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Vectors

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Vectors and scalars are the basic building blocks in linear algebra. We introduce vectors from a geometric and an algebraic point of view. We also define a field of scalars as an algebraic structure and highlight a few of the most important examples.

The word 'vector' comes from the Latin word meaning "to carry". The mathematical term was introduced in 1846 by W.R. Hamilton.

1.0 Geometry of Vectors

WHAT, GEOMETRICALLY, IS A VECTOR? A ***vector*** $\vec{v}$ is characterized by its magnitude (also known as its length) and its direction. A vector is often represented by an arrow from a point P to a point Q and denoted by $\vec{v} := \overrightarrow{PQ}$. In this situation, the point P is called the ***tail*** of the vector and the point Q is called the ***head***. The ***magnitude*** $\|\vec{v}\|$ of the vector $\vec{v}$ is a real number equal to the distance between the head and the tail. Two arrows $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ represent the same vector if the quadrilateral $PQRS$ is a parallelogram; the opposite sides are equal in length. Many physical quantities, such as displacement, velocity, acceleration, force, and momentum, are represented by vectors.

The International Organization for Standardization (ISO) recommends that a vector be typeset in bold italic font $\boldsymbol{v}$ or a non-bold italic font accented by a right arrow $\vec{v}$. Initially, we use both.

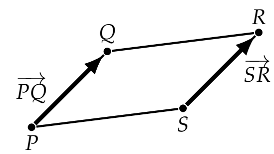


Figure 1.0: Equivalent vectors

HOW DO WE ADD VECTORS? The ***sum*** of two vectors $\vec{v}$ and $\vec{w}$ is the vector $\vec{v} + \vec{w}$ obtained by placing the tail of $\vec{w}$ at the head of $\vec{v}$ drawing the arrow from the tail of $\vec{v}$ to the head of $\vec{w}$.

1.0.0 Problem. A kayak is moving with velocity $\vec{v}$ and a speed of $5\ \mathrm{km}\cdot\mathrm{h}^{-1}$ relative to the water. The river has a current $\vec{w}$ with a speed of $4\ \mathrm{km}\cdot\mathrm{h}^{-1}$. What is the physical significance of the vector $\vec{v} + \vec{w}$?

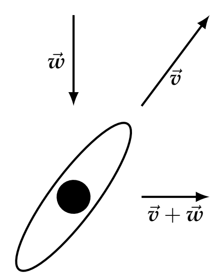
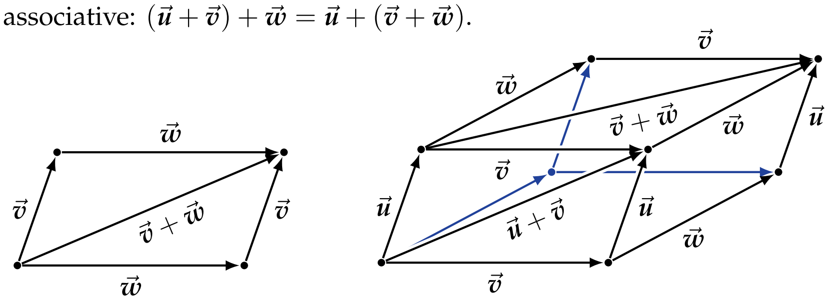


Figure 1.1: Kayaking into a current

Solution. The vector $\vec{v}$ indicates how the kayak is moving relative to the water and the vector $\vec{w}$ indicates how the water is moving relative to the riverbed. The sum $\vec{v} + \vec{w}$ is the velocity of the kayak relative to the riverbed. Although we add the velocity vectors, the magnitude of the sum is not necessarily the sum of the magnitudes. In this case, we have $1\ \mathrm{km}\cdot\mathrm{h}^{-1} \leqslant \|\vec{v} + \vec{w}\| \leqslant 9\ \mathrm{km}\cdot\mathrm{h}^{-1}$. $\square$

Calling this operation on vectors 'addition' implicitly suggests that it enjoys certain properties. From the parallelogram with sides equivalent to the vectors $\vec{v}$ and $\vec{w}$, we see that vector addition is commutative: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$. Similarly, from the parallelepiped (the 3-dimensional figure bounded by 6 parallelograms) with edges equivalent to the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, we see that vector addition is associative: $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$.



