Consensus Using a Network of Finite Memory Pólya Urns

Somya Singh, Fady Alajaji, *Senior Member, IEEE*, and Bahman Gharesifard, *Member, IEEE*

***Abstract*—We introduce a *finite memory* interacting Pólya urn process over a connected network which models consensus dynamics for interacting individuals. More specifically, each urn (individual) in the network is initially equipped with some red and black balls, with the fraction corresponding to the individual's opinion (or belief) on a certain color. At each time instant and for each urn, a ball is drawn from a "super-urn", which consists of all balls present in that urn and its neighboring urns; then reinforcing balls of the color just drawn are added to the urn for a limited period of M future time instants, where M denotes the memory parameter. Additionally, and important for our objective, as of time $t = M + 1$, we remove the balls which were present in the urns initially. By examining the structure of the resulting underlying reducible Markov process, we show that individuals eventually reach consensus in the sense that they all achieve identical probabilities of drawing a red ball. Moreover, when the network has homogeneous reinforcement parameters, we construct a class of linear dynamical systems with time delay whose trajectory gives the probability of drawing a red ball for each node i at a time instant t. We examine the asymptotic behavior of such a network and exactly determine its consensus value. Our simulation confirms our theoretical findings by demonstrating the asymptotic behavior of draw variables of the network in some case studies.**

***Index Terms*—Opinion and consensus dynamics, multi-agent systems, stochastic reinforcement processes with finite memory, Pólya urn networks, absorbing Markov chains.**

I. Introduction

Stochastic reinforcement processes, which include Pólya urn processes, have been widely used in the modeling and analysis of social networks [1]–[9], opinion dynamics [10], [11] and epidemic spread [12], [13]. In particular, Pólya urn models naturally exhibit "consensus like" properties due to their underlying reinforcement mechanism and have been studied in the literature.

For example in [7], it is shown that a complete network of interacting two-color Pólya urns with same reinforcement and interaction parameters synchronize asymptotically. Unlike [7], there are some consensus models in which instead of considering a network of Pólya urns, the opinions (or beliefs) of agents in a network change in time using Pólya reinforcement. One such example is [6] in which draw variables from a single multi-color Pólya urn are used to update the opinion of an agent (or individual) in a social network. In this network, consensus is achieved due to the exchangeability property of the draw variables of the Pólya process. Moreover, Pólya urn reinforcement is analogous to preferential attachment models in the sense that the probability of adding balls of a certain color to a Pólya urn is larger if there are more balls of the same color present in the urn (for preferential attachment, the probability of adding an edge to a vertex is larger, if its degree is higher, see [14]–[16]). The usage of preferential attachment models to study consensus phenomena in social networks (e.g., [17], [18]) makes Pólya processes a good choice to model consensus problems. Consensus typically refers to agreement among a population. More precisely, and presented here for the simplest possible setting, if V represents a group of agents who can access beliefs of a limited number of agents in V, prescribed by a graph $\mathcal{G}$ with vertex set V of size N, and $x_i(t) \in \mathbb{R}$ represents the belief of agent $i \in V$ at time t, then consensus is achieved when

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$$

for all $i, j \in V$.

One of the most commonly used consensus algorithms is where each agent's opinion or belief is set to be a weighted average of the opinion of its neighbors. In other words, the opinion of agents is governed by the linear dynamical system $x(k+1) = Wx(k)$, where $x(k) = (x_1(k), x_2(k), \ldots, x_N(k))^T$ is the column vector containing the opinion of all N agents at time k and W is a fixed weighted adjacency matrix for the underlying graph $\mathcal{G}$, taken to be constant in time for the time being (see [19] for an example). In more complex problems the graph $\mathcal{G}$ itself can depend on the opinion of the agents; for instance, in Hegselmann-Krause dynamics [20], [21] agents update their state using the opinion of agents with similar view points. An elementary calculation demonstrates that, in this simple setting, network connectivity can fully characterize

Manuscript received March 21, 2022; revised May 12, 2022; accepted May 13, 2022. Date of publication May 24, 2022; date of current version June 1, 2022. This work was supported in part by NSERC of Canada. Recommended by Senior Editor V. Ugrinovskii. *(Corresponding author: Somya Singh.)*

Somya Singh and Fady Alajaji are with the Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N8, Canada (e-mail: 17ss262@queensu.ca; fa@queensu.ca).

Bahman Gharesifard is with the Department of Electrical and Computer Engineering, University of California at Los Angeles, Los Angeles, CA 90095 USA (e-mail: gharesifard@ucla.edu).

Digital Object Identifier 10.1109/LCSYS.2022.3177428

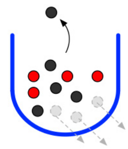

Fig. 1. Illustration of finite memory Pólya urn draws.

whether consensus is achieved, and the speed of convergence is directly related to the spectral properties of the corresponding adjacency matrix. Consensus dynamics is a well-studied subject, with a large volume of literature devoted to it, e.g., see [5], [22], [23] and references within. *Our objective herein is not to provide sharper conditions for consensus, but to develop and analyze alternative dynamics for this purpose via a novel Pólya-based model with rich properties, in particular, heterogeneity considerations.*

In our model, we equip each individual in the network with a *finite memory two-color* Pólya urn (see [24]), where the two colors represent two competing opinions or brands, and devise a draw protocol which only uses local information and provably achieves consensus, where $x_i(t)$ represents the probability of drawing a red ball at time t. The *finite memory* property adds temporal lag in the consensus problem which we later identify with dynamical systems with time-delay. The effect of time delay in consensus problems has been widely studied, e.g., see [25]–[27].

