Privacy-Aware Guessing Efficiency

Shahab Asoodeh, Mario Diaz, Fady Alajaji, and Tamás Linder

***Abstract*—We investigate the problem of guessing a discrete random variable Y under a privacy constraint dictated by another correlated discrete random variable X, where both guessing efficiency and privacy are assessed in terms of the probability of correct guessing. We define $\hbar(P_{XY}, \varepsilon)$ as the maximum probability of correctly guessing Y given an auxiliary random variable Z, where the maximization is taken over all $P_{Z|Y}$ ensuring that the probability of correctly guessing X given Z does not exceed ε. We show that the map $\varepsilon \mapsto \hbar(P_{XY}, \varepsilon)$ is strictly increasing, concave, and piecewise linear, which allows us to derive a closed form expression for $\hbar(P_{XY}, \varepsilon)$ when X and Y are connected via a binary-input binary-output channel. For $\{(X_i, Y_i)\}_{i=1}^n$ being pairs of independent and identically distributed binary random vectors, we similarly define $\underline{h}_n(P_{X^nY^n}, \varepsilon)$ under the assumption that Z^n is also a binary vector. Then we obtain a closed form expression for $\underline{h}_n(P_{X^nY^n}, \varepsilon)$ for sufficiently large, but nontrivial values of ε.**

I. Introduction and Preliminaries

Given private information, represented by a random variable X, non-private observable information, say Y, is generated via a fixed channel $P_{Y|X}$. Consider two communicating agents Alice and Bob, where Alice observes Y and wishes to disclose it to Bob as accurately as possible in order to receive a payoff, but in such a way that X is kept almost private from him. Given the joint distribution P_{XY}, Alice chooses a random mapping $P_{Z|Y}$, a so-called privacy filter, to generate a new random variable Z, called the *displayed data*, such that Bob can *guess* Y from Z with as small error probability as possible while Z cannot be used to efficiently guess X.

The tradeoff between utility and privacy was addressed from an information-theoretic viewpoint in [1]–[5], where both utility and privacy were measured in terms of information-theoretic quantities. In particular, in [2] both utility and privacy were measured in terms of the mutual information I. Specifically, the so-called *rate-privacy function* $g(P_{XY}, \varepsilon)$ was defined as the maximum of $I(Y; Z)$ over all $P_{Z|Y}$ such that $I(X; Z) \leq \varepsilon$. In the most stringent privacy setting $\varepsilon = 0$, called *perfect privacy*, it was shown that $g(P_{XY}, 0) > 0$ if and only if X is weakly independent of Y, that is, if the set of vectors $\{P_{X|Y}(\cdot|y) : y \in \mathcal{Y}\}$ is linearly dependent. In [4], an equivalent result was obtained in terms of the singular values of the operator $f \mapsto \mathbb{E}[f(X)|Y]$. Although a connection between this information-theoretic privacy measure and a coding theorem is established in [2] and [6], the use of mutual information as a privacy measure is not satisfactorily motivated in an *operational* sense. To find a measure of privacy with a clear operational meaning, in this paper we take an estimation-theoretic approach and define both privacy and utility measures in terms of the probability of guessing correctly.

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The authors are with the Department of Mathematics and Statistics, Queen's University, Canada. Emails: {asoodehshahab, fady, linder}@mast.queensu.ca, 13madt@queensu.ca.

Given discrete random variables $U \in \mathcal{U}$ and $V \in \mathcal{V}$, the probability of correctly guessing U given V is defined as

$$\mathsf{P_c}(U|V) := \max_g \Pr(U = g(V)) = \sum_{v \in \mathcal{V}} \max_{u \in \mathcal{U}} P_{UV}(u, v),$$

where the first maximum is taken over all functions $g : \mathcal{V} \to \mathcal{U}$. It is easy to show that $\mathsf{P_c}$ satisfies the data processing inequality, i.e., $\mathsf{P_c}(U|W) \leq \mathsf{P_c}(U|V)$ for U, V and W which form the Markov chain $U \multimap V \multimap W$. Thus, we measure privacy in terms of $\mathsf{P_c}(X|Z)$ which quantifies the advantage of an adversary observing Z in guessing X in a single shot attempt.

A similar operational measure of privacy was recently proposed in [7], where $P_{Z|X}$ is said to be ε-private if $\log \frac{\mathsf{P_c}(U|Z)}{\mathsf{P_c}(U)} \leq \varepsilon$ for *all* auxiliary random variables U satisfying $U \multimap X \multimap Z$. This requirement guarantees that no *randomized* function of X can be efficiently estimated from Z, which leads to a strong privacy guarantee. In [8], maximal correlation [9] was proposed as another measure of privacy. Operational interpretations corresponding to this privacy measure are given in [10] for the discrete case and in [11] for a continuous setup.

To quantify the conflict between utility and privacy, we define the *privacy-aware guessing function* $\hbar$ as

$$\hbar(P_{XY}, \varepsilon) := \sup_{\substack{P_{Z|Y}: X \multimap Y \multimap Z, \\ \mathsf{P_c}(X|Z) \leq \varepsilon}} \mathsf{P_c}(Y|Z). \qquad (1)$$

Due to the data processing inequality, we can restrict the privacy threshold ε to the interval $[\mathsf{P_c}(X), \mathsf{P_c}(X|Y)]$, where $\mathsf{P_c}(X)$ is the probability of correctly guessing X in the absence of any side information. For ε close to $\mathsf{P_c}(X)$, the privacy guarantee $\mathsf{P_c}(X|Z) \leq \varepsilon$ intuitively means that it is nearly as hard to guess X observing Z as it is without observing Z.

