Two-Way Gaussian Channels with an Intelligent Jammer

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Abstract— **We consider the existence and structure of (zero-sum game) Nash equilibria for a two-way channel in the presence of an intelligent jammer capable of tapping the channel signals in both directions. We assume that the source and noise signals are all Gaussian random variables, where the source signals are independent of each other while the noise signals are arbitrarily correlated. We show that for fixed jammer power constraints, a Nash equilibrium exists with respect to the system wide mean square error (MSE), and equilibrium jamming policies are always Gaussian. We derive the equilibrium policies in closed form under various system parameters. Finally for two system scenarios, we analytically determine the optimal power allocation levels the jammer can deploy in each channel link, when allowed to operate under an overall power budget.**

I. Introduction

The question of how a communication system performs in the worst case scenario is one of vital importance, for if one can guarantee a suitable operation in the worst case then the system will operate at least as well in all other cases. In determining the worst case faced by a system, it is useful to personify the noise as an intelligent "jammer" actively working to make the system operate poorly. Further, in many scenarios, there may indeed be a malicious agent who intends to suppress communication and control in a decentralized system. These setups motivate the use of game theoretic methods in communications and networked control applications.

One of the earliest works to consider such a problem is by Başar [1], where a one-way additive Gaussian noise channel is studied in the presence of an intelligent jammer. Başar establishes complete solutions to a zero-sum formulation, proving the optimality of linear/affine/Gaussian policies under various setups and assumptions. Further relevant studies include [2]-[3].

The problem stated above focuses on one-way (or point-to-point) communications. However, modern communication systems are increasingly decentralized and multi-terminal for better utilization of limited channel resources, e.g., see [4] for a general overview. The simplest networked system is the two-way channel first introduced by Shannon in [5]. In this paper, we consider the case of a Gaussian two-way channel with an intelligent jammer. In such a channel, each user transmits and receives signals simultaneously. This allows each encoder to interactively adapt the current input to its own message and all previously received signals, hence rendering it more resilient to channel noise. The reader is referred to [5]-[11] and the references therein for coding theorems and channel capacity results for two-way channels. In particular, it is shown in [11] that zero-delay linear (scalar) coding and decoding achieve the Shannon theoretical limit for a two-way Gaussian source-channel system with independent sources and are hence optimal. We herein focus on the same scalar two-way Gaussian system in the presence of a jammer capable of accessing the channel's signals in both directions. This can also be considered as a one shot control problem.

The identification of optimal linear/affine/Gaussian policies for decentralized systems involving Gaussian variables under quadratic criteria (such as in linear quadratic control (LQG)) is a recurring problem in stochastic networked control and estimation theory (see [4, Chap. 11] for a review). These certainly include the classical problem of communicating a scalar Gaussian source over a Gaussian channel [12]-[15], where linear encoding policies are optimal, which also extends to the vector case under certain conditions [16]-[20]. For non-classical decentralized stochastic control problems, Witsenhausen's counterexample [21] shows that optimal policies for LQG systems may be non-linear and this suboptimality also extends to various decentralized LQG problems as reviewed in [4, Chap. 11] and [22].

For game-theoretic formulations, somewhat surprisingly, optimality and linearity again coincide for a large class of setups: in Witsenhausen's counterexample, if the noise variable is viewed as the maximizer and the encoders/decoders (or the controllers) act as the minimizers, then affine policies may be optimal [23]-[24]. For a setup similar to [1], but with the game being played only between an encoder and a jammer (with the decoder being a Bayesian decision maker), it is shown in [25] that the worst additive channel noise is Gaussian and the optimal encoder is linear. This result may be viewed as a Stackelberg extension of the Nash setup given in [1] (for a detailed discussion on the distinction between Nash vs. Stackelberg equilibria in signaling games, see [26] and [27]), where the receiver is a follower and the encoder/jammer pair is a leader.