Let us now describe our Pólya-urn based process. As mentioned, given the network $\mathcal{G}$, we equip each agent with an urn, initially consisting of balls of two colors, red and black. The "belief" of each urn (or individual) at a given time instant is the probability that a red ball is chosen in the drawing process. However, the draw process utilizes the spatial interconnections in $\mathcal{G}$; in particular, the urns (or individuals) interact through what we call "super-urns". A super-urn of urn i consists of all the balls which are present in urn i and its neighboring urns; see [28], [29] where this notion was first introduced. We additionally assume that the urns have finite memory, a concept originally developed in [24] in the context of using a single Pólya urn process to model burst noise propagation in communication channels. In a finite memory Pólya urn, along with addition of new reinforcing balls to the urn at each time step $t \geq M+1$, one removes the balls which were added to the urn at time $t - M$, where M is the urn's memory parameter, see Fig. 1. Unlike the classical Pólya process, the drawing process for a Pólya urn with memory M forms a Markov chain with 2^M states. Other Markovian versions of the Pólya process are studied in [30], [31].

Finite memory Pólya urn processes on a network were already introduced in [32]. However, in this model, the random vector of drawing variables forms an *irreducible* Markov chain and hence there are no absorbing states for the Markov process. In this letter, we modify this process in order to make the Markov chain *reducible*. To wit, for a memory M network, at time $t = M + 1$, we remove the balls which were present in the urns at time $t = 0$, i.e., we remove the initial conditions. The significance of removing initial conditions is that the individuals forget about their inherent beliefs as of time $t \geq M + 1$. This modified version of the finite memory Pólya network has two absorbing states which enables all the individuals to reach a consensus value, as long as the underlying network is connected.

Organization: In Section II, we describe our network of finite memory Pólya urns and show that for a memory M, the vector of draw variables forms a time invariant Mth order Markov chain. In Section III, we study the structure of this Markov process and show that our network achieves consensus. In Section IV, we present an alternate method to show consensus for a homogeneous connected network. In this method, we obtain a class of linear dynamical systems with time delay which gives the probability of drawing a red ball from the super-urns at any time t. We then obtain the consensus value of this connected homogeneous network by studying the asymptotic properties of these delayed linear dynamical systems. Section V contains simulation studies. We conclude this letter in Section VI.

II. The Model

A finite memory Pólya urn is a modified version of classical Pólya urn where at each time all reinforcing balls added to the urn M time steps before are removed [24]. We consider an undirected connected network $\mathcal{G}_N$ consisting of N nodes, each equipped with a finite memory Pólya urn (where each urn represents an individual in the network) with memory M. By connected here we mean that there exists a path between urns i and j for all $i, j \in \{1, 2, \ldots, N\}$. At time $t = 0$, urn i contains R_i red balls and B_i black balls, $i = 1, \ldots, N$. We let $T_i = R_i + B_i$ be the total number of balls in the ith urn at time $t = 0$, and assume that no urn is empty at time $t = 0$, i.e., $T_i > 0$ for all i. We let $U_{i,t}$ denote the ratio of red balls in urn i at time t, with its initial value (at time $t = 0$) given by $U_{i,0} = R_i/T_i$. At each time $t \geq 1$, we draw a ball from the "super-urn" of every urn. If a red ball is drawn from the super-urn of an urn i, we add $\Delta_{r,i}$ red balls to the urn i; similarly, we add $\Delta_{b,i}$ black balls to urn i if a black ball is drawn from the super-urn of urn i. $\Delta_{r,i}$ and $\Delta_{b,i}$ are called the reinforcement parameters and the individuals in the network (represented by urns) update their belief according to these parameters at every time step.

Since, the urns in our network have a finite memory M, after adding reinforcing balls at time $t \geq M + 1$, we remove the reinforcing balls which were added to the urns at time $t - M$. Furthermore, at time $t = M + 1$, we remove the balls which were present in the urn at time $t = 0$, i.e., after M draws, we permanently remove the initial T_i balls from urn i. Let $Z_{i,t}$ be the indicator function of the ball drawn from the super-urn of urn i at time $t \geq 1$, i.e.,

$$Z_{i,t} = \begin{cases} 1 & \text{if a red ball is drawn from the super-urn} \\ & \text{of urn } i \text{ at time } t, \\ 0 & \text{if a black ball is drawn from the super-urn} \\ & \text{of urn } i \text{ at time } t. \end{cases} \tag{1}$$