We derive functional properties of the map $\varepsilon \mapsto \hbar(P_{XY}, \varepsilon)$. In particular, we show that it is strictly increasing, concave, and piecewise linear. Piecewise linearity (Theorem 1), which is the most important and technically difficult result in the paper, allows us to derive a tight upper bound on $\hbar(P_{XY}, \varepsilon)$ for general P_{XY}. As a consequence of concavity, we derive a closed form expression for $\hbar(P_{XY}, \varepsilon)$ for any $\varepsilon \in [\mathsf{P_c}(X), \mathsf{P_c}(X|Y)]$ when X and Y are both binary. It is shown (Theorem 2) that either the Z-channel or the *reverse* Z-channel achieves $\hbar(P_{XY}, \varepsilon)$ in this case depending on the backward channel.

We also consider the vector case for a pair of binary random vectors (X^n, Y^n) under an additional constraint that Z^n is a binary random vector. Here, Z^n is revealed publicly and the goal is to guess Y^n under the privacy constraint $\mathsf{P_c}(X^n|Z^n) \leq \varepsilon^n$. This model can be viewed as a privacy-constrained version of the *correlation distil-*

lation problem studied in [12]. Suppose Alice and Bob respectively observe Y^n and Z^n, where $\{(Y_i, Z_i)\}_{i=1}^n$ is independent and identically distributed (i.i.d.) according to the joint distribution P_{YZ}, and assume that they are to design non-constant Boolean functions f and g such that $\Pr(f(Y^n) = g(Z^n))$ is maximized. A dimension-free upper bound for this probability was given in [12]. Now suppose P_{YZ} is not given and Alice is to design $P_{Z|Y}$ (for a fixed $\mathcal{Y}$-marginal) that maximizes $\mathsf{P_c}(f(Y^n)|Z^n)$ for a given function f while $\mathsf{P_c}(X^n|Z^n) \le \varepsilon^n$. We show (Theorem 3) that if $\{(X_i, Y_i)\}_{i=1}^n$ is i.i.d. according to P_{XY} with $|\mathcal{X}| = |\mathcal{Y}| = 2$ and $P_{Y|X}$ is a binary symmetric channel, then the maximum of $\mathsf{P_c}(Y^n|Z^n)$ under the privacy constraint $\mathsf{P_c}(X^n|Z^n) \le \varepsilon^n$ admits a closed form expression for sufficiently large but nontrivial ε. This then provides a lower bound for the privacy-constrained correlation distillation problem due to the trivial fact that $\mathsf{P_c}(f(Y^n)|Z^n) \ge \mathsf{P_c}(Y^n|Z^n)$ for any function f.

We omit the proof of most of the results due to space limitations. The proofs are available in [13].

II. Scalar Case

Suppose X and Y are discrete random variables with finite alphabets $\mathcal{X} = \{1, \dots, M\}$ and $\mathcal{Y} = \{1, \dots, N\}$, respectively, and with joint distribution $\mathsf{P} = \{P_{XY}(x, y), x \in \mathcal{X}, y \in \mathcal{Y}\}$, whose marginals over $\mathcal{X}$ and $\mathcal{Y}$ are $(p_1, \dots, p_M)$ and $(q_1, \dots, q_N)$, respectively. Let X represent the private data and Y represent a non-private measurement of X, which, upon passing it via a privacy filter $P_{Z|Y}$, is publicly displayed as Z. In order to quantify the conflict between privacy with respect to X and utility with respect to Y, the so-called rate-privacy function $g(\mathsf{P}, \varepsilon)$ was introduced in [2]. In what follows, we use Arimoto's mutual information to generalize this definition.

A. The Utility-Privacy Function of Order (ν, μ)

Let $H_\nu(X)$ and $H_\nu^{\mathsf{A}}(X|Z)$ denote respectively the Rényi entropy of order ν and Arimoto's conditional entropy of order ν [14], defined for $\nu > 1$ as

$$H_\nu(X) := \frac{1}{1-\nu} \log \left(\sum_{x \in \mathcal{X}} P_X^\nu(x) \right),$$

and

$$H_\nu^{\mathsf{A}}(X|Z) := \frac{\nu}{1-\nu} \log \left(\sum_{z \in \mathcal{Z}} \left[\sum_{x \in \mathcal{X}} P_{XZ}^\nu(x, z) \right]^{1/\nu} \right).$$

We define (by continuity) $H_1(X) = H(X)$, $H_1^{\mathsf{A}}(X|Z) = H(X|Z)$, $H_\infty(X) = -\log \mathsf{P_c}(X)$, and $H_\infty^{\mathsf{A}}(X|Z) = -\log \mathsf{P_c}(X|Z)$. *Arimoto's mutual information of order* $\nu \ge 1$ is defined as (see, e.g., [14])

$$I_\nu^{\mathsf{A}}(X;Z) := H_\nu(X) - H_\nu^{\mathsf{A}}(X|Z).$$

Thus $I_1^{\mathsf{A}}(X;Z) = I(X;Z)$.