$\vec{v}+\vec{w} = \vec{w}+\vec{v}$ $\qquad$ $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

Figure 1.2: Geometry of commutativity and associativity for vector addition

Vector addition also determines some special vectors. A ***zero vector*** $\vec{\mathbf{0}}$ is any vector with zero magnitude.

In making a definition, we emphasize the ***definiendum***, that which is being defined, by switching between italic and non-italic fonts and using a bold typeface.

1.0.1 Proposition (Property of the zero vector)**.** *The zero vector is the unique vector such that, for all vectors $\vec{v}$, we have $\vec{v}+\vec{\mathbf{0}} = \vec{v}$.*

Proof. Since the magnitude of a zero vector $\vec{\mathbf{0}}$ is zero, the points corresponding to its head and tail coincide. The definition of vector addition implies that, for all vectors $\vec{v}$, we have $\vec{v}+\vec{\mathbf{0}} = \vec{v}$.

To establish uniqueness, suppose that $\vec{z}$ is also a zero vector. The property of a zero vector and the commutativity of vector addition give $\vec{z} = \vec{z}+\vec{\mathbf{0}} = \vec{\mathbf{0}}+\vec{z} = \vec{\mathbf{0}}$, so any two zero vectors are equal. □

1.0.2 Proposition (Property of additive inverses)**.** *For all vectors $\vec{v}$, there exists a unique vector $-\vec{v}$, called the* **additive inverse** *of $\vec{v}$, such that $\vec{v}+(-\vec{v}) = \vec{\mathbf{0}}$.*

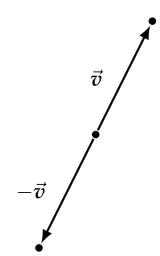


Figure 1.3: A vector $\vec{v}$ and its additive inverse $-\vec{v}$

Proof. The claim has two parts.

(existence) Let $\vec{w}$ be a vector with the same magnitude as $\vec{v}$ and the opposite direction; when $\vec{v} := \overrightarrow{PQ}$, we have $\vec{w} = \overrightarrow{QP}$. Vector addition implies that $\vec{v}+\vec{w} = \vec{\mathbf{0}}$, so $\vec{v}$ has an additive inverse.

(uniqueness) Suppose that $\vec{u}$ and $\vec{w}$ are both additive inverses of the vector $\vec{v}$. The properties of the zero vector and an additive inverse, together with the commutativity and associativity of vector addition, establish that

$$\vec{u} = \vec{u}+\vec{\mathbf{0}} = \vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w} = (\vec{v}+\vec{u})+\vec{w} = \vec{\mathbf{0}}+\vec{w} = \vec{w}+\vec{\mathbf{0}} = \vec{w}\,,$$

so any two additive inverses of the vector $\vec{v}$ are equal. □

HOW DO WE MULTIPLE A VECTOR BY A NUMBER? The ***scalar multiple*** of a vector $\vec{v}$ by a number c is another vector $c\,\vec{v}$ simply denoted by juxtaposition. For any two vectors $\vec{v}$ and $\vec{w}$, and any two numbers b

and c, scalar multiplication satisfies the following properties:

(compatibility with multiplication of scalars) $\quad b\,(c\,\vec{v}) = (bc)\,\vec{v}$
(existence of a multiplicative identity) $\quad 1\,\vec{v} = \vec{v}$
(distributivity over vector addition) $\quad c\,(\vec{v} + \vec{w}) = c\,\vec{v} + c\,\vec{w}$
(distributivity over scalar addition) $\quad (c + d)\,\vec{v} = c\,\vec{v} + d\,\vec{v}$

Scalar multiplication by a positive number rescales (stretches or contracts) the magnitude by the given factor without changing the direction. In constrast, scalar multiplication by a negative number gives a vector in the opposite direction.