Since the drawing mechanism is applied simultaneously to all super-urns, the draw variables $Z_{i,t}$ and $Z_{i',t}$ are conditionally independent given all past draws in the network for any two urns $i \neq i'$, i.e., at any time t,

$$P\Big(Z_{1,t}, \dots, Z_{N,t} | \{Z_{1,k}\}_{k=1}^{t-1}, \dots, \{Z_{N,k}\}_{k=1}^{t-1}\Big) = \prod_{i=1}^{N} P\Big(Z_{i,t} | \{Z_{1,k}\}_{k=1}^{t-1}, \dots, \{Z_{N,k}\}_{k=1}^{t-1}\Big). \quad (2)$$

The ratio of red balls in urn i for time $t \leq M$ is given by

$$U_{i,t} = \frac{U_{i,0} + \sum_{n=1}^{t-1} \Delta_{r,i} Z_{i,n}}{1 + \sum_{n=1}^{t-1} \Delta_{r,i} Z_{i,n} + \sum_{n=1}^{t-1} (1 - \Delta_{b,i}) Z_{i,n}}, \quad (3)$$

and for $t \geq M + 1$ is given by

$$U_{i,t} = \frac{\sum_{n=t-M}^{t-1} \Delta_{r,i} Z_{i,n}}{\sum_{n=t-M}^{t-1} \Delta_{r,i} Z_{i,n} + \sum_{n=t-M}^{t-1} \Delta_{b,i}(1 - Z_{i,n})}. \quad (4)$$

Defining $Z_t = (Z_{1,t}, Z_{2,t}, \dots, Z_{N,t})$ as the network wide draw tuple, we arrive at the following result.

Lemma 1: The stochastic process $\{Z_t\}_{t=1}^{\infty}$ is a time invariant Mth order Markov chain.

Proof: Let $a_t = (a_{1,t}, \dots, a_{N,t}) \in \{0, 1\}^N$. Using (4) and by virtue of the conditional independence stated in (2), for $t \geq M + 1$ we have that

$$\begin{aligned} &P[Z_{t+1} = a_{t+1} | Z_t = a_t, \dots, Z_1 = a_1] \\ &= \prod_{i=1}^{N} \left(\frac{a_{i,t+1}\Big(\sum_{j \in \mathcal{N}_i'} \sum_{n=t-M+1}^{t} \Delta_{r,j} a_{j,n}\Big)}{\sum_{j \in \mathcal{N}_i'} \sum_{n=t-M+1}^{t} (\Delta_{r,j} a_{j,n} + \Delta_{b,j}(1 - a_{j,n}))} \right. \\ &\quad \left. + \frac{(1 - a_{i,t+1})\Big(\sum_{j \in \mathcal{N}_i'} \sum_{n=t-M+1}^{t} \Delta_{b,j}(1 - a_{j,n})\Big)}{\sum_{j \in \mathcal{N}_i'} \sum_{n=t-M+1}^{t} (\Delta_{r,j} a_{j,n} + \Delta_{b,j}(1 - a_{j,n}))} \right), \end{aligned} \quad (5)$$

where $\mathcal{N}_i'$ is the set of all neighbors of urn i and the urn i itself. As a result, we have that for all $t \geq M + 1$,

$$\begin{aligned} &P[Z_{t+1} = a_{t+1} | Z_t = a_t, \dots, Z_1 = a_1] \\ &= P[Z_{t+1} = a_{t+1} | Z_t = a_t, \dots, Z_{t-M+1} = a_{t-M+1}] \\ &= P[Z_{M+1} = a_{t+1} | Z_M = a_t, \dots, Z_1 = a_{t-M+1}]. \end{aligned} \quad (6)$$

Hence the process $\{Z_t\}_{t=1}^{\infty}$ is a time invariant Mth order Markov chain. ■

III. Consensus in General Networks

We next study the structure of this Markov process with memory M. Setting $\mathbf{W}_t := (Z_t, Z_{t+1}, \dots, Z_{t+M-1})$, Lemma 1 states that $\{\mathbf{W}_t\}_{t=1}^{\infty}$ is a Markov chain of order one. Note that for a network of size N and memory M, the Markov chain $\{\mathbf{W}_t\}_{t=1}^{\infty}$ has 2^{MN} states. Before stating the next theorem, we define the following: Recall that $\mathcal{N}_i'$ is the set of all neighbors of urn i and the urn i itself. We define

$$\mathcal{N}_i^{(\ell)} := \bigcup_{k \in \mathcal{N}_i^{(\ell-1)}} \mathcal{N}_k^{(\ell-1)},$$

where $\ell \geq 1$, and $\mathcal{N}_k^{(0)} := \mathcal{N}_k'$. Note that in a connected network of urns $\mathcal{G}_N$, for every urn $i \in \{1, 2, \dots, N\}$, there exists $n \geq 1$ such that

$$\mathcal{N}_i' \cup \mathcal{N}_i^{(1)} \cup \mathcal{N}_i^{(2)} \cup \dots \cup \mathcal{N}_i^{(n)} = \mathcal{G}. \quad (7)$$

We now use (7) to classify the states of the Markov chain $\{\mathbf{W}_t\}_{t=1}^{\infty}$ as either absorbing or transient.