Definition 1. *For a given joint distribution* P *and a pair* (ν, μ)*,* $\nu, \mu \in [1, \infty]$*, the utility-privacy function of order* (ν, μ) *is*

$$g^{(\nu,\mu)}(\mathsf{P}, \varepsilon) := \max_{P_{Z|Y} \in \mathscr{D}^\nu(\mathsf{P}, \varepsilon)} I_\mu^{\mathsf{A}}(Y;Z),$$

where

$$\mathscr{D}^\nu(\mathsf{P}, \varepsilon) := \{P_{Z|Y} : X -\!\!\circ\!\!- Y -\!\!\circ\!\!- Z, I_\nu^{\mathsf{A}}(X;Z) \le \varepsilon\}.$$

Note that $\mathscr{D}^\nu(\mathsf{P}, \varepsilon)$ cannot be empty since all channels $P_{Z|Y}$ with Z independent of X satisfy $I_\nu^{\mathsf{A}}(X;Z) = 0$, and so they belong to $\mathcal{D}^\nu(\mathsf{P}, \varepsilon)$ for any $\varepsilon \ge 0$. Using a similar technique as in [15], one can show that $\varepsilon \mapsto g^{(\nu,\mu)}(\mathsf{P}, \varepsilon)$ is strictly increasing for any $\nu, \mu \ge 1$. It is also worth mentioning that an application of Minkowski's inequality implies that the map $P_{Z|Y} \mapsto \exp\left\{\frac{(\nu-1)}{\nu} I_\nu^{\mathsf{A}}(Y;Z)\right\}$ is convex for $\nu \ge 1$, and thus the maximum in the definition of $g^{(\nu,\mu)}(\mathsf{P}, \varepsilon)$ is achieved at the boundary of the feasible set where $I_\nu^{\mathsf{A}}(X;Z) = \varepsilon$. We denote $g^{(\infty,\infty)}(\mathsf{P}, \varepsilon)$ and $g^{(1,1)}(\mathsf{P}, \varepsilon)$ respectively by $g^\infty(\mathsf{P}, \varepsilon)$ and $g(\mathsf{P}, \varepsilon)$. Since $I_\infty(Y;Z) = \log \frac{\mathsf{P_c}(Y|Z)}{\mathsf{P_c}(Y)}$, $g^\infty(\mathsf{P}, \varepsilon)$ can be equivalently described as the smallest $\Gamma \ge 0$ such that $\mathsf{P_c}(Y|Z) \le \mathsf{P_c}(Y)2^\Gamma$, for every $P_{Z|Y}$ satisfying $\mathsf{P_c}(X|Z) \le \mathsf{P_c}(X)2^\varepsilon$. We note that for small ε the condition $I_\infty^{\mathsf{A}}(X;Z) \le \varepsilon$ intuitively means that it is nearly as hard for an *adversary* observing Z to predict X as it is without Z. Therefore, $g^\infty(\mathsf{P}, 0)$ quantifies the efficiency of guessing Y from Z such that $\mathsf{P_c}(X|Z) = \mathsf{P_c}(X)$. It is thus interesting to obtain a necessary and sufficient condition for P under which $g^\infty(\mathsf{P}, 0) > 0$. We obtain such a condition for the special case of binary X and Y in the next section.

In general, the map $\nu \mapsto I_\nu^{\mathsf{A}}(X;Z)$ is not monotonic[1] and hence $P_{Z|Y}$ might belong to $\mathscr{D}^\nu(\mathsf{P}, \varepsilon)$ but not to $\mathscr{D}^\mu(\mathsf{P}, \varepsilon)$ for $\mu < \nu$. Nevertheless, the following lemma allows us to obtain upper and lower bounds for $g^{(\nu,\mu)}(\mathsf{P}, \cdot)$ in terms of $g^\infty(\mathsf{P}, \cdot)$.

Lemma 1. *Let* (X, Y) *be a pair of random variables having joint distribution* P *and* $\nu, \mu \in (1, \infty)$*. Then*

$$g^{(\nu,\mu)}(\mathsf{P}, \varepsilon) \le g^\infty(\mathsf{P}, \psi(\nu, \varepsilon)) + H_\mu(Y) - H_\infty(Y),$$

where $\psi(\nu, \varepsilon) := \frac{\nu-1}{\nu}\varepsilon + \frac{1}{\nu} H_\infty(X)$*. Furthermore, we have for* $\varepsilon \ge H_\nu(X) - H_\infty(X)$ *that*

$$g^{(\nu,\mu)}(\mathsf{P}, \varepsilon) \ge \frac{\mu}{\mu-1} g^\infty(\mathsf{P}, \varphi(\nu, \varepsilon)) - \frac{1}{\mu-1} H_\infty(Y),$$

where $\varphi(\nu, \varepsilon) := \varepsilon - H_\nu(X) + H_\infty(X)$*.*

This lemma shows that the family of functions $g^{(\nu,\mu)}(\mathsf{P}, \varepsilon)$ for $\nu, \mu > 1$ can be bounded from above and below by $g^\infty(\mathsf{P}, \delta)$, where δ depends on ε and ν. The case $\nu = \mu = 1$ is studied in [2]. As a result, in the following section we only focus on $g^\infty(\mathsf{P}, \varepsilon)$. It turns out that it is easier to study $\hbar(\mathsf{P}, \varepsilon)$, defined in (1), instead. It is straightforward to verify that

$$g^\infty(\mathsf{P}, \varepsilon) = \log \frac{\hbar(\mathsf{P}, 2^\varepsilon \mathsf{P_c}(X))}{\mathsf{P_c}(Y)},$$

and hence all the results for $\hbar(\mathsf{P}, \varepsilon)$ can be translated to results for $g^\infty(\mathsf{P}, \varepsilon)$. In particular, perfect privacy $g^\infty(\mathsf{P}, 0)$ corresponds to $\hbar(\mathsf{P}, \mathsf{P_c}(X))$. Notice that $\hbar(\mathsf{P}, \mathsf{P_c}(X)) > \mathsf{P_c}(Y)$ is equivalent to $g^\infty(\mathsf{P}, 0) > 0$. As opposed to $I_\nu(X;Z)$ with $1 \le \nu < \infty$, $I_\infty(X;Z) = 0$ does not