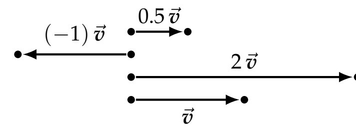


Figure 1.4: Scalar multiplies of $\vec{v}$

1.0.3 Definition (Parallel vectors)**.** The vector $\vec{v}$ is ***parallel*** to the vector $\vec{w}$ if there exists a real number c such that $c\,\vec{v} = \vec{w}$.

Exercises

1.0.4 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The sum of two vectors is a vector.
ii. The sum of nonzero vectors is never the zero vector.
iii. Scalar multiplication allows one to multiple two vectors.
iv. Parallel vectors have the same direction.

1.0.5 Problem. A ***unit*** vector has magnitude equals 1. Demonstrate that every nonzero vector is parallel to a unit vector.

1.0.6 Problem. Prove that the additive inverse of the zero vector is the zero vector: $\vec{\mathbf{0}} = -\vec{\mathbf{0}}$.

1.0.7 Problem. For any vector $\vec{v}$, prove that scalar multiplication by 0 is the zero vector, $0\,\vec{v} = \vec{\mathbf{0}}$, and every vector is parallel to the zero vector.

1.0.8 Problem. For any vector $\vec{v}$, prove that scalar multiplication by -1 produces the additive inverse of $\vec{v}$: $(-1)\,\vec{v} = -\vec{v}$.

1.0.9 Problem. Prove that vector addition is cancellative: for all vector $\vec{u}$, $\vec{v}$, and $\vec{w}$, the equation $\vec{u} + \vec{v} = \vec{w} + \vec{v}$ always implies that $\vec{u} = \vec{w}$.

1.0.10 Problem. Vector subtraction is defined by adding the additive inverse: $\vec{v} - \vec{w} := \vec{v} + (-\vec{w})$. Describe this operation geometrically.

1.0.11 Problem. Prove that vector subtraction is anti-commutative: for all vectors $\vec{v}$ and $\vec{w}$, we have $\vec{v} - \vec{w} = -(\vec{w} - \vec{v})$.

1.0.12 Problem. Demonstrate that vector subtraction is non-associative. Draw three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ such that $(\vec{u} - \vec{v}) - \vec{w} \neq \vec{u} - (\vec{v} - \vec{w})$.

1.0.13 Problem. For any integer m that is greater than 2, exhibit m nonzero vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ all with the same magnitude such that $\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_m = \vec{\mathbf{0}}$.

1.0.14 Problem. Show that the scalar multiple of the zero vector by any integer equals the zero vector: $m\,\vec{\mathbf{0}} = \vec{\mathbf{0}}$.

1.0.15 Problem. The 13 points appearing in Figure 1.5 are part of a hexagonal lattice; any three nearby points form an equilateral triangle of the same size. Express each of the following vectors in terms of the vector $\vec{v} := \overrightarrow{OD}$ and the vector $\vec{w} := \overrightarrow{OL}$.

i. $\overrightarrow{GD}$
ii. $\overrightarrow{DC}$
iii. $\overrightarrow{GF}$
iv. $\overrightarrow{CJ}$
v. $\overrightarrow{DC} + \overrightarrow{FI} + \overrightarrow{JG}$

