|\bar{\mathbf{X}}) \triangleq \sup\{r > 0 : E \text{ is } r\text{-achievable}\}.$$

III. Forward β-Cutoff Rate

Definition 3 Fix $\beta < 0$. $R_0 \geq 0$ is a forward β-achievable rate for the general hypothesis testing problem if

$$D_e(E|\mathbf{X}\|\bar{\mathbf{X}}) \geq \beta(E - R_0)$$

for every $E > 0$. The forward β-cutoff rate is defined as the supremum of all forward β-achievable rates, and is denoted by $R_0^{(f)}(\beta|\mathbf{X}\|\bar{\mathbf{X}})$. Our main result is the following.

Theorem 1 (Forward β-cutoff rate formula). Fix $\beta < 0$. For the general hypothesis testing problem,

$$R_0^{(f)}(\beta|\mathbf{X}\|\bar{\mathbf{X}}) = \liminf_{n\to\infty} \frac{1}{n} D_{\frac{1}{1-\beta}}(X^n\|\bar{X}^n),$$

where

$$D_\alpha(X^n\|\bar{X}^n) \triangleq \frac{1}{\alpha-1}\log\left(\sum_{x^n\in\mathcal{X}^n}[P_{X^n}(x^n)]^\alpha[P_{\bar{X}^n}(x^n)]^{1-\alpha}\right)$$

is the n-dimensional α-divergence.

The techniques used in our proof are a mixture of the techniques used in [1] for deriving the forward and reverse β-cutoff rates for source coding. However, some new techniques are also needed to obtain the result.

References

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[1] This research was supported in part by the Natural Sciences and Engineering Research Council of Canada, the Premier's Research Excellence Award of Ontario, and the National Science Council of Taiwan.