In view of the discussion above, our paper provides further conditions on when affine and Gaussian policies may constitute equilibria for such decentralized quadratic Gaussian optimization problems. In particular, we show that for a two-way networked system with Gaussian noise and scalar variables with an intelligent jammer, an essentially unique zero-sum Nash equilibrium exists and the equilibrium policies are affine/Gaussian. We derive the closed form of the equilibrium policies under various system parameters.

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Thus, our paper provides a two-way (and thus a decentralized) generalization, in the sense that there exists a team of encoders/decoders against a single jammer, of the findings of [1] where a single encoder/decoder pair is present against a jammer. The nature of two-way channels adds significant complexity to the problem. The correlation between the noise signals on the channel requires special analysis in different situations, and ultimately results in different jamming policies depending on the noise correlation and variance. Finding the equilibrium jamming policy amounts to solving for the fixed point of a best response function. In the two way setup, this is now a function of two variables, not just one, which complicates solving for the fixed point. Furthermore, we investigate the optimal power allocation the jammer can employ for each channel direction under a given overall budget; this problem has no counterpart in the one-way setup.

The rest of this paper is organized as follows. In Section II, we formulate the problem. In Section III-A, we examine the common setup for the two-way channel noise variables and derive full closed form solutions for the equilibrium policies. In Section III-B, we analyze using a slightly different approach a special "degenerate" case under which the noise variables coincide in each channel direction. We compare the results to previous work and provide a qualitative analysis in Section III-C. In Section III-D, we investigate the optimal power allocation levels when the jammer is allowed to choose its power constraints subject to an overall budget. A sketch of the proof for the main theorem is presented in Section IV and concluding remarks are drawn in Section V.

II. Problem Setup

Consider two terminals T_1 and T_2 attempting to exchange Gaussian independent signals U_1 and U_2, respectively, where $U_i \sim \mathcal{N}(0,1)$ has zero mean and unit variance for $i=1,2$, across a two-way additive Gaussian noise channel as depicted in Fig. 1. More specifically, each terminal T_i observes signal U_i and uses transmitter policy $\gamma_i : \mathbb{R} \to \mathbb{R}$ to generate signal X_i subject to the power constraint

$$E[(\gamma_i(U_i))^2] \leq c_i, \quad i=1,2. \tag{1}$$

The two-way channel inputs are X_1, X_2 and its outputs are

$$Y_i = X_1 + X_2 + Z_i, \quad i=1,2 \tag{2}$$

where Z_1 and Z_2 are Gaussian random variables (which are independent of (U_1,U_2)) with zero mean and covariance matrix

$$\Sigma = \begin{pmatrix} E[Z_1^2] & E[Z_1 Z_2] \\ E[Z_2 Z_1] & E[Z_2^2] \end{pmatrix} = \begin{pmatrix} \zeta_1 & \zeta_{1,2} \\ \zeta_{1,2} & \zeta_2 \end{pmatrix} \tag{3}$$

where $\zeta_{1,2}$ takes values in $[-(\zeta_1\zeta_2)^{\frac{1}{2}}, (\zeta_1\zeta_2)^{\frac{1}{2}}]$.

We furthermore assume the existence of a third party, the jammer. The latter taps the channel in both directions and sees signals Y_1 and Y_2; in return, its sends an adversarial signal ν_i to each terminal T_i using the jamming policy $\nu_i = \beta_i(y_1,y_2)$, $i=1,2$, where each β_i is in general a random mapping. Let $\mathcal{M}$ denote the space of the pairs (μ_1,μ_2), where μ_i is the probability measure associated with jamming signal ν_i under the power constraint

$$E[\nu_i^2] \leq k_i, \quad i=1,2. \tag{4}$$

Terminal T_i then receives signal $Q_i = Y_i + \nu_i$, which it uses together with side information X_i (i.e., its own signal sent to Terminal T_j) to reconstruct U_j via $\hat{U}_j$ under decoding policy $\delta_i : \mathbb{R}\times\mathbb{R} :\to \mathbb{R}$, $i \neq j$, $i,j=1,2$.