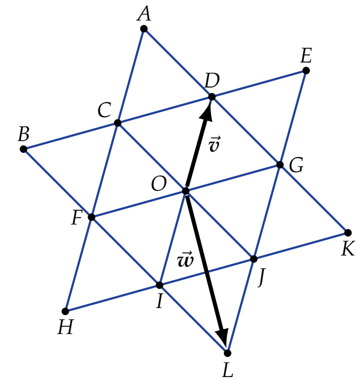


Figure 1.5: Part of a hexagonal lattice

1.0.16 Problem. Show that the binary relation "is parallel" on the set of all nonzero vectors is reflexive, symmetric, and transitive. More precisely, prove that, for all nonzero vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, we have

(reflexivity) The vector $\vec{v}$ is parallel to $\vec{v}$.
(symmetry) The vector $\vec{v}$ is parallel to $\vec{w}$ if and only if the vector $\vec{w}$ is parallel to $\vec{v}$.
(transitivity) If $\vec{u}$ is parallel to $\vec{v}$ and $\vec{v}$ is parallel to $\vec{w}$, then $\vec{u}$ is parallel to $\vec{w}$.

1.1 Points and Lines

HOW ARE POINTS AND VECTORS RELATED? By fixing a distinguished point, every point corresponds a unique vector. More explicitly, let O be a fixed reference point called the ***origin***. The point P corresponds to the ***position vector*** $\overrightarrow{OP}$ and the vector $\vec{v}$ corresponds to the point that lies at its head when its tail is at the origin. These two maps compose, in either order, to the identity. Specifically, the point P corresponds to the vector $\overrightarrow{OP}$ and this vector corresponds to the point P. Conversely, the vector $\vec{v}$ corresponds to the point at its head when its tail is at the origin and this point corresponds to $\vec{v}$.

The word "origin" is borrowed from the Latin word meaning "ancestry", "coming into being", or "source". In mathematics, it refers to a fixed point from which measurement or motion commences.

1.1.0 Problem. In a triangle, show that the line segment joining the midpoints of two sides is parallel to the third and has half the length.

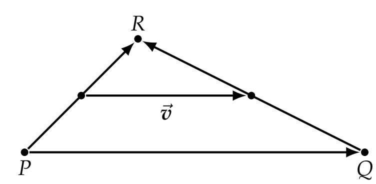


Figure 1.6: Joining the midpoints of two sides in a triangle

Solution. If P, Q, and R are the vertices of the triangle, then we have $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$. For brevity, choose P to be the origin. The midpoint of the line segment $\overline{PR}$ corresponds to the vector $\frac{1}{2}\,\overrightarrow{PR}$ and the midpoint of the line segment $\overline{QR}$ corresponds to the vector $\overrightarrow{PQ} + \frac{1}{2}\,\overrightarrow{QR}$. Hence, the vector $\vec{v}$ from the first midpoint to the second midpoint is

$$\vec{v} = \left(\overrightarrow{PQ} + \tfrac{1}{2}\,\overrightarrow{QR}\right) - \left(\tfrac{1}{2}\,\overrightarrow{PR}\right) = \left(\overrightarrow{PQ} + \tfrac{1}{2}\,\overrightarrow{QR}\right) - \tfrac{1}{2}\left(\overrightarrow{PQ} + \overrightarrow{QR}\right) = \tfrac{1}{2}\,\overrightarrow{PQ}\,.$$

We conclude that the line segment joining these midpoints is parallel to the line segment $\overline{PQ}$ and has half its length. □

1.1.1 Definition. The ***centroid*** of a nonempty finite collection of points is the point corresponding to the mean of their position vectors.

The ***mean*** of m vectors is the scalar multiple of their sum by $1/m$.

