Capacity of Finite-State Two-Way Channels

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***Abstract*—This paper addresses the capacity problem for a class of finite-state two-way channels (FS-TWCs). Specifically, inner and outer bounds for the channel capacity of FS-TWCs are derived, and they are combined to characterize the capacity region in a limiting expression for some special FS-TWCs. Although such an expression is often incomputable, it is illustrated via an example that a reduction to single-letter form is possible as long as the system variables exhibit stationarity and the associated "average channel" satisfies certain symmetry properties.**

I. Introduction

Channels with state [1, Section 4], being simplified models to capture intrinsic impairments of signal transmissions such as inter-symbol interference (ISI), burst errors, unwanted interference, and correlated fading, have been gaining importance in developing next-generation communication systems. In particular, a considerable amount of research, e.g., [2]–[12], has been devoted to the determination of channel capacity for one-way channels (OWCs) under various scenarios, ranging from specific state and noise dynamics to the presence of feedback and/or multiple users. Closed-form expressions of channel capacity and computable bounds [12], [13] were also derived for some cases. Apart from the OWC setup, this paper considers a version of Shannon's two-way communication channels [14] with state, in order to understand the limits of full-duplex transmission under more realistic channel assumptions.

Although the capacity region for discrete-memoryless two-way channels (DM-TWCs) is generally unknown in single-letter form, the recent work in [15] has characterized several channel symmetry properties under which Shannon's (non-adaptive) random coding inner bound is tight. In addition, Shannon's DM-TWCs were studied in various directions such as error exponent analysis [16], joint source-channel coding [17], Q-graph coding for common-output DM-TWCs [18], and the existence of a helper [19], a jammer [20], or an eavesdropper [21]. Capacity results for continuous Gaussian and Poisson TWCs were also given in [22] and [23], respectively, and a practical adaptive linear coding scheme for Gaussian TWCs was proposed in [24]. Compared to these findings, relatively little is known about TWCs with state.

In the literature, the first TWC with state appeared in the original work of Shannon [14, Section 16], where he derived a capacity region in a limiting form [14, Theorem 6] for a class of TWCs with state that exhibits a "recoverable" property, i.e., where there exists a finite-length joint channel input that can reset the TWC to its initial state. The dirty paper coding problem on TWCs in [25] viewed external interferences as channel state. A capacity result was derived assuming that the state is additive Gaussian interference and that the state is partially and non-causally known to each transmitter. Another related channel model is the TWCs with memory in [15, Section IV] whose channel outputs are deterministic functions of channel inputs and a jointly stationary and ergodic noise process. Under some mild conditions, the capacity region was determined.

This paper considers a class of finite-state TWCs (FS-TWCs) inspired by prior work on FS-OWCs. We note that the proposed channel model does not cover all possible FS-TWCs, but it is sufficiently rich to study as a first step. The contribution of the paper includes a series of transmissibility results for the FS-TWCs with/without the statistical information of the initial state. Here, the derivations of the achievability part are based on Gallager's maximum likelihood (ML) decoding [1] for the codebook that is obtained by concatenating multiple (adaptive) code trees [26]. The converse part follows standard steps, in which a dependency graph technique [26] is employed to identify the required conditional independence. Moreover, we combine the two parts to obtain the channel capacity for some special cases in a limiting expression. Although this expression is often incomputable, we show via an example that it is possible to reduce it to a single-letter form. However, such a reduction heavily relies on a stationarity property for the state process and on a channel symmetry property [15]; this illustrates the difficulty in obtaining the capacity region in single-letter form for FS-TWCs.

The rest of this paper is organized as follows. In Section II, a channel model of FS-TWCs is proposed. Section III develops inner and outer bounds for the capacity region for the FS-TWC with different channel state information; their capacity regions are also established for special cases. In Section IV, an example of the FS-TWC whose capacity region can be obtained in single-letter form is presented, followed by some comments on the related TWC models with state. Finally, conclusions are drawn in Section V.

II. Preliminaries

Throughout this paper, all alphabets are assumed to be finite. Random variables are denoted using capital letters, and we use lower case letters to represent their realization from an alphabet. We also adopt the convention that $W^n \triangleq (W_1, W_2, \dots, W_n)$, where W_i's are random variables taking values from an identical alphabet $\mathcal{W}$. Moreover, the indices

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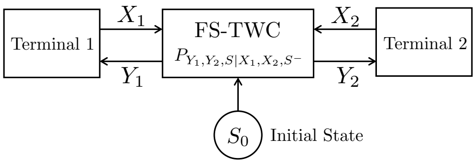

Fig. 1. The proposed FS-TWC model, where S^- represents the channel state determined by the previous transmission and it influences the outputs at the current transmission. Before transmission, an initial state, s_0, will be given by nature or generated according to some probability distribution P_{S_0}.