The overall MSE of the system is given by

$$\begin{aligned} &R(\gamma_1,\gamma_2,\delta_1,\delta_2,\mu_1,\mu_2) \\ &\qquad = \frac{1}{2}\sum_{i=1}^{2}\left(\int_{-\infty}^{\infty} E[(\hat{U}_i - U_i)^2|\nu_i]d\mu_i(\nu_i)\right). \end{aligned} \tag{5}$$

Let Γ_t^i, Γ_r^i be the set of admissible (as specified above) transmitter and receiver policies for terminal T_i, $i=1,2$. Naturally, the objective of terminals T_1 and T_2 is to choose their encoding/decoding policies so that the system MSE is minimized, while the jammer aims at designing its policies in order to maximize MSE.

Definition 2.1: A policy tuple $(\gamma_1^*,\gamma_2^*,\delta_1^*,\delta_2^*,\mu_1^*,\mu_2^*)$ is a Nash equilibrium if

$$\begin{aligned} R(\gamma_1^*,\gamma_2^*,\delta_1^*,\delta_2^*,\mu_1,\mu_2) &\leq R(\gamma_1^*,\gamma_2^*,\delta_1^*,\delta_2^*,\mu_1^*,\mu_2^*) \\ &\leq R(\gamma_1,\gamma_2,\delta_1,\delta_2,\mu_1^*,\mu_2^*). \end{aligned} \tag{6}$$

$\forall \gamma_i \in \Gamma_t^i, \delta_i \in \Gamma_r^i, i=1,2, (\mu_1,\mu_2) \in \mathcal{M}$.

We will separately consider a special case of this problem which can be viewed as a *degenerate case*: the case when the same noise variable affects both channel directions; i.e., $Z_1 = Z_2$. This yields that $Y_1 = Y_2$ and the jammer sees two identical signals. As this case changes the course of our analysis, we will treat it separately. We first consider the *non-degenerate* case, where the elements of Σ are not all identical. Note that this includes the case when the noise variables are fully correlated, as long as they have different variances. We next specify regions of interest in terms of the system parameters. Setting $C = c_1 + c_2$, consider the case with

$$k_i \geq C + \frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1+\zeta_2-2\zeta_{1,2}}, \quad i=1,2.$$

Then an admissible policy for the jammer to use for terminal T_i is given by

$$\beta_i(y_1,y_2) = a_{i,i}y_i + a_{i,j}y_j \tag{7}$$

where $i \neq j$,

$$a_{i,i} = -\frac{\zeta_j - \zeta_{1,2}}{\zeta_1+\zeta_2-2\zeta_{1,2}} \qquad a_{i,j} = -\frac{\zeta_i - \zeta_{1,2}}{\zeta_1+\zeta_2-2\zeta_{1,2}}$$

Under this policy, $a_{i,i} + a_{i,j} = -1$ hence the signal received at terminal T_i is purely noise: it contains no trace of either X_i or X_j signals. Therefore the MSE has a maximal unity value regardless of the transmitter/receiver policies

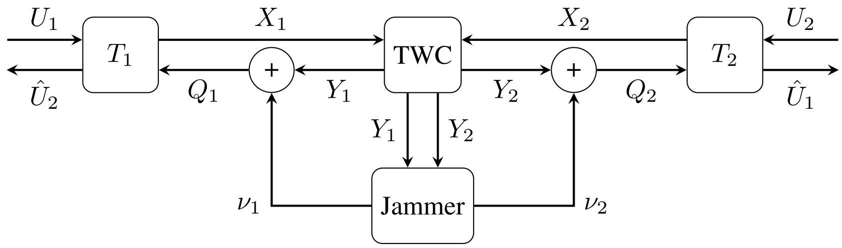

Fig. 1: System diagram of a two-way channel with an intelligent jammer.

which is considered as an uninteresting case (in terms of finding a Nash equilibrium). Thus, for each $i = 1, 2$, we divide our analysis into the regions

$$R_1^i = \left\{ k_i \geq C + \frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1 + \zeta_2 - 2\zeta_{1,2}} \right\}, \tag{8}$$

$$R_2^i = \left\{ k_i < C + \frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1 + \zeta_2 - 2\zeta_{1,2}} \right\}. \tag{9}$$