Theorem 1: The Markov chain $\{\mathbf{W}_t\}_{t=1}^{\infty}$ has two absorbing states which are state $\mathbf{0}$ with all entries zero and state $\mathbf{1}$ with all entries one. The remaining states are transient, i.e., $\{\mathbf{W}_t\}_{t=1}^{\infty}$ is an absorbing Markov chain.

Proof: We denote a state of the Markov chain $\{\mathbf{W}_t\}_{t=1}^{\infty}$ by the following length-NM tuple

$$a := ((a_{11}, a_{21}, \dots, a_{N1}), \dots, (a_{1M}, a_{2M}, \dots, a_{NM}))$$

where $a \in \{0, 1\}^{NM}$. Let $\mathbf{0}$ (resp., $\mathbf{1}$) be the state for which $a_{ij} = 0$ (resp., $a_{ij} = 1$) for all $i \in \{1, \dots, N\}$ and $j \in \{1, 2, \dots, M\}$. Using (5), we obtain that

$$P(\mathbf{W}_{t+1} = \mathbf{0} | \mathbf{W}_t = \mathbf{0}) = \prod_{i=1}^{N} P(Z_{i,t+1} = 0 | \mathbf{W}_t = \mathbf{0}) = 1.$$

Similarly, $P(\mathbf{W}_{t+1} = \mathbf{1} | \mathbf{W}_t = \mathbf{1}) = 1$. Hence $\mathbf{0}$ and $\mathbf{1}$ are both absorbing states of the Markov chain $\{\mathbf{W}_t\}_{t=1}^{\infty}$. We now show that the remaining states of $\mathbf{W}_t$ are transient. It is enough to show that for any state $b \notin \{\mathbf{0}, \mathbf{1}\}$, there exists a time t_b such that

$$P(\mathbf{W}_{t_b} = \mathbf{0} | \mathbf{W}_1 = b) > 0. \quad (8)$$

To show this, we construct a finite length path from state b to state $\mathbf{0}$ which occurs with positive probability.

- Suppose the Markov chain is in state $\mathbf{W}_t = b$ at time t, with

  $$b := ((b_{11}, b_{21}, \dots, b_{N1}), \dots, (b_{1M}, b_{2M}, \dots, b_{NM})).$$

  Note that there exists a component $b_{ij} = 0$ for some $i \in \{1, 2, \dots, N\}$ and $j \in \{1, 2, \dots, M\}$.
- Let

  $$b' := \big((b'_{11}, b'_{21}, \dots, b'_{N1}), \dots, (b'_{1M}, b'_{2M}, \dots, b'_{NM})\big)$$

  be a state of the Markov chain $\{\mathbf{W}_t\}_{t=0}^{\infty}$ with $b'_{kj} = b_{kj}$ for all $k \in \{1, 2, \dots, N\}$ and $j \in \{1, 2, \dots, M-1\}$. Also, $b'_{kM} = 0$ for all $k \in \mathcal{N}_i'$ and $b'_{kM} = b_{kM}$ for all $k \notin \mathcal{N}_i'$. We will now show that we can go from state b to b' in a single time step, i.e.,

  $$P(\mathbf{W}_{t+1} = b' | \mathbf{W}_t = b) > 0.$$

  Note that

  $$\begin{aligned} &P(\mathbf{W}_{t+1} = b' | \mathbf{W}_t = b) \\ &= P\big(Z_{t+M} = (b'_{1M}, b'_{2M}, \dots, b'_{NM}) | \mathbf{W}_t = b\big) \\ &= \prod_{k=1}^{N} P(Z_{k,t+M} = b'_{k,M} | \mathbf{W}_t = b). \end{aligned} \quad (9)$$

  At time $t+M-1$, after making the draws and adding and removing corresponding balls, the super urn of $k \in \mathcal{N}_i'$ contains a black ball because $Z_{i,t+j-1} = b_{i,j} = 0$ for some $j \in \{1, 2, \dots, M\}$. At time $t+M$, it is possible to draw a black ball from the super urn of k with probability

  $$P(Z_{k,t+M} = b'_{KM} = 0 | \mathbf{W}_t = b) > 0.$$

For the super urn of $k \notin \mathcal{N}_i'$, at time $t+M$, it is possible to draw a ball which was added to the urn k at time $t+M-1$, i.e.,

$$P(Z_{k,t+M} = b'_{KM} = b_{KM} | \mathbf{W}_t = b) > 0.$$

Hence each term of the product in (9) is strictly positive.

- If $b' = \mathbf{0}$, then we are done. Otherwise, at the next time step, i.e., at time $t+M+1$, we draw a black ball from super urns of $j \in \mathcal{N}_i^{(1)}$ (it is possible to draw a black ball from such a super urn because $Z_{k,t+M} = 0$ for all $k \in \mathcal{N}_i'$). For super urns of $j \notin \mathcal{N}_i^{(1)}$, it is possible to draw a ball which was added to the urn j at time $t+M$.
- Repeating the above procedure, by the virtue of (7), we will eventually hit (with positive probability) the state $\mathbf{0}$ at some time. ■

Using this structure of the Markov chain $\{\mathbf{W}_t\}_{t=1}^{\infty}$, we now obtain the consensus result for our connected network $\mathcal{G}_N$ of finite memory Pólya urns.