$j, j' \in \{1,2\}$, where $j \neq j'$, are reserved for denoting terminals. The message M_j to be sent by terminal j via N channel uses takes values from the message set $\mathcal{M}_j = \{1, 2, \ldots, 2^{NR_j}\}$ randomly and uniformly, where $R_j \geq 0$ is the transmission rate (bits/channel use) and NR_j is assumed to be a non-negative integer for simplicity. We also assume that the messages M_1 and M_2 are independent.

A. Channel Model

An FS-TWC with channel input alphabets $\mathcal{X}_j$, output alphabets $\mathcal{Y}_j$, $j = 1, 2$, and channel state alphabet $\mathcal{S}$ is described by the conditional probability distribution

$$P_{Y_1,Y_2,S|X_1,X_2,S^-}(y_1, y_2, s|x_1, x_2, s^-) \quad (1)$$

for all $x_j \in \mathcal{X}_j$, $y_j \in \mathcal{Y}_j$, and $s, s^- \in \mathcal{S}$, with a time-invariant Markov property that at each time instant $n \geq 1$, we have that

$$\begin{aligned} &P_{Y_{1,n},Y_{2,n},S_n|X_1^n,X_2^n,S^{n-1},S_0}(y_{1,n}, y_{2,n}, s_n|x_1^n, x_2^n, s^{n-1}, s_0) \\ &\quad = P_{Y_1,Y_2,S|X_1,X_2,S^-}(y_{1,n}, y_{2,n}, s_n|x_{1,n}, x_{2,n}, s_{n-1}), \quad (2) \end{aligned}$$

where $x_{j,n}$ and $y_{j,n}$ represent the realizations of channel input and output of terminal j at time-n, respectively.

In this channel model, we neither assign an initial state s_0 nor impose a probability distribution P_{S_0} for the initial state S_0 to make our channel model general. However, when using the FS-TWC, we always assume that an initial state s_0 is given and determined by nature or generated according to some P_{S_0} independently of the terminals' message. In the latter case, we only consider $P_{S_0}(s_0) > 0$ for all $s_0 \in \mathcal{S}$. Furthermore, we also assume that the two terminals cannot access the initial state (as shown in Fig. 1), but the terminals might know the probability distribution P_{S_0}. Below, we introduce some special cases of the FS-TWCs. Note that our objective is not to analyze each case, but to show some extensions from FS-OWCs.

B. Special Cases of the FS-TWCs

At first glance, the above channel model may look restrictive (as the outputs depend only on the most recent channel state), but in fact it generates a rich family of FS-TWCs, some of which we list in what follows.

Definition 1. *An FS-TWC is said to be indecomposable if for every $\epsilon > 0$, there exists an integer N_0 such that for $N \geq N_0$,*

$$\begin{aligned} &|P_{S_N|X_1^N,X_2^N,S_0}(s_N|x_1^N, x_2^N, s_0) \\ &\quad - P_{S_N|X_1^N,X_2^N,S_0}(s_N|x_1^N, x_2^N, s_0')| \leq \epsilon \quad (3) \end{aligned}$$

for all $s_0, s_0', s_N \in \mathcal{S}$, $x_1^N \in \mathcal{X}_1^N$, and $x_2^N \in \mathcal{X}_2^N$.

We remark that the FS-TWC with independent and identical distributed (i.i.d.) states $(S_0, S_1, \ldots,)$ belongs to this class. For indecomposable channels, the effect of the unknown initial state s_0 dies out eventually.

Definition 2. *An FS-TWC is said to have ISI if*

$$P_{Y_1,Y_2,S|X_1,X_2,S^-} = P_{Y_1,Y_2|X_1,X_2,S^-} P_{S|X_1,X_2,S^-} \quad (4)$$

for all possible arguments of the conditional probabilities, where $P_{S|X_1,X_2,S^-}$ models the ISI effect.

An FS-TWC model without ISI can be obtained by setting $P_{S|X_1,X_2,S^-}(s|x_1, x_2, s^-) = P_{S|S^-}(s|s^-)$ for all $x_j \in \mathcal{X}_j$, $y_j \in \mathcal{Y}_j$, and $s, s^- \in \mathcal{S}$. If the ISI-free FS-TWC is additionally indecomposable, then $\lim_{N\to\infty} P_{S_N|S_0}(s|s_0) = \pi_S(s)$ for all $s_0 \in \mathcal{S}$ and the probability distribution π_S is the unique stationary distribution of the underlying state process. Note that the uniqueness of π_S is due to the finite alphabet and the ISI-free assumptions and the properties in (2) and (3). An ISI-free indecomposable FS-TWC is called stationary if $P_{S_0} = \pi_S$.