1.1.2 Problem. Demonstrate that the centroid is independent of the choice of origin.

Solution. Let P and Q be two choices of origin. For any point R, we have $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$. Given m points $R_1, R_2 \dots, R_m$, let C denote the centroid. It follows that

$$\begin{aligned}\overrightarrow{PC} &= \tfrac{1}{m}(\overrightarrow{PR_1} + \overrightarrow{PR_2} + \cdots + \overrightarrow{PR_m})\\ &= \tfrac{1}{m}\big((\overrightarrow{PQ} + \overrightarrow{QR_1}) + (\overrightarrow{PQ} + \overrightarrow{QR_2}) + \cdots + (\overrightarrow{PQ} + \overrightarrow{QR_m})\big)\\ &= \overrightarrow{PQ} + \tfrac{1}{m}(\overrightarrow{QR_1} + \overrightarrow{QR_2} + \cdots + \overrightarrow{QR_m}).\end{aligned}$$

Thus, we see that $\overrightarrow{QC} = \overrightarrow{PC} - \overrightarrow{PQ} = \frac{1}{m}(\overrightarrow{QR_1} + \overrightarrow{QR_2} + \cdots + \overrightarrow{QR_m})$. □

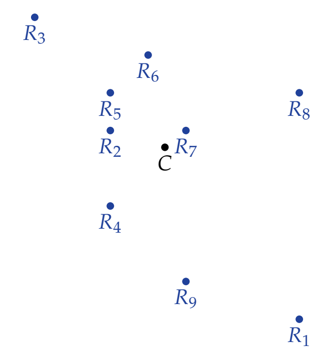


Figure 1.7: The centroid of 9 points

How do vectors describe the points on a line? Let P and Q be distinct points and let O denote the origin. The points on the line through P and Q correspond to a simple collection of vectors. Since the point P is corresponds to the vector $\overrightarrow{OP}$ and the vector $\overrightarrow{PQ}$ is parallel to the line through P and Q, each vector

$$\vec{\ell}(t) := \overrightarrow{OP} + t\,\overrightarrow{PQ} = \overrightarrow{OP} + t\,(\overrightarrow{OQ} - \overrightarrow{OP}) = (1-t)\,\overrightarrow{OP} + t\,\overrightarrow{OQ},$$

where t is a real number, corresponds to exactly one point on this line. In particular, the vector $\vec{\ell}(0)$ corresponds to the point P and the vector $\vec{\ell}(1)$ corresponds to Q. When $0 \leqslant t \leqslant 1$, the vector $\vec{\ell}(t)$ corresponds to a point on the line segment $\overline{PQ}$.

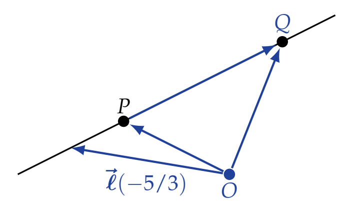


Figure 1.8: Vector equation of a line

1.1.3 Problem. Let P, Q, and R be three distinct points and let O be the origin. Show that the points P, Q, and R are collinear (all lie on a single line) if and only if there exists three real numbers a, b, and c, not all zero, such that $a + b + c = 0$ and $a\,\overrightarrow{OP} + b\,\overrightarrow{OQ} + c\,\overrightarrow{OR} = \vec{\mathbf{0}}$.

Solution. To establish this 'if and only if"'statement, we prove two separate implications: the 'if' and the 'only if'.

$\Rightarrow$: Suppose that the three points are collinear. Since the point R lies on the line through P and Q, there exists a real number t such that $\overrightarrow{OR} = (1-t)\,\overrightarrow{OP} + t\,\overrightarrow{PQ}$. Setting $a := (1-t)$, $b := t$, and $c := -1$, we have $a + b + c = 0$ and $a\,\overrightarrow{OP} + b\,\overrightarrow{OQ} + c\,\overrightarrow{OR} = \vec{\mathbf{0}}$.

The expression $A := B$ means that A is, by definition, equal to B.