Theorem 2: For a general connected network $\mathcal{G}_N$,

$$\lim_{t\to\infty} P(Z_{i,t} = 1) = \lim_{t\to\infty} P(Z_{j,t} = 1) \quad \text{for all} \quad i, j \in \{1, 2, \ldots, N\}.$$

Proof: We denote the limiting distributions of the Markov chain $\{\mathbf{W}_t\}_{t=1}^{\infty}$ by Π, where the entries of Π are denoted by $\pi_{k_1 k_2 \cdots k_M}$ with $k_j = (k_{1j}, k_{2j}, \ldots, k_{Nj}) \in \{0,1\}^N$ for $j \in \{1, 2, \ldots, M\}$. The subscript $k_1 k_2 \cdots k_M$ denotes the state of the Markov chain. By Theorem 1, the limiting distribution of $\{\mathbf{W}_t\}_{t=1}^{\infty}$ is given by $\Pi = (1-\pi, 0, \ldots, 0, \pi)$, $0 \le \pi \le 1$, where first and the last states of the Markov chain are the absorbing states (corresponding to states $\mathbf{0}$ and $\mathbf{1}$, respectively). Since $\{\mathbf{W}_t\}_{t=1}^{\infty}$ is a reducible Markov chain, there is no unique limiting distribution. Also, the limiting distribution can vary depending on $\mathbf{W}_1$, i.e., the initial state of the Markov chain. The marginal limiting distribution for an urn i is given by:

$$\begin{aligned}\lim_{t\to\infty} P(Z_{i,t} = 1) &= \sum_{\substack{k_{ij}=1 \\ j\in\{1,2,\ldots,M\}}} \pi_{k_1 k_2 \cdots k_M} \\ &= \pi + 0 = \pi. \end{aligned} \tag{10}$$

Hence,

$$\lim_{t\to\infty} P(Z_{i,t} = 1) = \pi = \lim_{t\to\infty} P(Z_{j,t} = 1)$$

for $i, j \in \{1, \ldots, N\}$, proving the claim. ■

In the proof of Theorem 2, we observe that the consensus value is given by π which is the asymptotic belief of each individual in the network $\mathcal{G}_N$; however it is hard to analytically solve for π in terms of the initial state for a general network.

IV. Consensus in Homogeneous Networks

In this section, we present an alternate approach to show consensus for homogeneous connected networks by constructing a class of linear dynamical systems with time delay. We further derive the exact consensus value obtained in such networks by examining the asymptotic behavior of these dynamical systems. By homogeneous here, we mean that all reinforcement parameters are identical, i.e., $\Delta_{r,i} = \Delta_{b,i} = \Delta$ for all $i \in \{1, 2, \ldots, N\}$. However we allow the initial composition (i.e., $U_{i,0}$) to be different among the urns (i.e., Even though the individuals update their beliefs with the same reinforcement parameters, their initial beliefs can be different). Rewriting (5) with the homogeneous conditions, we obtain that for $t \ge M+1$,

$$\begin{aligned} &P[Z_{i,t} = 1 | Z_{t-1}, Z_{t-2}, \ldots, Z_{t-M+1}] \\ &\quad = \frac{\sum_{j\in\mathcal{N}_i'} \sum_{n=t-M}^{t-1} Z_{j,n}}{(1+d_i)M}. \end{aligned} \tag{11}$$

where d_i is the degree of urn i in the network $\mathcal{G}_N$. Now, taking expectation of both sides with respect to the random variables $Z_{t-1}, Z_{t-2}, \cdots, Z_{t-M+1}$ in (11), we obtain

$$P(Z_{i,t} = 1) = \frac{\sum_{j\in\mathcal{N}_i'} \sum_{n=t-M}^{t-1} P(Z_{j,n} = 1)}{(1+d_i)M}. \tag{12}$$

We further define $P(Z_{i,t} = 1) := P_i(t)$, to write (12) as a discrete time linear dynamical system given by

$$P_i(t) = \frac{\sum_{j\in\mathcal{N}_i'} \sum_{n=t-M}^{t-1} P_j(n)}{(1+d_i)M}. \tag{13}$$

In (13), $P_i(t)$ depends on $P_i(t-1), \ldots, P_i(t-M)$ in a linear fashion, and therefore, in the homogeneous case, we obtain a linear dynamical system with time delay. We next write the dynamical system (13) in matrix form. Define

$$P(t) := (P_1(t), \ldots, P_N(t))^T \quad \text{and}$$
$$X_{t,M} := (P(t), \ldots, P(t-M+1))^T.$$

Let

$$B_{N,M} = \frac{1}{M}(I_N + D)^{-1}(I_N + A),$$

where I_N is the identity matrix of size N, D is a diagonal matrix for which ith diagonal entry is d_i, and A is the adjacency matrix of the connected network $\mathcal{G}_N$. We hence have the following linear dynamical system:

$$X_{t,M} = J_{N,M} X_{t-1,M} \tag{14}$$

where

$$J_{N,M} = \left[\begin{array}{c|c|c|c|c} B_{N,M} & B_{N,M} & B_{N,M} & \cdots & B_{N,M} \\ \hline \mathbf{I}_N & \mathbf{0}_N & \mathbf{0}_N & \cdots & \mathbf{0}_N \\ \hline \mathbf{0}_N & \mathbf{I}_N & \mathbf{0}_N & \cdots & \mathbf{0}_N \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \mathbf{0}_N & \mathbf{0}_N & \cdots & \mathbf{I}_N & \mathbf{0}_N \end{array}\right]$$

is a stochastic block matrix of size $NM \times NM$. It has M^2 blocks each of which is a square matrix of size N. In the matrix $J_{N,M}$, $\mathbf{I}_N$ is identity matrix of size N and $\mathbf{0}_N$ is a square matrix of size N with all entries 0.