C. Codes and Definitions for Channel Capacity

Due to the continuous exchange of information between the two channel terminals during transmission, each terminal can generate its channel input by adapting to the previously received signals in addition to its own message. This feature enables the use of adaptive channel codes in the following form.

Definition 3. *An (N, R_1, R_2) adaptive channel code C for the FS-TWCs consists of two message sets $\mathcal{M}_1$ and $\mathcal{M}_2$, two sequences of encoding functions $\boldsymbol{f}_1 \triangleq (f_{1,1}, f_{1,2}, \ldots, f_{1,N})$ and $\boldsymbol{f}_2 \triangleq (f_{2,1}, f_{2,2}, \ldots, f_{2,N})$ such that*

$$\begin{aligned} X_{1,1} &= f_{1,1}(M_1), \quad X_{1,n} = f_{1,n}(M_1, Y_1^{n-1}), \\ X_{2,1} &= f_{2,1}(M_2), \quad X_{2,n} = f_{2,n}(M_2, Y_2^{n-1}), \end{aligned}$$

for $n = 2, 3, \ldots, N$, and two decoding functions g_1 and g_2 such that $\hat{M}_2 = g_1(M_1, Y_1^N)$ and $\hat{M}_1 = g_2(M_2, Y_2^N)$ are the reconstructed messages at terminals 1 and 2, respectively.

Remark 1. The above encoder $\boldsymbol{f}_j$ can be represented by 2^{NR_j} rooted trees, where each tree is associated with one possible message (see [26, Section 4.1] for more details). As a code is selected and given to terminals before any channel use, the rooted trees only depend on its associated message. The tree on the left in Fig. 2 is an $N = 2$ adaptive codeword for some message m_1.

Given an FS-TWC and a channel code $(\boldsymbol{f}_1, \boldsymbol{f}_2, g_1, g_2)$, the joint probability distribution of all involved system variables conditional on $S_0 = s_0$ is given by

$$\begin{aligned} &P_{M_1,M_2,S^N,X_1^N,X_2^N,Y_1^N,Y_2^N|S_0}(m_1, m_2, s^N, x_1^N, x_2^N, y_1^N, y_2^N|s_0) \\ &= \frac{1}{2^{N(R_1+R_2)}} \prod_{n=1}^{N} P_{X_{1,n}|M_1,Y_1^{n-1}}(x_{1,n}|m_1, y_1^{n-1}) \\ &\qquad P_{X_{2,n}|M_2,Y_2^{n-1}}(x_{2,n}|m_2, y_2^{n-1}) \\ &\qquad P_{Y_1,Y_2,S|X_1,X_2,S^-}(y_{1,n}, y_{2,n}, s_n|x_{1,n}, x_{2,n}, s_{n-1}) \end{aligned}$$

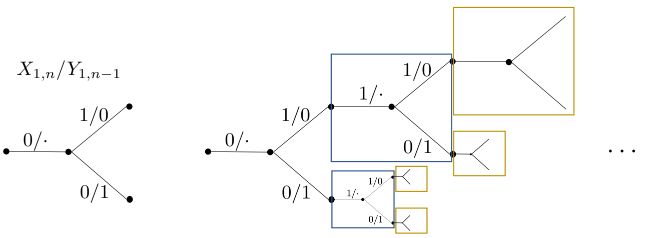

Fig. 2. The tree on the left represents a length-two (i.e., $N=2$) adaptive codeword for some message m_1, where $\mathcal{X}_1=\mathcal{Y}_1=\{0,1\}$; we concatenate this type of codes (each with length L) multiple times to generate a code as shown on the right for the proof of achievability. The boxes of the same color contain an identical tree; the trees in the boxes of different colors are generated randomly and independently.

where the equality holds due to (2). When P_{S_0} is known, one can further obtain $P_{S_0,M_1,M_2,S^N,X_1^N,X_2^N,Y_1^N,Y_2^N}$.

Let $\bar{P}_{e,s_0}^{(N)}(C)=\Pr\Big(\hat{M}_1\neq M_1,\hat{M}_2\neq M_2|S_0=s_0\Big)$ denote the average error probability of using the above channel code C for an FS-TWC given initial state s_0.