[1] It is relatively easy to show that if X is uniformly distributed, then $I_\nu^{\mathsf{A}}(X;Z)$ coincides with Sibson's mutual information of order ν [14] which is known to be increasing in ν [16, Theorem 4]. Consequently, $\nu \mapsto I_\nu^{\mathsf{A}}(X;Z)$ is increasing over $(1, \infty]$ if X is uniformly distributed.

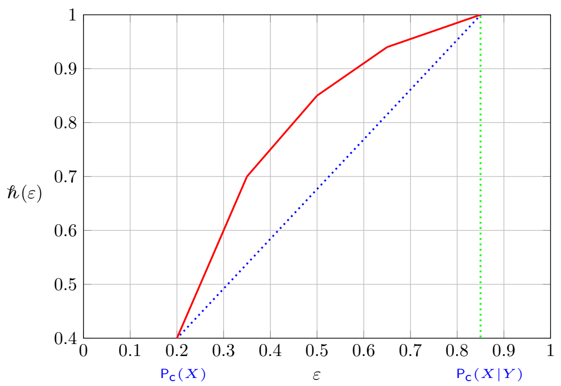


Fig. 1. Typical graph of $\hbar(\varepsilon)$. The dotted line represents the chord connecting $(p, \hbar(p))$ and $(\mathsf{P_c}(X|Y), 1)$ which can be viewed as a trivial lower bound for $\hbar(\cdot)$.

necessarily imply the independence of X and Z (unless X is uniformly distributed). In particular, the weak independence[2] argument from [2, Lemma 10] (see also [4]) cannot be applied for g^{∞}. For the sake of brevity, we simply write $\hbar(\varepsilon)$ for $\hbar(\mathsf{P}, \varepsilon)$ when there is no risk of confusion.

B. Privacy-Aware Guessing Function

It is clear from (1) that $\mathsf{P_c}(Y) \leq \hbar(\varepsilon) \leq 1$, and $\hbar(\varepsilon) = 1$ if and only if $\varepsilon \geq \mathsf{P_c}(X|Y)$. A direct application of the Support Lemma [17, Lemma 15.4] shows that it is enough to consider random variables Z supported on $\mathcal{Z} = \{1, \ldots, N+1\}$. Thus, the privacy filter $P_{Z|Y}$ can be realized by an $N \times (N+1)$ stochastic matrix F. Let $\mathcal{F}$ be the set of all such matrices. Then both utility $\mathcal{U}(\mathsf{P}, F) = \mathsf{P_c}(Y|Z)$ and privacy $\mathcal{P}(\mathsf{P}, F) = \mathsf{P_c}(X|Z)$ are functions of $F \in \mathcal{F}$ and we can express $\hbar(\varepsilon)$ as

$$\hbar(\varepsilon) = \max_{\substack{F \in \mathcal{F}, \\ \mathcal{P}(\mathsf{P}, \varepsilon) \leq \varepsilon}} \mathcal{U}(\mathsf{P}, F).$$

It can be verified that $F \mapsto \mathcal{P}(\mathsf{P}, F)$ and $F \mapsto \mathcal{U}(\mathsf{P}, F)$ are continuous convex functions over $\mathcal{F}$. It can also be shown that the set

$$\mathcal{R} := \{(\mathcal{P}(\mathsf{P}, F), \mathcal{U}(\mathsf{P}, F)) : \ F \in \mathcal{F}\}$$

is convex. Furthermore, since the graph of $\hbar(\varepsilon)$ is the upper boundary of $\mathcal{R}$, we conclude that $\varepsilon \mapsto \hbar(\varepsilon)$ is concave, and so it is strictly increasing and continuous on $[\mathsf{P_c}(X), \mathsf{P_c}(X|Y)]$. As a consequence, for every $\varepsilon \in [\mathsf{P_c}(X), \mathsf{P_c}(X|Y)]$ there exists G such that $\mathcal{P}(\mathsf{P}, G) = \varepsilon$ and $\mathcal{U}(\mathsf{P}, G) = \hbar(\varepsilon)$. We call such a privacy filter G optimal at ε.

The following theorem reveals that $\hbar(\cdot)$ is a piecewise linear function, as depicted in Fig. 1.