We next establish the asymptotic behavior of the linear dynamical system with time delay in (14).

Theorem 3: If $\mathcal{G}_N$ is a connected homogeneous network with memory M, then we have that

$$\lim_{t\to\infty} X_{t,M}(i) = \pi \text{ for } i \in \{1, 2, \ldots, NM\}, \tag{15}$$

where

$$\pi = \sum_{j=1}^{M} \frac{(M-j+1)}{M} \sum_{i=1}^{N} v_{1,i} X_{M,M}(i),$$

$X_{t,M}(i)$ is the ith entry of the column vector $X_{t,M}$, and

$$\bar{v} = ((v_{1,1}, \dots, v_{1,N}), \dots, (v_{M,1}, \dots, v_{M,N}))$$

is the l_1-normalized left eigenvector of the matrix $J_{N,M}$ in (14) corresponding to eigenvalue $\lambda = 1$. Moreover, $(v_{1,1}, \dots, v_{1,N})$ is a left eigenvector of $B_{N,1}$ also corresponding to eigenvalue $\lambda = 1$.

Proof: Since $\mathcal{G}_N$ is connected, $B_{N,M}$ is a primitive matrix. Since all the entries of $J_{N,M}$ are non-negative and all the blocks in the first row of $J_{N,M}$ are primitive, for some positive integer $k > 0$, all the blocks of $J_{N,M}^k$ will be sum of positive powers of $B_{N,M}$. Since, $B_{N,M}$ is a primitive matrix, there exists $h > k$ such that $J_{N,M}^h$ has all positive entries. Hence $J_{N,M}$ is a primitive matrix. Since $J_{N,M}$ is a stochastic and a primitive matrix, it is a transition probability matrix for an irreducible Markov chain and the normalized left eigenvector of the matrix $J_{N,M}$, which we denote by

$$\bar{v} = ((v_{1,1}, \dots, v_{1,N}), \dots, (v_{M,1}, \dots, v_{M,N})),$$

is the unique stationary distribution for this Markov chain. We can write (14) as

$$X_{t,M} = J_{N,M}^{t-M} X_{M,M}$$

Taking $t \to \infty$ in the above equation, we obtain

$$\lim_{t\to\infty} X_{t,M} = V_{NM} X_{M,M} \tag{16}$$

where the limit in $X_{t,M}$ is taken entry wise and V_{NM} is a square matrix of size NM with each row given by $\bar{v}$.

The symmetry in the structure of the block matrix $J_{N,M}$ makes it possible to find a useful relationship between the entries of the left eigenvector of $J_{N,M}$ corresponding to eigenvalue $\lambda = 1$ in terms of the memory parameter M and the matrix B_{N1}. The equation $\bar{v} J_{N,M} = \bar{v}$ yields the following recursive relations:

$$v_{j,i} = \frac{(M-j+1)}{M} v_{1,i}, \; j = 1, \dots, M, i = 1, \dots, N. \tag{17}$$

We obtain (15) by substituting (17) in (16). ■

V. Simulation Results

In this section, we present simulations to illustrate the consensus behavior of our network of urns (For a complete list of parameters used for generating all figures, refer to the link: https://www.dropbox.com/sh/ojvmeo79wbbdv3g/AAA5onqqo0TrCU7I0iteuzRGa?dl=0). In Fig. 2, we define the empirical sum of urn i at time t as

$$I_t(i) = \frac{1}{t} \sum_{n=1}^{t} Z_{i,n} \tag{18}$$

For each time instant t, the empirical sum $I_t(i)$ for node i (i.e., urn or agent/individual i) is computed 100 times and the arithmetic mean value is plotted against time.

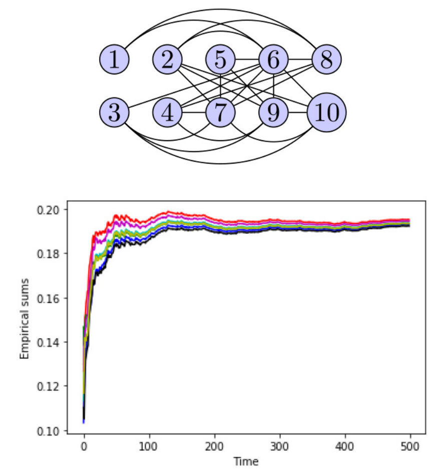

Fig. 2. Empirical sum for first seven urns in a network with 10 urns with memory $M = 1$. Initial ratio of red balls, Δ_r's and Δ_b's are all taken to be different. We observe that asymptotically the empirical sum of all the network urns approach a consensus value.