$\Leftarrow$: Suppose that there are real numbers a, b, and c, not all zero, such that $a + b + c = 0$ and $a\,\overrightarrow{OP} + b\,\overrightarrow{OQ} + c\,\overrightarrow{OR} = \vec{\mathbf{0}}$. If c were equal to 0, then we would have $a = -b$ and $\overrightarrow{OP} = \overrightarrow{OQ}$, which would show that $P = Q$ contradicting the hypothesis that three points are distinct. Hence, we must have $c \neq 0$. It follows that $\frac{a}{c} + \frac{b}{c} + 1 = 0$ and $\overrightarrow{OR} = -\frac{a}{c}\,\overrightarrow{OP} - \frac{b}{c}\,\overrightarrow{OQ} = \left(\frac{b}{c} + 1\right)\overrightarrow{OP} - \frac{b}{c}\,\overrightarrow{OQ}$. Setting $t := -\frac{b}{c}$, we have $\overrightarrow{OR} = (1-t)\,\overrightarrow{OP} + t\,\overrightarrow{OQ}$, which proves that the point R is on the line through P and Q. □

1.1.4 Definition. A ***median*** of a triangle is a line through a vertex and the midpoint of the opposite side.

1.1.5 Problem. Let P, Q, and R be three points that are not collinear. Show that the centroid of the points P, Q, and R lies on each median in the triangle PQR.

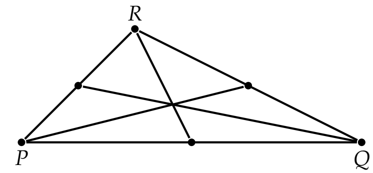


Figure 1.9: Medians in a triangle

Solution. Let O be the origin. By definition, the centroid of the three points is the point corresponding to the vector $\frac{1}{3}(\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR})$. Because the point P and the midpoint of the opposite side $\overline{QR}$ correspond to the vectors $\overrightarrow{OP}$ and $\frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OR})$ respectively, the line through these points is parameterized by the vectors

$$\vec{\ell}(t) := (1-t)\,\overrightarrow{OP} + t\big(\tfrac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OR})\big)\,.$$

We see that $\vec{\ell}\big(\frac{2}{3}\big) = \frac{1}{3}\,\overrightarrow{OP} + \frac{2}{3}\big(\frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OR})\big) = \frac{1}{3}(\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR})$, so the centroid of the triangle PQR lies on the median through the vertex P. By permuting the labels for the vertices, we conclude that the centroid lies on each median. $\square$

Exercises

1.1.6 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. Points and vectors are identical.
ii. The midpoint of a line segment is the centroid of its endpoints.
iii. Any three points line on a line.

1.1.7 Problem. Given four points A, B, C, and D, establish that the following three conditions are equivalent:

a. $\overrightarrow{AB} = \overrightarrow{DC}$,
b. $\overrightarrow{AD} = \overrightarrow{BC}$,
c. the line segments $\overline{AC}$ and $\overline{BD}$ have the same midpoint.

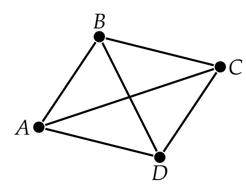


Figure 1.10: Diagonals in a parallelogram

1.1.8 Problem. Show that the diagonals in a parallelogram bisect each other (intersect at their midpoints). Moreover, this intersection point is the centroid of the four vertices of the parallelogram.

1.1.9 Problem. Prove that the midpoints of the four sides of an arbitrary quadrilateral form a parallelogram.

1.1.10 Problem. Show that the centroid of the four vertices of a tetrahedron (a solid with four vertices joined by six lines that bound the tetrahedron's four triangular faces) is the intersection of the following seven lines:

i. it is 3/4 of the way from the vertex of the tetrahedron along the line segment joining the vertex to the centroid of the opposite face;

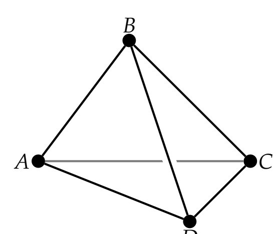


Figure 1.11: Tetrahedron

ii. it is also the midpoint of the line segment joining the midpoints of any pair of opposite edges.