III. Main Results

A. Equilibrium Policies in the Non-Degenerate Case

We consider a system where $(k_1, k_2) \in R_2^1 \times R_2^2$. If either k_i were to belong to the R_1^i region, then the jammer can just send ν_i according to the policy described in (7) and the transmitter and receiver policies are irrelevant. Define

$$\omega = C(\zeta_1 + \zeta_2 - 2\zeta_{1,2}) + \zeta_1\zeta_2 - \zeta_{1,2}^2. \tag{10}$$

We then have the following theorem.

Theorem 3.1: Fix $(k_1, k_2) \in R_2^1 \times R_2^2$. Then there exist four saddle-point solutions $(\gamma_1^*, \gamma_2^*, \delta_1^*, \delta_2^*, \mu_1^*, \mu_2^*)$ depending on whether the transmitter uses $\sqrt{c_i}$ or $-\sqrt{c_i}$. Assuming both transmitters use the positive amplification, the equilibrium policies for the system are given by

$$\gamma_i^*(u_i) = \sqrt{c_i} u_i \tag{11}$$

$$\delta_i^*(q_i, u_i) = \alpha_i(q_i - (1 + a_{i,i}^* + a_{i,j}^*)\sqrt{c_i} u_i) \tag{12}$$

$$\beta_i^*(y_1, y_2) = a_{i,i}^* y_i + a_{i,j}^* y_j + \eta_i \tag{13}$$

where $\eta_i \sim \mathcal{N}(0, b_i^*)$ is a Gaussian signal with zero mean and variance b_i^* that is independent of the system signals,

$$\alpha_i = \sqrt{c_j} * (1 + a_{i,i}^* + a_{i,j}^*) * ((1 + a_{i,i}^* + a_{i,j}^*)^2 c_j + (1 + a_{i,i}^*)^2 \zeta_i + (a_{i,j}^*)^2 \zeta_j + (1 + a_{i,i}^*)(a_{i,j}^*)\zeta_{1,2} + b_i^*)^{-1}, \tag{14}$$

and the coefficients $a_{i,i}^*, a_{i,j}^*, b_i^*$ are detailed in Table I depending on the relationship between ζ_i and $\zeta_{1,2}$.

Furthermore, the zero-sum Nash equilibria are essentially unique up to the changes of the signs of the encoding/decoding coefficients.

Proof: A sketch of the proof is presented in Section IV. ■

	Jammer Coefficients ($i, j = 1, 2$, $i \neq j$)
$\zeta_i > \zeta_{1,2}$	$a_{i,i}^* = \frac{-k_i\omega^{\frac{1}{2}} + (C + \zeta_{1,2})(k_i(C + \zeta_i - k_i))^{\frac{1}{2}}}{(C + \zeta_i)\omega^{\frac{1}{2}}}$ $a_{i,j}^* = -\left(\frac{k_i(C + \zeta_i - k_i)}{\omega}\right)^{\frac{1}{2}}$ $b_i^* = 0$
$\zeta_i < \zeta_{1,2}$	$a_{i,i}^* = \frac{-k_i\omega^{\frac{1}{2}} - (C + \zeta_{1,2})(k_i(C + \zeta_i - k_i))^{\frac{1}{2}}}{(C + \zeta_i)\omega^{\frac{1}{2}}}$ $a_{i,j}^* = +\left(\frac{k_i(C + \zeta_i - k_i)}{\omega}\right)^{\frac{1}{2}}$ $b_i^* = 0$
$\zeta_i = \zeta_{1,2}$	$a_{i,i}^* = -\left(\frac{k_i}{C + \zeta_i}\right)$ $a_{i,j}^* = 0$ $b_i^* = k_i\left(1 - \frac{k_i}{C + \zeta_i}\right)$

TABLE I: Jammer coefficients for different relationships between the noise variance and covariance.