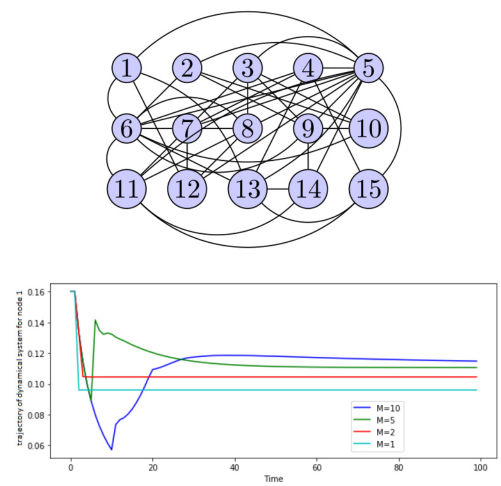

Fig. 3. A 15-node connected homogeneous network of finite memory Pólya urns. We have $\Delta = 5$ and even through the network is homogeneous, the initial ratio of red balls ($U_{i,0}$ for urn i) are different for all the urns.

We observe in Fig. 2 that our network exhibits a consensus behavior with the empirical sum for all the urns eventually reaching the same value (we plot the empirical sums for only 7 urns in Fig. 2 for better visibility of the curves). In this figure, the values of the Δ_r's and Δ_b's are taken to be in the range 5 to 15. We indeed remark that in the long run, the empirical beliefs of the urns (agents) about the red colored balls align to a value of about 20%; i.e., the agents eventually gravitate towards favoring the viewpoint represented by the black colored balls.

In Fig. 3, we plot the trajectory of the delayed linear dynamical systems obtained in (13) for different values of the memory parameter M which illustrates the variation of the consensus value with memory for a connected homogeneous network. As seen in Section IV, we can verify the consensus value of a homogeneous connected network using Theorem 3 by computing the fixed points using (15). As an illustration, in the 15-node network of Fig. 3, we used the following vector of initial ratio of red balls

$$\rho = \left(U_{1,0}, U_{2,0} \ldots, U_{15,0}\right) = (0.16, 0.08, 0.12, 0.04, 0.04, 0.04, 0.24, 0.04, 0.16, 0.12, 0.24, 0.08, 0.16, 0.2, 0.12).$$

Also for $M = 1$, letting v denote the normalized left eigenvector of the matrix $B_{15,1}$ corresponding to eigenvalue 1, we obtain (via computations carried up to the nearest two digits) that

$$v = (0.04, 0.04, 0.06, 0.06, 0.13, 0.1, 0.07, 0.06, 0.06, 0.04, 0.04, 0.04, 0.07, 0.04, 0.04).$$

Then, using (15), the fixed point is given by $\langle \rho, v\rangle = 0.0988$ (where $\langle \cdot, \cdot\rangle$ is the standard inner product). This fixed point is the same as shown by the light blue curve for $M = 1$ in Fig. 3 (the same behavior is observed for $M = 2, 5$ and 10); hence, these simulations indeed verify Theorem 3.

VI. Conclusion

In this letter, we demonstrated that a connected network of finite memory Pólya urns can be used to model opinion dynamics in a social network. Using the properties of the underlying Markov process, we proposed a provably correct consensus dynamics using this model. For the case with homogeneous reinforcement parameters across individuals, we provided a delayed dynamical system that can be used alternatively to study the asymptotic properties of this model and determine explicitly the consensus value. Future work includes exploring extensions of our proposed finite memory Pólya system to model more complex dynamics for social networks, including self-appraisal models, game-theoretic settings and comparisons to other classical consensus models such as the DeGroot model [22].

References

[1] B. Skyrms and R. Pemantle, "A dynamic model of social network formation," *Proc. Nat. Acad. Sci.*, vol. 97, no. 16, pp. 9340–9346, 2000.

[2] R. Pemantle, "A survey of random processes with reinforcement," *Probab. Surv.*, vol. 4, pp. 1–79, Feb. 2007.

[3] F. Kuhn, Y. Moses, and R. Oshman, "Coordinated consensus in dynamic networks," in *Proc. Annu. ACM Symp. Principles Distrib. Comput.*, 2011, pp. 1–10.

[4] R. O. Saber and R. M. Murray, "Consensus protocols for networks of dynamic agents," in *Proc. Amer. Control Conf.*, 2003, pp. 951–956.

[5] A. Tahbaz-Salehi and A. Jadbabaie, "On consensus over random networks," in *Proc. 44th Annu. Allerton Conf. Commun. Control Comput.*, 2006, pp. 1315–1321.

[6] A. Fazeli and A. Jadbabaie, "On consensus in a correlated model of network formation based on a Polya urn process," in *Proc. 50th IEEE Conf. Decis. Control Eur. Control Conf.*, 2011, pp. 2341–2346.

[7] P. D. Pra, P. Y. Louis, and I. G. Minelli, "Synchronization via interacting reinforcement," *J. Appl. Probab.*, vol. 51, no. 2, pp. 556–568, 2014.