Definition 4. *A rate pair (R_1,R_2) is said to be achievable for an FS-TWC when P_{S_0} is unavailable if there exists a sequence of (N,R_1,R_2) codes such that*

$$\lim_{N\to\infty}\bar{P}_{e,s_0}^{(N)}(C)=0 \text{ for all } s_0\in\mathcal{S}. \tag{5}$$

Definition 5. *The capacity region $\mathcal{C}_1$ of an FS-TWC with unknown P_{S_0} is the closure of all achievable rate pairs.*

When P_{S_0} is known, the average error probability is defined as $\bar{P}_{e}^{(N)}(C)=\sum_{s_0\in\mathcal{S}}P_{S_0}(s_0)\bar{P}_{e,s_0}^{(N)}(C)$ and the achievability condition in (5) changes to

$$\lim_{N\to\infty}\bar{P}_{e}^{(N)}(C)=0. \tag{6}$$

Letting $\mathcal{C}_2$ denote the corresponding capacity region, we have that $\mathcal{C}_1\subseteq\mathcal{C}_2$ since the condition in (5) implies the condition in (6). Moreover, any inner (resp., outer) bound of $\mathcal{C}_1$ (resp., $\mathcal{C}_2$) is clearly an inner (resp., outer) bound of $\mathcal{C}_2$ (resp., $\mathcal{C}_1$). In the next section, we develop inner and outer bounds for $\mathcal{C}_1$ and $\mathcal{C}_2$ followed by some results on the capacity regions.

III. Capacity Regions for FS-TWCs

Here we present achievability results based on Gallager's analysis [1, Section 5] with ML decoding for the codebook constructed by concatenating multiple adaptive code trees [26, Section 2]. An illustration of this type of code is given in Fig. 2. Such a proof technique has been seen in the context of DM-TWCs [26] and FS-OWCs with/without feedback [8]. We also present outer bounds, the derivation of which mainly follows standard steps and a dependency graph [26, Section 2] is employed to reveal the required conditional independence.

A. Capacity Bounds for $\mathcal{C}_1$

Theorem 1 (Inner Bound for $\mathcal{C}_1$)**.** *For any $L\in\mathbb{N}$, the convex hull of the set of non-negative rate pairs (R_1,R_2) satisfying*

$$R_1<\frac{1}{L}\min_{s_0\in\mathcal{S}}I(\boldsymbol{A}_1;\mathbf{Y}_2|\boldsymbol{A}_2,S_0=s_0)-\frac{\log_2|\mathcal{S}|}{L}, \tag{7a}$$
$$R_2<\frac{1}{L}\min_{s_0\in\mathcal{S}}I(\boldsymbol{A}_2;\mathbf{Y}_1|\boldsymbol{A}_1,S_0=s_0)-\frac{\log_2|\mathcal{S}|}{L}, \tag{7b}$$

is achievable, where $\boldsymbol{A}_j=[\underline{A}_{j,1},\underline{A}_{j,2},\dots,\underline{A}_{j,L}]$ denotes an L-level tree code and the sub-vector $\underline{A}_{j,l}$ is of length $|\mathcal{Y}_j|^{l-1}$ for $1\le l\le L$, each component in $\underline{A}_{j,l}$ takes value in $\mathcal{X}_j$, $\mathbf{Y}_j=(Y_{j,1},Y_{j,2},\dots,Y_{j,L})$, $j=1,2$, and the joint probability distribution of all involved random variables is marginalized from

$$\begin{aligned}&P_{\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{Y}_1,\boldsymbol{Y}_2|S_0}(\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y}_1,\boldsymbol{y}_2|s_0)\\&=P_{\boldsymbol{A}_1}(\boldsymbol{a}_1)P_{\boldsymbol{A}_2}(\boldsymbol{a}_2)\\&\prod_{l=1}^{L}P_{X_{1,l}|\boldsymbol{A}_1,Y_1^{l-1}}(x_{1,l}|\boldsymbol{a}_1,y_1^{l-1})P_{X_{2,l}|\boldsymbol{A}_2,Y_2^{l-1}}(x_{2,l}|\boldsymbol{a}_2,y_2^{l-1})\\&\quad\cdot P_{Y_1,Y_2,S|X_2,X_2,S^-}(y_{1,l},y_{2,l},s_l|x_{1,l},x_{2,l},s_{l-1}).\end{aligned} \tag{8}$$

The proof of Theorem 1 follows Gallager's approach in [27, Section 4], which is similar to the proof of the achievability result for DM-TWCs [27, Section 4] and for FS-OWCs in [8]. We omit the details here for the sake of brevity.