Theorem 1. *The function $\hbar : [\mathsf{P_c}(X), \mathsf{P_c}(X|Y)] \to \mathbb{R}$ is piecewise linear, i.e., there exist $K \geq 1$ and thresholds $\mathsf{P_c}(X) = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_K = \mathsf{P_c}(X|Y)$ such that $\hbar$ is linear on $[\varepsilon_{i-1}, \varepsilon_i]$ for all $1 \leq i \leq K$.*

Consider the map $\mathcal{H} : \mathcal{F} \to [0,1] \times [0,1]$ given by $\mathcal{H}(\mathsf{P}, F) = (\mathcal{P}(\mathsf{P}, F), \mathcal{U}(\mathsf{P}, F))$. Let $\mathcal{D} := \{D \in \mathcal{M}_{N \times N+1} : \|D\| = 1\}$, where $|| \cdot ||$ denotes the Euclidean norm on $\mathcal{M}_{N \times (N+1)}$, the set of real matrices of size $N \times (N+1)$. For $G \in \mathcal{F}$ define

$$\mathcal{D}(G) := \{D \in \mathcal{D} : G + tD \in \mathcal{F} \text{ for some } t > 0\}.$$

The proof of the previous theorem is heavily based on the following technical, yet intuitive, result: for every $G \in \mathcal{F}$, there exists $\delta > 0$ such that $\mathcal{H}$ is linear on $[G, G + \delta D]$ for *every* $D \in \mathcal{D}(G)$.

The proof technique allows us to derive the slope of $\hbar$ on $[\varepsilon_{i-1}, \varepsilon_i]$, given the family of optimal filters at a single point $\varepsilon \in [\varepsilon_{i-1}, \varepsilon_i]$. For example, since the family of optimal filters at $\varepsilon = \mathsf{P_c}(X|Y)$ is easily obtainable, it is then possible to compute $\hbar$ on the last interval. In the binary case, this observation and the concavity of $\hbar$ allow us to show that $\hbar$ is linear on its entire domain $[\mathsf{P_c}(X), \mathsf{P_c}(X|Y)]$.

C. Binary Case

Assume now that X and Y are both binary. Let $\mathsf{BIBO}(\alpha, \beta)$ denote a binary input binary output channel from X to Y with $P_{Y|X}(\cdot|0) = (\bar{\alpha}, \alpha)$ and $P_{Y|X}(\cdot|1) = (\beta, \bar{\beta})$, where $\bar{x} := 1 - x$ for $x \in [0,1]$. Notice that if $X \sim \mathsf{Bernoulli}(p)$ with $p \in [\frac{1}{2}, 1)$, then $\mathsf{P_c}(X) = p$ and hence $\hbar(p)$ corresponds to the maximum of $\mathsf{P_c}(Y|Z)$ under perfect privacy $\mathsf{P_c}(X|Z) = p$. Furthermore, if $P_{Y|X} = \mathsf{BIBO}(\alpha, \beta)$ with $\alpha, \beta \in [0, \frac{1}{2})$, then we have $\mathsf{P_c}(X|Y) = \max\{\bar{\alpha}\bar{p}, \beta p\} + \bar{\beta} p$. Notice that if $\bar{\alpha}\bar{p} \leq \beta p$, then $\mathsf{P_c}(X|Y) = \mathsf{P_c}(X) = p$.

The binary symmetric channel with crossover probability α, denoted by $\mathsf{BSC}(\alpha)$, and also the Z-channel with crossover probability β, denoted by $\mathsf{Z}(\beta)$, are both examples of $\mathsf{BIBO}(\alpha, \beta)$, corresponding to $\alpha = \beta$ and $\alpha = 0$, respectively. Let $q := \Pr(Y = 1)$. We say that perfect privacy yields a non-trivial utility if $\mathsf{P_c}(Y|Z) > \mathsf{P_c}(Y)$ for some Z such that $\mathsf{P_c}(X|Z) = \mathsf{P_c}(X)$, or equivalently, if $\hbar(p) > \max\{\bar{q}, q\}$. The following lemma determines $\hbar(p)$ in the non-trivial case $\bar{\alpha}\bar{p} > \beta p$.

Lemma 2. *Let $X \sim \mathsf{Bernoulli}(p)$ with $p \in [\frac{1}{2}, 1)$ and $P_{Y|X} = \mathsf{BIBO}(\alpha, \beta)$ with $\alpha, \beta \in [0, \frac{1}{2})$ such that $\bar{\alpha}\bar{p} > \beta p$. Then*

$$\hbar(p) = \begin{cases} 1 - \zeta q & \text{if } \alpha\bar{\alpha}\bar{p}^2 < \beta\bar{\beta}p^2, \\ q & \text{otherwise,} \end{cases}$$

where $q = \alpha\bar{p} + \bar{\beta} p$ and

$$\zeta := \frac{\bar{\alpha}\bar{p} - \beta p}{\bar{\beta} p - \alpha\bar{p}}. \tag{2}$$

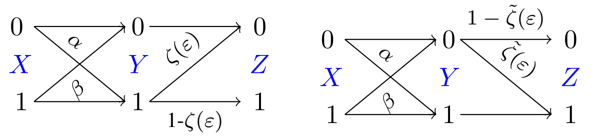
Notice that $1 - \zeta q > \bar{q}$ if and only if $\zeta < 1$, which occurs if and only if $p \in (\frac{1}{2}, 1)$. Also, it is straightforward to show that $1 - \zeta q > q$ if and only if $\alpha\bar{\alpha}\bar{p}^2 < \beta\bar{\beta}p^2$. In particular, we have the following necessary and sufficient condition for non-trivial utility under perfect privacy.