[8] S. Motsch and E. Tadmor, "Heterophilious dynamics enhances consensus," *SIAM Rev.*, vol. 56, no. 4, pp. 577–621, 2014.

[9] P. Jia, A. Mirtabatabaei, N. E. Friedkin, and F. Bullo, "On the dynamics of influence networks via reflected appraisal," in *Proc. Amer. Control Conf.*, 2013, pp. 1249–1254.

[10] A. Jadbabaie, A. Makur, E. Mossel, and R. Salhab, "Inference in opinion dynamics under social pressure," 2022, *arXiv:2104.11172*.

[11] J. Gaitonde, J. Kleinberg, and É. Tardos, "Adversarial perturbations of opinion dynamics in networks," in *Proc. 21st ACM Conf. Econ. Comput.*, 2020, pp. 471–472.

[12] D. Soriana-Paños, Q. Guo, V. Latora, and J. Gómez-Gardeñes, "Explosive transitions induced by interdependent contagion-consensus dynamics in multiplex networks," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 99, Jun. 2019, Art. no. 62311.

[13] S. F. Ruf, K. Paarporn, and P. E. Paré, "Going viral: Stability of consensus-driven adoptive spread," *IEEE Trans. Netw. Sci. Eng.*, vol. 7, no. 3, pp. 1764–1773, Jul.–Sep. 2020.

[14] N. Berger, C. Borgs, J. T. Chayes, and A. Saberi, "On the spread of viruses on the Internet," in *Proc. ACM-SIAM Symp. Discrete Algorithms*, vol. 5, 2005, pp. 301–310.

[15] N. Berger, C. Borgs, J. T. Chayes, and A. Saberi, "Asymptotic behavior and distributional limits of preferential attachment graphs," *Ann. Probab.*, vol. 42, no. 1, pp. 1–40, 2014.

[16] A. Collevecchio, C. Cotar, and M. LiCalzi, "On a preferential attachment and generalized Pólya's urn model," *Ann. Appl. Probab.*, vol. 23, no. 3, pp. 1219–1253, 2013.

[17] Y. Hongyong, L. Lan, and Z. Siying, "Consensus of synchronization-preferential scale-free networks," *Phys. Procedia*, vol. 3, no. 5, pp. 1913–1920, 2010.

[18] Z. Wu, Z.-H. Guan, and X. Wu, "Consensus problem in multi-agent systems with physical position neighbourhood evolving network," *Physica A, Stat. Mech. Appl.*, vol. 379, no. 2, pp. 681–690, 2007.

[19] A. Tahbaz-Salehi and A. Jadbabaie, "Small world phenomenon, rapidly mixing Markov chains, and average consensus algorithms," in *Proc. 46th IEEE Conf. Decis. Control*, 2007, pp. 276–281.

[20] B. Touri, *Product of Random Stochastic Matrices and Distributed Averaging*. Berlin, Germany: Springer, 2012.

[21] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis, and simulation," *J. Artif. Soc. Soc. Simul.*, vol. 5, no. 3, pp. 1–33, 2002.

[22] M. H. DeGroot, "Reaching a consensus," *J. Amer. Stat. Assoc.*, vol. 69, no. 345, pp. 118–121, 1974.

[23] J. Haslegrave and M. Puljiz, "Reaching consensus on a connected graph," *J. Appl. Probab.*, vol. 54, no. 1, pp. 181–198, 2017.

[24] F. Alajaji and T. Fuja, "A communication channel modeled on contagion," *IEEE Trans. Inf. Theory*, vol. 40, no. 6, pp. 2035–2041, Nov. 1994.

[25] G. R. de Campos and A. Seuret, "Improved consensus algorithms using memory effects," in *Proc. 50th IEEE Conf. Decis. Control Eur. Control Conf.*, 2011, pp. 982–987.

[26] F. M. Atay, "The consensus problem in networks with transmission delays," *Philos. Trans. Roy. Soc. London A, Math. Phys. Eng. Sci.*, vol. 371, no. 1999, 2013, Art. no. 20120460.

[27] W. Yu, G. Chen, M. Cao, and W. Ren, "Delay-induced consensus and quasi-consensus in multi-agent dynamical systems," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 60, no. 10, pp. 2679–2687, Oct. 2013.

[28] A. Banerjee, P. Burlina, and F. Alajaji, "Image segmentation and labeling using the Pólya urn model," *IEEE Trans. Image Process.*, vol. 8, pp. 1243–1253, 1999.

[29] M. Hayhoe, F. Alajaji, and B. Gharesifard, "A Pólya contagion model for networks," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 4, pp. 1998–2010, Dec. 2018.

[30] G. Aletti and I. Crimaldi, "The rescaled Pólya urn: Local reinforcement and chi-squared goodness of fit test," 2020, *arXiv:1906.10951*.

[31] D. Pfeifer, "Pólya-Lundberg process," in *Encyclopedia of Statistical Sciences*. Hoboken, NJ, USA: Wiley, 2006.

[32] S. Singh, F. Alajaji, and B. Gharesifard, "A finite memory interacting Pólya contagion network and its approximating dynamical systems," *SIAM J. Control Optim.*, vol. 60, no. 2, pp. 347–369, 2022.