Remark 2. The inequalities in (7) can be rewritten in terms of the causally conditional directed information [8]:

$$R_1<\frac{1}{L}\min_{s_0\in\mathcal{S}}I(\boldsymbol{A}_1\to\mathbf{Y}_2||\boldsymbol{X}_2,S_0=s_0)-\frac{\log_2|\mathcal{S}|}{L}, \tag{9a}$$
$$R_2<\frac{1}{L}\min_{s_0\in\mathcal{S}}I(\boldsymbol{A}_2\to\mathbf{Y}_1||\boldsymbol{X}_1,S_0=s_0)-\frac{\log_2|\mathcal{S}|}{L}. \tag{9b}$$

Here, $I(U^N\to V^N||W^N,S)\triangleq\sum_{i=1}^{N}I(U^i;V_i|W^i,V^{i-1},S)$. Moreover, if the FS-TWC is indecomposable, then the minimum over all $s_0\in\mathcal{S}$ can be replaced with the maximum. Note that $L=1$ corresponds to the case of non-adaptive coding.

Theorem 2 (Outer Bound for $\mathcal{C}_1$)**.** *Any achievable rate pair (R_1,R_2) must satisfy*

$$0\le R_1\le\frac{1}{N}I(\boldsymbol{A}_1\to\mathbf{Y}_2||\boldsymbol{X}_2,S_0=s_0)+\epsilon_{N,s_0} \tag{10a}$$
$$0\le R_2\le\frac{1}{N}I(\boldsymbol{A}_2\to\mathbf{Y}_1||\boldsymbol{X}_1,S_0=s_0)+\epsilon_{N,s_0} \tag{10b}$$

for all $s_0\in\mathcal{S}$, $N\ge1$, and some joint probability distribution of the form in (8)*, where $\lim_{N\to\infty}\epsilon_{N,s_0}=0$ for all $s_0\in\mathcal{S}$.*

Proof: The proof follows the standard approach. First,

$$\begin{aligned}&NR_1=H(M_1)=H(M_1|S_0=s_0)\\&=I(M_1;M_2,\boldsymbol{A}_2,X_2^N,Y_2^N|S_0=s_0)\\&\qquad+H(M_1|M_2,\boldsymbol{A}_2,X_2^N,Y_2^N,S_0=s_0)\\&\le I(M_1;M_2,\boldsymbol{A}_2,X_2^N,Y_2^N|S_0=s_0)+N\epsilon_{N,s_0} &(11)\\&=I(M_1,\boldsymbol{A}_1;M_2,\boldsymbol{A}_2,X_2^N,Y_2^N|S_0=s_0)+N\epsilon_{N,s_0} &(12)\\&=I(\boldsymbol{A}_1;X_2^N,Y_2^N|S_0=s_0)+N\epsilon_{N,s_0} &(13)\end{aligned}$$

$$= \sum_{n=1}^{N} I(\boldsymbol{A}_1; X_{2,n}, Y_{2,n} | X_2^{n-1}, Y_2^{n-1}, S_0 = s_0) + N\epsilon_{N,s_0}$$

$$= \sum_{n=1}^{N} I(\underline{A}_1^n; Y_{2,n} | X_2^n, Y_2^{n-1}, S_0 = s_0) + N\epsilon_{N,s_0} \quad (14)$$

$$= I(\boldsymbol{A}_1 \to \boldsymbol{Y}_2 || \boldsymbol{X}_2, S_0 = s_0) + N\epsilon_{N,s_0}, \quad (15)$$

where Fano's inequality [28] is applied in (11), (12) and (13) hold since $\boldsymbol{A}_1$ and M_1 have a one-to-one correspondence and (X_2^N, Y_2^N, S_0) d-separates [26] $\boldsymbol{A}_1$ from $(M_2, \boldsymbol{A}_2)$, (14) holds since $(X_2^{n-1}, Y_2^{n-1}, S_0)$ d-separates $\underline{A}_{n+1}^N$ from $X_{2,n}$, and we set $Y_2^N = \boldsymbol{Y}_2$ and $X_2^N = \boldsymbol{X}_2$ in the last line. By symmetry, we also have that $NR_2 \leq I(\boldsymbol{A}_2 \to \boldsymbol{Y}_1 || \boldsymbol{X}_1, S_0 = s_0) + N\epsilon_{N,s_0}$, which completes the proof. ■

Let $\mathcal{R}_{1,L}$ denote the achievable region of Theorem 1 for a given $L \in \mathbb{N}$. A limiting expression of $\mathcal{C}_1$ is given below.

Theorem 3. *For ISI-free indecomposable and stationary FS-TWCs, we have that* $\mathcal{C}_1 = \lim_{L\to\infty} \mathcal{R}_{1,L}$.