B. Equilibrium Policies in the Degenerate Case:

We next consider the degenerate case of having identical noise signals in both channel directions ($Z_1 = Z_2$). In this case, Y_1 and Y_2 are the same signal and hence the jammer essentially has access to one unique signal. We address this scenario by considering the problem where the jamming signal is given by $\nu_i = \beta_i(y_i)$, where again β_i is in general a random mapping. However, we must redefine our regions of interest accordingly. The game becomes uninteresting in the sense that the signal is fully cancelled at the receiver when we have $\beta_i(y_i) = -y_i$. The lowest power constraint which admits this policy is $k_i = C + \zeta_i$, therefore we define

$$R_1^i = \{k_i \geq C + \zeta_i\} \tag{15}$$

$$R_2^i = \{k_i < C + \zeta_i\} \tag{16}$$

we obtain the following result.

Theorem 3.2: Fix $(k_1, k_2) \in R_2^1 \times R_2^2$. Then there exist four saddle-point solutions $(\gamma_1^*, \gamma_2^*, \delta_1^*, \delta_2^*, \mu_1^*, \mu_2^*)$ depending on if the transmitters use $\pm\sqrt{c_i}$. If the transmitters use positive amplification, the equilibrium policies are

$$\gamma_i^*(u_i) = \sqrt{c_i}u_i \tag{17}$$

$$\beta_i^*(y_i) = a_i y_i + \eta_i \tag{18}$$

$$\delta_i^*(q_i, u_i) = \frac{\sqrt{c_j}}{c_j + \zeta_i - a_i c_i}\left(q_i - (1+a_i)(\sqrt{c_i}u_i)\right) \tag{19}$$

where

$$a_i = -\frac{k_i}{C + \zeta_i} \tag{20}$$

and

$$\eta_i \sim \mathcal{N}\left(0, \left(1 - \frac{k_i}{C+\zeta_i}\right)k_i\right). \tag{21}$$

The proof follows a similar approach as the one for the non-degenerate case.

This theorem can be considered as the solution to the problem of two separate jammers, each trying to jam the channel individually without sharing any information. However, we can also apply this result to the degenerate case. Given access to only one signal, the separate jammer's optimal policy consists of sending

$$\beta^*(y) = -\left(\frac{k_i}{C+\zeta_i}\right)y + \eta_i.$$

However, since our single jammer has access to the same signal ($Y_1 = Y_2$), any policy of the form

$$\beta_i(y_1, y_2) = a_{i,1}y_1 + a_{i,2}y_2 + \eta_i$$

which satisfies

$$a_{i,1} + a_{i,2} = -\left(\frac{k_i}{C+\zeta_i}\right)$$

will indeed be an equilibrium policy. We can then see there is actually an infinite number of equilibrium jamming policies which all produce the same output signal ν_i.

C. Discussion

Let us consider a special case with the parameters $c_2 = 0, \zeta_1 \to \infty, \zeta_{1,2} = 0$. These values correspond to shutting down the channel in the direction $T_2 \to T_1$, reducing the system to a one-way system going from T_1 to T_2, which falls under the analysis considered in [1]. Note that the notation used in each paper is different, but each variable in [1] has a counterpart in our work. Staying consistent with our notation, we next show that our results agree with the results of [1]. Under our analysis, the boundary between the R_1^2 and R_2^2 regions now becomes

$$\left. C + \frac{\zeta_1\zeta_2 - \zeta_{1,2}^2}{\zeta_1 + \zeta_2 - 2\zeta_{1,2}}\right|_{\zeta_1\to\infty, \zeta_{1,2}=0, c_2=0} = c_1 + \zeta_2$$

which is identical to the definition of the $R2$ region in [1] under the following notational equivalences $k^2 = k_2, c^2 = c_1, \xi_1 = \zeta_2, \sigma = 0$, where the left-hand side (LHS) terms in each identity are from [1]. The results in [1] state that for a k_2 (i.e., k^2 in [1]) value in the R_2^2 (i.e., $R2$ in [1]) region, the equilibrium jamming policy is given by