Corollary 1. *Let $X \sim \mathsf{Bernoulli}(p)$ with $p \in [\frac{1}{2}, 1)$ and $P_{Y|X} = \mathsf{BIBO}(\alpha, \beta)$ with $\alpha, \beta \in [0, \frac{1}{2})$ such that $\bar{\alpha}\bar{p} > \beta p$. Then $g^{\infty}(\mathsf{P}, 0) > 0$ if and only if $\alpha\bar{\alpha}\bar{p}^2 < \beta\bar{\beta}p^2$ and $p \in (\frac{1}{2}, 1)$.*

Remark that the condition $\alpha\bar{\alpha}\bar{p}^2 < \beta\bar{\beta}p^2$ can be equivalently written as

$$P_{X|Y}(0|1) P_{X|Y}(0|0) < P_{X|Y}(1|0) P_{X|Y}(1|1).$$

[2]Using a similar proof as in [2], it can be shown that $g^{(\nu,\mu)}(\mathsf{P}, 0) > 0$ for $\nu, \mu \in [1, \infty)$ if and only if X is weakly independent of Y.



(a) $\alpha\bar{\alpha}\bar{p}^2 < \beta\bar{\beta}p^2$ (b) $\alpha\bar{\alpha}\bar{p}^2 \geq \beta\bar{\beta}p^2$

Fig. 2. The optimal privacy filters for $P_{Y|X} = \mathsf{BIBO}(\alpha, \beta)$.

The following theorem establishes the linear behavior of $\hbar$ when $P_{Y|X} = \mathsf{BIBO}(\alpha, \beta)$.

Theorem 2. *Let $X \sim \mathsf{Bernoulli}(p)$ for $p \in [\frac{1}{2}, 1)$ and $P_{Y|X} = \mathsf{BIBO}(\alpha, \beta)$ with $\alpha, \beta \in [0, \frac{1}{2})$. If $\bar{\alpha}\bar{p} > \beta p$, then for any $\varepsilon \in [p, \bar{\alpha}\bar{p} + \bar{\beta}p]$, we have the following:*

- *If $\alpha\bar{\alpha}\bar{p}^2 < \beta\bar{\beta}p^2$, then*
$$\hbar(\varepsilon) = 1 - \zeta(\varepsilon)q,$$
where $q = \alpha\bar{p} + \bar{\beta}p$ and
$$\zeta(\varepsilon) := \frac{\bar{\alpha}\bar{p} + \bar{\beta}p - \varepsilon}{\bar{\beta}p - \alpha\bar{p}}. \tag{3}$$
Furthermore, $\hbar(\varepsilon)$ is achieved by the Z-channel $\mathsf{Z}(\zeta(\varepsilon))$ (as shown in Fig. 2).
- *If $\alpha\bar{\alpha}\bar{p}^2 \geq \beta\bar{\beta}p^2$, then*
$$\hbar(\varepsilon) = 1 - \tilde{\zeta}(\varepsilon)\bar{q},$$
where
$$\tilde{\zeta}(\varepsilon) := \frac{\bar{\alpha}\bar{p} + \bar{\beta}p - \varepsilon}{\bar{\alpha}\bar{p} - \beta p}.$$
Moreover, $\hbar(\varepsilon)$ is achieved by a reverse Z-channel with crossover probability $\tilde{\zeta}(\varepsilon)$ (as shown in Fig. 2).

Proof Sketch. Recall that $\varepsilon \mapsto \hbar(\varepsilon)$ is concave, and thus its graph lies above the segment connecting $(p, \hbar(p))$ to $(\mathsf{P_c}(X|Y), 1)$. In particular,
$$\hbar(\varepsilon) \geq \hbar(p) + (\varepsilon - p)\left[\frac{1 - \hbar(p)}{\mathsf{P_c}(X|Y) - p}\right].$$
By Lemma 2, the above inequality becomes
$$\begin{aligned}\hbar(\varepsilon) \quad \geq \quad & \hbar(p) + \frac{q(\varepsilon - p)}{\bar{\beta}p - \alpha\bar{p}} 1_{\{\alpha\bar{\alpha}\bar{p}^2 < \beta\bar{\beta}p^2\}} \\ & + \frac{\bar{q}(\varepsilon - p)}{\bar{\alpha}\bar{p} - \beta p} 1_{\{\alpha\bar{\alpha}\bar{p}^2 \geq \beta\bar{\beta}p^2\}}.\end{aligned} \tag{4}$$
Since $\varepsilon \mapsto \hbar(\varepsilon)$ is piecewise linear, its right derivative exists at $\varepsilon = \mathsf{P_c}(X|Y)$. Using the geometric properties of $\mathcal{H}$ used to prove Theorem 1, we can show that
$$\begin{aligned}\hbar'(\mathsf{P_c}(X|Y)) \quad = \quad & \frac{q}{\bar{\beta}p - \alpha\bar{p}} 1_{\{\alpha\bar{\alpha}\bar{p}^2 < \beta\bar{\beta}p^2\}} \\ & + \frac{\bar{q}}{\bar{\alpha}\bar{p} - \beta p} 1_{\{\alpha\bar{\alpha}\bar{p}^2 \geq \beta\bar{\beta}p^2\}},\end{aligned}$$
which is equal to the slope of the chord connecting $(p, \hbar(p))$ to $(\mathsf{P_c}(X|Y), 1)$ described in (4). The concavity of $\hbar(\cdot)$ thus implies that the inequality (4) is indeed equality. ∎

Under the hypotheses of the previous theorem, for every $\varepsilon \in [\mathsf{P_c}(X), \mathsf{P_c}(X|Y)]$ there exists a Z-channel that achieves $\hbar(\varepsilon)$. It can be shown that Z-channel is the *only* binary filter with this property. It is also worth mentioning that even if $P_{Y|X}$ is symmetric (i.e., $\alpha = \beta$), the optimal filter cannot be symmetric, unless X is uniform, in which case $\mathsf{BSC}(0.5\zeta(\varepsilon))$ is also optimal.