Proof: Since (10) holds for all $s_0 \in \mathcal{S}$, it must also hold when taking minimum over all $s_0 \in \mathcal{S}$. Letting $N \to \infty$ then results in that

$$0 \leq R_1 \leq \lim_{N\to\infty} \min_{s_0 \in \mathcal{S}} \frac{1}{N} I(\boldsymbol{A}_1 \to \boldsymbol{Y}_2 || \boldsymbol{X}_2, S_0 = s_0), \quad (16a)$$

$$0 \leq R_2 \leq \lim_{N\to\infty} \min_{s_0 \in \mathcal{S}} \frac{1}{N} I(\boldsymbol{A}_2 \to \boldsymbol{Y}_1 || \boldsymbol{X}_1, S_0 = s_0). \quad (16b)$$

Together with Remark 2, the set of rate pairs (R_1, R_2) that satisfies the above two inequalities is equal to $\lim_{L\to\infty} \mathcal{R}_{1,L}$.[1] Since $\lim_{L\to\infty} \mathcal{R}_{1,L}$ is achievable, the outer bound matches the achievable region asymptotically, i.e., $\mathcal{C}_1 = \lim_{L\to\infty} \mathcal{R}_{1,L}$. Note that a super-additivity of $\mathcal{R}_{1,L}$ [8] is used to ensure the existence of the limit. ■

B. Capacity Bounds for $\mathcal{C}_2$

Theorem 4 (Inner Bound for $\mathcal{C}_2$)**.** *For any* $L \in \mathbb{N}$*, the convex hull of the set of non-negative rate pairs* (R_1, R_2) *satisfying*

$$R_1 < \frac{1}{L} I(\boldsymbol{A}_1; \boldsymbol{Y}_2 | \boldsymbol{A}_2) \quad (17a)$$

$$R_2 < \frac{1}{L} I(\boldsymbol{A}_2; \boldsymbol{Y}_1 | \boldsymbol{A}_1) \quad (17b)$$

is achievable, where $\boldsymbol{A}_j$ *and* $\boldsymbol{Y}_j$ *are defined in the same way as those in Theorem 1, and the joint probability distribution of all involved random variables is given by (with* (8)*)*

$$P_{S_0,\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{Y}_1,\boldsymbol{Y}_2}(s_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2) = P_{S_0}(s_0)$$
$$\cdot P_{\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{Y}_1,\boldsymbol{Y}_2|S_0}(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2 | s_0). \quad (18)$$

We omit the proof here (it is similar to the proof of Theorem 1). The key is to consider a new channel law $P_{Y_1,Y_2|X_1,X_2}$ that is obtained by averaging the FS-TWC with respect to all states, i.e., $P_{Y_1,Y_2|X_1,X_2} = \sum_{s_0} P_{Y_1,Y_2|X_1,X_2,S^-=s_0} P_{S_0}(s_0)$.

[1]A sequence of sets $\mathcal{A}_n \subseteq \mathbb{R}^d$ is said to converge to a set $\mathcal{A} \subseteq \mathbb{R}^d$ if $\limsup_n \mathcal{A}_n = \liminf_n \mathcal{A}_n$, i.e., $\cup_{n\geq 1} \cap_{m\geq n} \mathcal{A}_m = \cap_{n\geq 1} \cup_{m\geq n} \mathcal{A}_m$.

Remark 3. The inequalities in (17) can be rewritten as

$$R_1 < \frac{1}{L} I(\boldsymbol{A}_1 \to \boldsymbol{Y}_2 || \boldsymbol{X}_2), \quad (19a)$$

$$R_2 < \frac{1}{L} I(\boldsymbol{A}_2 \to \boldsymbol{Y}_1 || \boldsymbol{X}_1). \quad (19b)$$

Also, setting $L = 1$ produces a non-adaptive coding result for the FS-TWC with the knowledge P_{S_0}.

We next obtain an outer bound for $\mathcal{C}_2$ based on Theorem 2. Specifically, we "average" the conditions in (10) with respect to all initial states, i.e.,

$$\sum_{s_0 \in \mathcal{S}} P_{S_0}(s_0) \left[\frac{1}{N} I(\boldsymbol{A}_1 \to \boldsymbol{Y}_2 | \boldsymbol{X}_2, S_0 = s_0) + \epsilon_{N,s_0} \right]$$
$$= \frac{1}{N} I(\boldsymbol{A}_1 \to \boldsymbol{Y}_2 | \boldsymbol{X}_2, S_0) + \epsilon_N$$
$$\leq \frac{1}{N} I(\boldsymbol{A}_1 \to \boldsymbol{Y}_2 | \boldsymbol{X}_2) + \frac{\log_2 |\mathcal{S}|}{N} + \epsilon_N,$$

where the inequality is due to that $|I(\boldsymbol{A} \to \boldsymbol{B} | \boldsymbol{C}, D) - I(\boldsymbol{A} \to \boldsymbol{B} | \boldsymbol{C})| \leq \log |\mathcal{D}|$ [8], which yields the following bound.