$$\nu_2 = -\left(\frac{k_2}{c_1 + \zeta_2}\right)y_2 + \eta_2$$

where η_2 is given by (21) using $i = 2$ and $C = c_1$, while our results state that the policy will be of the form

$$\beta_2^*(y_1, y_2) = a_{2,1}y_1 + a_{2,2}y_2$$

where the values for $a_{2,1}, a_{2,2}$ are specified in the first row of Table I. These two results may at first not seem to agree: one is a combination of a negative feedback term and Gaussian noise, while the other is a linear combination of the two received signals. However, setting $\zeta_1 \to \infty$ yields

$$\lim_{\zeta_1\to\infty} a_{2,2} = \lim_{\zeta_1\to\infty}\left(\frac{-k_2\omega^{\frac{1}{2}} + (c_1)(k_2(c_1+\zeta_2-k_2))^{\frac{1}{2}}}{(c_1+\zeta_2)\omega^{\frac{1}{2}}}\right)$$
$$= -\frac{k_2}{c_1+\zeta_2}$$

and transforms signal Y_1 into pure noise. Thus in the jamming policy, the term $a_{2,1}Y_1$ acts as a zero mean Gaussian random variable (which is independent of the other system signals) with variance given by

$$\lim_{\zeta_1\to\infty} E[(a_{2,1}Y_1)^2] = \lim_{\zeta_1\to\infty}\left(\frac{k_2(c_1+\zeta_2-k_2)}{c_1(\zeta_2+\zeta_1)+\zeta_1\zeta_2}\right)(c_1+\zeta_1)$$
$$= k_2\left(1 - \frac{k_2}{c_1+\zeta_2}\right).$$

Therefore, we conclude that the two results indeed coincide.

In general, when faced with a Gaussian system, be it a one-way or two-way, there are certain traits that appear in the jamming policies. There is an R_1-type region where the jammer has too much power and can fully cancel the transmitted signal before it reaches the receiver, making the game trivial. When the jammer cannot fully cancel the signal, its equilibrium policies are either linear, or affine by combining a linear policy with Gaussian noise. For a two way channel, the choice of linear or affine in the jamming policy is determined by the covariance matrix of the noise variables, Σ. If $\zeta_i \neq \zeta_{1,2}$ then a linear policy is used. If $\zeta_i = \zeta_{1,2}$ (in either the degenerate or non-degenerate case) then an affine policy is used. The reasoning for this involves an in depth examination of the best response function used in the proof of Theorem 3.1. In essence, if the function admits a fixed point the policy is linear, and if it does not admit a fixed point an affine policy is used.

We also note the generality given to the jammer's information and policy structure. We allow the jammer full access to the channel and full knowledge of the system dynamics. Although we require a power constraint on the jammer we do not impose any further assumptions on the policy structure itself. Simple linear/affine jamming policies arise naturally as the optimal choice. This allows for a more rich analysis of possible jamming policies, but makes transitioning the problem from a one-shot to a finite horizon setup difficult due to the complexity.

This is in contrast to what may be called "denial of service" jamming [28] where the jammer either does nothing or shuts down the channel completely. Such models can also impose further restrictions on the jammer's information [29] by only allowing the jammer to see if a signal is transmitted, not the signal value itself. Such models allow simplicity to move from one shot to finite horizon problems. Yet the main work of this paper, namely determining the structure of equilibrium policies in the simplest network system, does not play a roll in these systems.

D. Jamming Power Allocation

Theorems 3.1 and 3.2 fully describe the problem when the jamming power levels k_1 and k_2 are fixed. While maintaining constraint (4), we now allow

$$k_1 + k_2 \leq K \tag{22}$$

and consider the power allocation of (k_1, k_2) that maximizes the MSE. The jammer can then fix power levels k_1 and k_2 at their optimal levels and then apply its policies using the previous theorems. We derive analytical results for the best jamming power allocation for the degenerate case and uncorrelated noise, as these setups produce symmetry in the jamming policies which makes the solution easier to derive.