1.1.11 Problem. Consider six points A, P, B, Q, C, R that lie alternately on two distinct lines that intersect at the point O; see Figure 1.12. If the vector $\overrightarrow{AQ}$ is parallel to the vector $\overrightarrow{BR}$ and $\overrightarrow{BP}$ is parallel to $\overrightarrow{CQ}$, then show that $\overrightarrow{AP}$ is parallel to $\overrightarrow{CR}$.

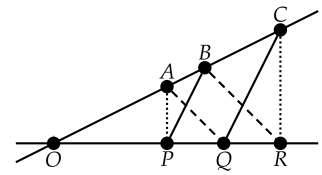


Figure 1.12: Six points on two lines

1.2 Algebra of Vectors

WHAT ARE SCALARS? The most primitive numbers $0, 1, 2, 3, \ldots$ are those used for counting. These numbers form the set of ***nonnegative integers*** denotes by $\mathbb{N}$. This set is equipped with two basic binary operations: addition and multiplication. For all $a, b, c \in \mathbb{N}$, we have the following seven familiar properties.

We reserve the Blackboard bold typeface for a few special sets.

(commutativity of addition)	$a + b = b + a$
(associativity of addition)	$(a + b) + c = a + (b + c)$
(existence of an additive identity)	$a + 0 = a$
(commutativity of multiplication)	$ab = ba$
(associativity of multiplication)	$(ab)c = a(bc)$
(existence of a multiplicative identity)	$1a = a$
(distributivity of multiplication over addition)	$a(b + c) = ab + ac$

In contrast with vectors, only one nonnegative integer (namely the number 0) has an additive inverse contained in $\mathbb{N}$. To guarantee that every number has an additive inverse, we enlarge our collection of numbers to the ***integers*** $\mathbb{Z}$ consisting of $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. Both addition and multiplication extend to this larger set. Besides the seven properties listed for $\mathbb{N}$, the addition operation on $\mathbb{Z}$ also satisfies:

The symbol $\mathbb{Z}$ comes from "Zahlen" the German word for numbers.

(existence of additive inverses) $\qquad a + (-a) = 0$

The only integers that have a multiplicative inverse (or reciprocal) contained in $\mathbb{Z}$ are 1 and -1. To ensure that every nonzero number has a multiplicative inverse, we consider the ***rational numbers*** $\mathbb{Q}$. A rational number can be expressed as a fraction a/b of two integers, where the numerator a is any integer and b is any nonzero integer. Both addition and multiplication extend to this larger set and these operations on $\mathbb{Q}$ acquire the extra property:

The symbol $\mathbb{Q}$ comes from "Quoziente" the Italian word for quotient.

(existence of multiplicative inverses) $\qquad a\left(\frac{1}{a}\right) = 1$ for all $a \neq 0$

The rational numbers are the prototypical scalars. More generally, a ***field*** $\mathbb{K}$ ***of scalars*** is a set with two operations, called addition and multiplication, that satisfy the nine properties listed for the rational numbers. For any two elements $a, b \in \mathbb{K}$, the result of addition is the sum $a + b$, and the result of multiplication is the product ab. Throughout this text, we use $\mathbb{K}$ to denote an arbitrary field of scalars.

The symbol $\mathbb{K}$ comes from "Körper" the German word for a field of scalars. The English term "field" was introduced in 1893 by E.H. Moore.

The real numbers $\mathbb{R}$ provide the second example of a field of scalars. The real numbers include all rational numbers and all irrational numbers, such as $\sqrt{2}$ and $\pi \approx 3.14159265\ldots$. Any real number can be determined by a possibly infinite decimal representation. The set $\mathbb{R}$ is visualized as an infinite number line, where the integers are equally spaced and each point on the line corresponds to a unique real number.