Theorem 5 (Outer Bound for $\mathcal{C}_2$)**.** *Any achievable rate pair* (R_1, R_2) *must satisfy*

$$0 \leq R_1 \leq \frac{1}{N} I(\boldsymbol{A}_1 \to \boldsymbol{Y}_2 || \boldsymbol{X}_2) + \frac{\log_2 |\mathcal{S}|}{N} + \epsilon_N \quad (20a)$$

$$0 \leq R_2 \leq \frac{1}{N} I(\boldsymbol{A}_2 \to \boldsymbol{Y}_1 || \boldsymbol{X}_1) + \frac{\log_2 |\mathcal{S}|}{N} + \epsilon_N \quad (20b)$$

for every $N \in \mathbb{N}$ *and some joint probability distribution given in* (18)*, where* $\lim_{N\to\infty} \epsilon_N = 0$.

Letting $N \to \infty$ in (20), we further obtain that

$$0 \leq R_1 \leq \lim_{N\to\infty} \frac{1}{N} I(\boldsymbol{A}_1 \to \boldsymbol{Y}_2 || \boldsymbol{X}_2), \quad (21a)$$

$$0 \leq R_2 \leq \lim_{N\to\infty} \frac{1}{N} I(\boldsymbol{A}_2 \to \boldsymbol{Y}_1 || \boldsymbol{X}_1), \quad (21b)$$

where the two limits exist due to a super-additive property for $I(\boldsymbol{A}_j \to \boldsymbol{Y}_{j'} || \boldsymbol{X}_{j'})$ for $j, j' = 1, 2$ with $j \neq j'$ [26]. For ISI-free indecomposable and stationary FS-TWCs, it can be shown that the region enclosed by the above two inequalities is equal to $\lim_{L\to\infty} \mathcal{R}_{2,L}$, where $\mathcal{R}_{2,L}$ denotes the achievable region described in Remark 3 for any fixed $L \in \mathbb{N}$ and its limit exists. We summarize this result in the following theorem.

Theorem 6. *For ISI-free indecomposable and stationary FS-TWCs, we have that* $\mathcal{C}_2 = \lim_{L\to\infty} \mathcal{R}_{2,L}$.

IV. A Case Study and Discussion

Similar to FS-OWCs with feedback, a capacity region in single-letter form is hard to obtain for FS-TWCs. Evaluating directed information is also difficult. However, using a stationarity property for the involved random variables can help us simplify and compute the limiting expression. As a first step, we derive a capacity result for a simple case: ISI-free indecomposable and stationary TWC with i.i.d. states.

Theorem 7. *The capacity region* $\mathcal{C}_2$ *of FS-TWCs with i.i.d. states such that* $P_{Y_1,Y_2|X_1,X_2} = \sum_s \pi_S(s) P_{Y_1,Y_2|X_1,X_2,S^-=s}$

satisfies the symmetry property in [15, Theorem 1] is the convex closure of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq I(X_1; Y_2|X_2) \text{ and } R_2 \leq I(X_2; Y_1|X_1), \tag{22}$$

where the joint probability distribution of all involved random variables is given by

$$P_{S,X_1,X_2,Y_1,Y_2|S_0=s_0} = P^*_{X_1} P_{X_2} P_{Y_1,Y_2,S|X_1,X_2,S^-=s_0}, \tag{23}$$

and $P^*_{X_1}$ is the common optimal channel input distribution for the one-way channels $P_{Y_2|X_1,X_2=x_2}$, $x_2 \in \mathcal{X}_2$.

Proof: We begin with the summation in (14) to obtain a simpler form of outer bound in this case:

$$\begin{aligned}
&\sum_{n=1}^{N} H(Y_{2,n}|X_2^n, Y_2^{n-1}, S_0 = s_0) \\
&\qquad\qquad -H(Y_{2,n}|\underline{A}_1^n, X_2^n, Y_2^{n-1}, S_0 = s_0) \\
&\leq \sum_{n=1}^{N} H(Y_{2,n}|X_{2,n}) - H(Y_{2,n}|X_{1,n}, X_{2,n}) \qquad (24) \\
&= \sum_{n=1}^{N} I(X_{1,n}; Y_{2,n}|X_{2,n}) \leq N \cdot I(X_1; Y_2|X_2), \qquad (25)
\end{aligned}$$

which results in that $R_1 \leq I(X_1; Y_2|X_2)$ as $N \to \infty$, where (24) holds since $(X_{1,n}, X_{2,n})$ d-separates $(\underline{A}_1^n, Y_2^{n-1}, S_0)$ from $Y_{2,n}$, and (25) holds due to a concavity of the conditional MI and we set $P_{X_1,X_2} = \frac{1}{N}\sum_{n=1}^{N} P_{X_{1,n},X_{2,n}}$. By symmetry, one also has that $R_2 \leq I(X_1; Y_2|X_2)$. Here, the joint probability distribution of all involved random variables is:

$$P_{X_1,X_2,Y_1,Y_2} = P_{X_1,X_2} \sum_{s \in \mathcal{S}} \pi_S(s) P_{Y_1,Y_2|X_1,X_2,S^-=s}.$$