Theorem 3.3: In the degenerate case ($Z_1 = Z_2$) the jammer's optimal power allocation is as follows. Note we will work with $\zeta_1 = \zeta_2 = \zeta$ since the values are identical. For $i, j = 1, 2$ with $i \neq j$, let

$$K_i^* = \frac{(C+\zeta)\left((c_j)^{\frac{1}{2}}(c_i+\zeta) - (c_i)^{\frac{1}{2}}(c_j+\zeta)\right)}{(c_j)^{\frac{3}{2}}} \tag{23}$$

$$\hat{k}_i = \frac{(c_j)^{\frac{3}{2}}(K_i^* + K)}{(c_i)^{\frac{3}{2}} + (c_j)^{\frac{3}{2}}} \tag{24}$$

If $K < |\min_{i=1,2} K_i^*|$, then the allocation is given by

$$(k_1, k_2) = \begin{cases} (K, 0) & \text{if } c_1(c_2+\zeta)^2 \leq c_2(c_1+\zeta)^2 \\ (0, K) & \text{if } c_1(c_2+\zeta)^2 > c_2(c_1+\zeta)^2 \end{cases}$$

If $K \geq |\min_{i=1,2} K_i^*|$, the jammer allocates according to

$$(k_1, k_2) = \begin{cases} \left(\min(\hat{k}_1, C+\zeta), K - k_1\right) \\ \text{if } c_1(c_2+\zeta)^2 \leq c_2(c_1+\zeta)^2 \\ \left(K - k_2, \min(\hat{k}_2, C+\zeta_2)\right) \\ \text{if } c_1(c_2+\zeta)^2 > c_2(c_1+\zeta)^2 \end{cases}$$

Two illustrating examples for Theorem 3.3 are given in Figures 2 and 3. Here we fix two different degenerate systems with different budgets K and vary k_1 from 0 to K while $k_2 = K - k_1$. We then plot the equilibrium MSE at the various allocations. In Figure 2, the overall budget K is small enough that the jammer employs an "all or nothing" strategy and maximizes by giving all power to channel one. In Figure 3, the budget K is larger so that the optimal strategy involves splitting the power between the two channels.

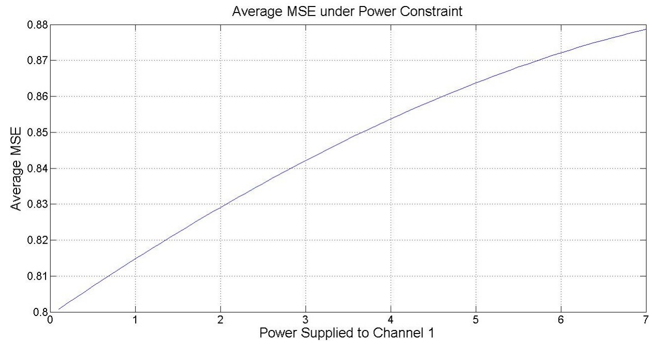

Fig. 2: Equilibrium MSE vs power supplied to channel 1 for a fixed budget with $K < |\min_{i=1,2} K_i^*|$.

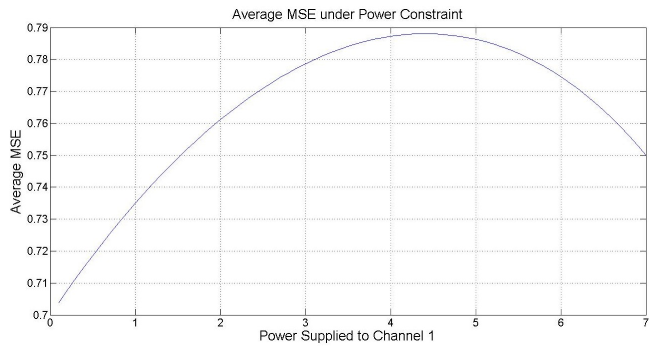

Fig. 3: Equilibrium MSE vs power supplied to channel 1 for a fixed budget with $K \geq |\min_{i=1,2} K_i^*|$.