III. I.I.D. Binary Symmetric Vector Case

We next study privacy aware guessing for a pair of binary random *vectors* (X^n, Y^n) with $X^n, Y^n \in \{0, 1\}^n$. Recall that in this case it is sufficient to consider auxiliary random variables having supports of cardinality $2^n + 1$. However, this condition may be practically inconvenient. Moreover, in the scalar binary case examined in the last section we observed that a binary Z was sufficient to achieve $\hbar(\varepsilon)$. Hence, it is natural to require the privacy filters to produce also binary random vectors, i.e., $Z^n \in \{0, 1\}^n$, which leads to the following definition. Recall that the data processing inequality implies that $\mathsf{P_c}(X^n) \leq \mathsf{P_c}(X^n|Z^n) \leq \mathsf{P_c}(X^n|Y^n)$ and hence we can assume $\mathsf{P_c}(X^n) \leq \varepsilon^n \leq \mathsf{P_c}(X^n|Y^n)$.

Definition 2. *For a given pair of binary random vectors (X^n, Y^n), we define $\underline{h}_n(\varepsilon)$ for $\varepsilon \in [\mathsf{P_c}^{1/n}(X^n), \mathsf{P_c}^{1/n}(X^n|Y^n)]$, as*
$$\underline{h}_n(\varepsilon) := \max\ \mathsf{P_c}^{1/n}(Y^n|Z^n), \tag{5}$$
where the maximum is taken over all (not necessarily memoryless) channels $P_{Z^n|Y^n}$ such that $Z^n \in \{0, 1\}^n$, $X^n \multimap Y^n \multimap Z^n$, and $\mathsf{P_c}(X^n|Z^n) \leq \varepsilon^n$.

Note that this definition does not make any assumption about the privacy filters $P_{Z^n|Y^n}$ except that $Z^n \in \{0, 1\}^n$. From an implementation point of view, the simplest privacy filter is a memoryless one such that Z_k is a noisy version of Y_k for $k = 1, \ldots, n$. More precisely, we are interested in a *single* BIBO channel $P_{Z|Y}$ which, given Y_k, generates Z_k according to
$$P_{Z^n|Y^n}(z^n|y^n) = \prod_{k=1}^{n} P_{Z|Y}(z_k|y_k).$$
Now, let $h_n^{\mathsf{i}}(\varepsilon)$ be defined as $\max \mathsf{P_c}^{1/n}(Y^n|Z^n)$, where the maximum is taken over such memoryless privacy filters satisfying $\mathsf{P_c}(X^n|Z^n) \leq \varepsilon^n$. Let $\oplus$ denote mod 2 addition. In what follows, we study $\underline{h}_n$ and h_n^{i} for the following setup:

a) $X_1, \ldots, X_n$ are i.i.d. $\mathsf{Bernoulli}(p)$ random variables with $p \geq \frac{1}{2}$,

b) $Y_k = X_k \oplus V_k$ for $k = 1, \ldots, n$, where $V_1, \ldots, V_n$ are i.i.d. $\mathsf{Bernoulli}(\alpha)$ random variables independent of X^n, such that $\alpha < \frac{1}{2}$.

We first determine $h_n^{\mathsf{i}}(\varepsilon)$ for this model and show that (as expected) $h_n^{\mathsf{i}}(\varepsilon)$ is independent of n. According to this model, $\mathsf{P_c}(X^n) = p^n$ and $\mathsf{P_c}(X^n|Y^n) = \bar{\alpha}^n$, and thus $p \leq \varepsilon \leq \bar{\alpha}$.

Proposition 1. *If (X^n, Y^n) satisfies a) and b) with $p \in [\frac{1}{2}, 1)$ and $\alpha \in [0, \frac{1}{2})$ such that $\bar{\alpha} > p$, then*
$$h_n^{\mathsf{i}}(\varepsilon) = \hbar(\varepsilon) = 1 - \zeta(\varepsilon)q,$$
for all $\varepsilon \in [p, \bar{\alpha}]$, where $\zeta(\varepsilon)$ is given in (3) and $q = \alpha\bar{p} + \bar{\alpha}p$.

Note that the proposition reduces to Theorem 2 for $n = 1$. However, for $n \geq 2$, we have $h_n^{\mathsf{i}}(\varepsilon) < \underline{h}_n(\varepsilon) \leq$

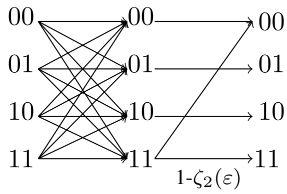


Fig. 3. The optimal privacy filter for $\underline{h}_2(\varepsilon)$ for $\varepsilon \in [\varepsilon_{\mathsf{L}}, \bar{\alpha})$, where $\zeta_2(\varepsilon)$ is defined in (6).