We next invoke the same proof of [15, Theorem 1] to show that independent channel inputs of the form $P^*_{X_1} P_{X_2}$ can attain the outer bound, where $P^*_{X_1}$ is the common optimal channel input distribution for the marginal channels $P_{Y_2|X_1,X_2=x_2}$, $x_2 \in \mathcal{X}_2$. Finally, choosing $L = 1$ and considering the joint probability distribution of the form (23) in (19) completes the proof. ■

Remark 4. Theorem 7 is the two-way counterpart of the capacity result for OWCs with i.i.d. state (see [29, Section 7.4]) where the state is unavailable at the sender and the receiver; in this case, coding for an "average channel" achieves channel capacity. However, as TWCs allow adaptive coding, a channel symmetry property is further required for the average channel to yield a capacity region in single-letter form.

To end this section, we distinguish our channel model from other models. In [15], a TWC model with memory is described by the following equations:

$$Y_{1,n} = F_1(X_{1,n}, X_{2,n}, Z_{1,n}), \tag{26a}$$

$$Y_{2,n} = F_2(X_{1,n}, X_{2,n}, Z_{2,n}), \tag{26b}$$

where F_1 and F_2 are deterministic and time-invariant functions while $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ is a two-dimensional stationary and ergodic noise process which is independent of the terminals'

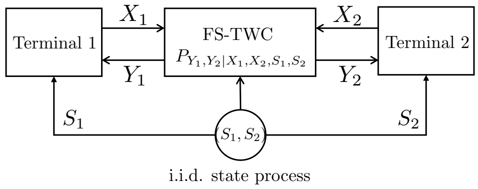

Fig. 3. Block diagram of the FS-TWCs in [25], where S_1 and S_2 are assumed to be non-causally known the terminal 1 and 2, respectively.

messages M_1 and M_2. We compare this channel model with the following special setting in our channel model (in (1))

$$P_{Y_1,Y_2,S|X_1,X_2,S^-} = P_{Y_1,Y_2|X_1,X_2,S^-} P_{S|S^-} \tag{27}$$

with $P_{S|S^-}$ being the transition kernel of a stationary and time-invariant Markov chain. Given the marginal channels with state $P_{Y_j|X_1,X_2,S^-}$, $j = 1, 2$, the functional representation lemma [29, Appendix B] ensures the existence of functions F_{j,s^-}, indexed by $s^- \in \mathcal{S}$, and corresponding Z'_{j,s^-}'s such that $Y_j = F_{j,s^-}(X_1, X_2, Z'_{j,s^-})$. Our channel model is more general than (26) in the sense that the channel input-output relationship can be time-varying deterministic functions and that we do not make any assumptions on Z'_{j,s^-}'s. Still, a detailed examination is required to differentiate the channel models in (26) and (27).

Furthermore, the two-way writing on dirty paper [25] looks at the system in Fig. 3; the state process $\{S_{1,n}, S_{2,n}\}_{n\geq 0}$ is assumed to be an i.i.d. process and partial state information is non-causally available at each terminal. Although the partial state information enables a more general adaptive channel code than the one in Definition 3, the channel model in [25] is a special case of ours realized by setting $S = (S_1, S_2)$ in the above i.i.d. example. Indeed, capacity regions and bounds for this model can be further investigated under different configurations, e.g., the state information is partially/fully/causally/non-causally known at encoders/decoders, which we leave for future research. As a side remark, the results in [25] yield a capacity region in single-letter form due to the Gaussianity of the i.i.d. additive interferences (states) and noise variables and the fact that adaptive coding is useless for Gaussian TWCs [22].

V. Conclusion

We determined two capacity regions in a limiting expression for a class of FS-TWCs depending on whether or not the stationary distribution of the initial state is known. In particular, when this statistical information is available and the states are i.i.d., the optimal coding scheme for this FS-TWC is simply the optimal coding scheme for the associated average DM-TWC. Via an example, we also found that the capacity region in single-letter form for FS-TWCs requires much more than a stationarity condition on the channel states, which highlights the difficulty of fully solving the capacity problem. Future research includes deriving computable bounds, finding more conditions for single-letterization, and investigating the FS-TWCs under other scenarios such as compound TWCs.

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