Theorem 3.4: For the uncorrelated noise case ($\zeta_{1,2} = 0$), the jammer allocates as follows.
If $K < 2\left(C + \frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2}\right)$, the optimal allocation is the solution to the equation

$$\begin{aligned} &c_2(C+\zeta_1)^2 \frac{(-2\zeta_1\omega^{\frac{1}{2}})k_1 + (\omega - \zeta_1^2)x_1 + \zeta_1(\omega)^{\frac{1}{2}}(C+\zeta_1)}{x_1(\lambda_{1,4}k_1 + 2\lambda_{1,5}x_1 + \lambda_{1,6})^2} \\ &= c_1(C+\zeta_2)^2 \frac{(-2\zeta_2\omega^{\frac{1}{2}})k_2 + (\omega - \zeta_2^2)x_2 + \zeta_2(\omega)^{\frac{1}{2}}(C+\zeta_2)}{x_2(\lambda_{2,4}k_2 + 2\lambda_{2,5}x_2 + \lambda_6)^2} \end{aligned} \tag{25}$$

$$\begin{aligned} x_i &= (k_i(C+\zeta_i - k_i)^{\frac{1}{2}} \\ \lambda_{i,4} &= \zeta_i C^2 + \zeta_j(C+\zeta_i)^2 - \zeta_i\omega + c_j(\zeta_i^2 - \omega) \\ \lambda_{i,5} &= \zeta_i c_i(\omega)^{\frac{1}{2}} \\ \lambda_{i,6} &= (\zeta_i + c_j)(C+\zeta_i)\omega. \end{aligned}$$

If $K \geq 2\left(C + \frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2}\right)$, the optimal allocation is $k_i = C + \frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2}$, $i = 1, 2$ so that both channels are in the R_1^i region and there is no signal reaching either terminal.

IV. A Proof Skecth for Theorem 3.1

Due to space constraints, we outline the main approach to proving the theorem. We fix the equilibrium jamming policy and prove the RHS of inequality (6). When the jamming policy is fixed, each transmitter is equivalent to a one way additive channel with Gaussian noise, a well studied problem with a known optimal transmission and receiving policy.

We then fix the equilibrium transmission and receiving policy and show that the provided jamming policy satisfies the LHS inequality. Expanding the MSE equation in (5) yields a polynomial in ν_i. When $\zeta_i \neq \zeta_{1,2}$, the Cauchy Schwartz inequality then shows that the jammer's best policy is a linear policy which achieves the jamming power constraint in (4). However, the transmission and receiving policies are optimal with respect to a linear jamming policy, and such polices have a best response jamming policy which is also linear. This imposes a function on the space of linear jamming policies, and the fixed point of this functional is the equilibrium jamming policy.

When $\zeta_i = \zeta_{1,2}$ the best response function no longer has a fixed point. This means the equilibrium policy cannot be linear. However, it now becomes possible to fully cancel the degree one term in the MSE expansion, which reduces the problem to finding the equilibrium policy which cancels the term and achieves the power constraint. It can be shown such a policy is affine.

V. Concluding Remarks

The results established in this paper provide a full set of solutions for the communication system presented in Fig. 1 with independent Gaussian sources and arbitrarily correlated Gaussian noise signals. The results in many ways provide a natural extension of [1] to a two-way system, maintaining the existence of a Nash equilibrium and the optimality of linear/affine policies. In the special degenerate case, there is actually an infinite set of equilibrium jamming policies due to signals Y_1 and Y_2 being identical. There are a number of intricacies for the two-way system that make the analysis more complicated than the one-way case. The correlation between the noise signals now plays a role determining the jamming policy. Furthermore, a linear policy now has two parameters, not just one, which complicates solving for a fixed point policy. We also analyse the optimal jamming power allocation which adds a new dimension to the problem.

Extensions of this work include examining correlation between the sources, which would change how terminal T_i decides to decode with side information X_i, as well as considering non-Gaussian noise and source variables. The problem can also be generalized from a one shot to a finite horizon setup, although some limitations may be required on the jammer's policy structure or information to simplify the analysis.

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