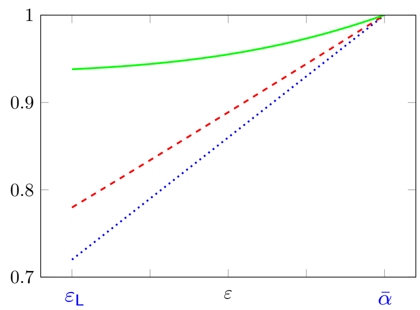


Fig. 4. The graphs of $\underline{h}_{10}$ (solid curve), $\underline{h}_2$ (dashed curve), and h^{i} (dotted line) given in Theorem 3 and Proposition 1 for i.i.d. (X^n, Y^n) with $X \sim \mathsf{Bernoulli}(0.6)$ and $P_{Y|X} = \mathsf{BSC}(0.2)$.

$\hbar(P_{X^nY^n}, \varepsilon)$, as implied by the following theorem. A channel W is said to be a 2^n-ary Z-channel, denoted by $\mathsf{Z}_n(\gamma)$, if the input and output alphabets are $\{0,1\}^n$ and $\mathsf{W}(a|a) = 1$ for $a \neq \mathbf{1}$, $\mathsf{W}(\mathbf{0}|\mathbf{1}) = \gamma$, and $\mathsf{W}(\mathbf{1}|\mathbf{1}) = \bar{\gamma}$, where $\mathbf{0} = (0, 0, \dots, 0)$ and $\mathbf{1} = (1, 1, \dots, 1)$.

Theorem 3. *Assume that (X^n, Y^n) satisfies a) and b) with $p \in [\frac{1}{2}, 1)$ and $\alpha \in [0, \frac{1}{2})$ such that $\bar{\alpha} > p$. Then, there exists $p \leq \varepsilon_{\mathsf{L}} < \bar{\alpha}$ such that*

$$\underline{h}_n^n(\varepsilon) = 1 - \zeta_n(\varepsilon) q^n,$$

for $\varepsilon \in [\varepsilon_{\mathsf{L}}, \bar{\alpha}]$, where $q = \alpha\bar{p} + \bar{\alpha}p$ and

$$\zeta_n(\varepsilon) := \frac{\bar{\alpha}^n - \varepsilon^n}{(\bar{\alpha}p)^n - (\alpha\bar{p})^n}. \tag{6}$$

Moreover, the channel $\mathsf{Z}_n(\zeta_n(\varepsilon))$ achieves $\underline{h}_n(\varepsilon)$ in this interval (see Fig. 3 for the case $n = 2$).

The memoryless privacy filter assumed in $h_n^{\mathsf{i}}(\varepsilon)$ is simple to implement. However, it is clear from Theorem 3 that this simple filter is not optimal even when (X^n, Y^n) is i.i.d. since $\underline{h}_n(\varepsilon)$ is a function of n, while $h_n^{\mathsf{i}}(\varepsilon)$ is not. The following corollary bounds the loss resulting from using a simple memoryless filter instead of an optimal one for $\varepsilon \in [\varepsilon_{\mathsf{L}}, \bar{\alpha}]$. Clearly, for $n = 1$, there is no gap as $\underline{h}_1(\varepsilon) = h_1^{\mathsf{i}}(\varepsilon)$.

Corollary 2. *Let (X^n, Y^n) satisfy a) and b) with $p \in [\frac{1}{2}, 1)$ and $\alpha \in [0, \frac{1}{2})$ such that $\bar{\alpha} > p$. If $p > \frac{1}{2}$ and $\alpha > 0$, then for $\varepsilon \in [\varepsilon_{\mathsf{L}}, \bar{\alpha}]$ and sufficiently large n*

$$\underline{h}_n(\varepsilon) - h_n^{\mathsf{i}}(\varepsilon) \geq (\bar{\alpha} - \varepsilon)[\Phi(1) - \Phi(n)], \tag{7}$$

where

$$\Phi(n) := \frac{q^n \bar{\alpha}^{n-1}}{(\bar{\alpha}p)^n - (\alpha\bar{p})^n}.$$

If $p = \frac{1}{2}$, then

$$h_n^{\mathsf{i}}(\varepsilon) \leq \underline{h}_n(\varepsilon) \leq h_n^{\mathsf{i}}(\varepsilon) + \frac{\alpha}{2\bar{\alpha}}, \tag{8}$$

for every $n \geq 1$ and $\varepsilon \in [\varepsilon_{\mathsf{L}}, \bar{\alpha}]$.

Since $\Phi(n) \downarrow 0$ as $n \to \infty$, (7) implies that, as expected, the gap between the performance of the optimal privacy filter and the optimal memoryless privacy filter increases as n increases. This observation is numerically illustrated in Fig. 4, where $\underline{h}_n(\varepsilon)$ is plotted as a function of ε for $n = 2$ and $n = 10$. Moreover, (8) implies that when $p = \frac{1}{2}$ and α is small, then $\underline{h}_n(\varepsilon)$ can be approximated by $h_n^{\mathsf{i}}(\varepsilon)$. Thus, we can approximate the optimal filter $\mathsf{Z}_n(\zeta_n(\varepsilon))$ with a simple memoryless filter given by $Z_k = Y_k \oplus W_k$, where $W_1, \dots, W_n$ are i.i.d. $\mathsf{Bernoulli}(0.5\zeta(\varepsilon))$ random variables that are independent of (X^n, Y^n).

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