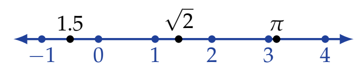


Figure 1.13: The real number line

What, algebraically, is a vector? For any positive integer n, the ***coordinate space*** $\mathbb{K}^n$ consists of all n-tuples (or lists with n entries) of scalars in $\mathbb{K}$. A vector $\vec{v} \in \mathbb{K}^n$ is typically expressed as a column of scalars. Addition and scalar multiplication are defined entrywise;

$$\text{for all } \vec{v} := \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{K}^n, \text{ all } \vec{w} := \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{K}^n, \text{ and all } c \in \mathbb{K}, \text{ we have } \vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix} \text{ and } c\,\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

The commutativity and associativity of addition in $\mathbb{K}^n$ are inherited from the commutativity and associativity of addition in $\mathbb{K}$.

The existence of an additive identity and additive inverses in $\mathbb{K}$ implies that

$$\vec{\mathbf{0}} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ and } \qquad -\vec{v} := \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix},$$

are the zero vector and additive inverse in $\mathbb{K}^n$ respectively. The existence of a multiplicative identity in $\mathbb{K}$ gives the multiplicative identity in $\mathbb{K}^n$. Finally, the compatibility of multiplications and the distributivity of scalar multiplication over both vector addition and scalar addition are derived from associativity and distributivity of multiplication in $\mathbb{K}$.

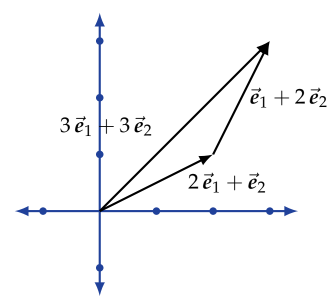


Figure 1.14: Vector addition

For each $1 \leqslant j \leqslant n$, the vector $\vec{e}_j$ has 1 in its j-th entry and zeros elsewhere. The ***standard basis*** is the list $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ of vectors in $\mathbb{K}^n$. With this notation, we have $\mathbb{K}^n = \{v_1\,\vec{e}_1 + v_2\,\vec{e}_2 + \cdots + v_n\,\vec{e}_n \mid v_j \in \mathbb{K}\}$. Choosing $\vec{\mathbf{0}}$ to be the origin, we see that the vector from the point $P := (p_1, p_2, \ldots, p_n)$ to the point $Q := (q_1, q_2, \ldots, q_n)$ is

$$\overrightarrow{PQ} = (q_1 - p_1)\,\vec{e}_1 + (q_2 - p_2)\,\vec{e}_2 + \cdots + (q_n - p_n)\,\vec{e}_n\,.$$

$$\vec{e}_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

In $\mathbb{K}^3$, the notation $\mathbf{i} := \vec{e}_1$, $\mathbf{j} := \vec{e}_2$, and $\mathbf{k} := \vec{e}_3$ is common.

Why consider vectors with more than three components? Almost all interesting applications depend on more than three things. For example, to describe the state of a physical system in 3-space, one needs the position and momentum vectors, a total of at least 6 components. To describe the stock market, one needs a vector consisting of all the prices of all stocks listed on the exchange. To describe a digital image to a computer, one needs to list the colour of each pixel. Linear algebra provides uniform tools not depending on the number of parameters—it works for all nonnegative integers n.

1.2.0 Problem. Decide whether the vector $\vec{v} := 2\vec{e}_1 + 3\,\vec{e}_2 + 5\,\vec{e}_3$ is parallel to $\vec{u} := -\vec{e}_1 - 1.5\,\vec{e}_2 - 2.5\,\vec{e}_3$ or $\vec{w} := 4\,\vec{e}_1 + 6\,\vec{e}_2 + 9\,\vec{e}_3$.

Solution. Since $\vec{u} = -\frac{1}{2}\vec{v}$, these vectors are parallel. However, $\vec{w}$ is not a scalar multiple of $\vec{v}$ because $\frac{2}{4} = \frac{3}{6} \neq \frac{5}{9}$. □

Exercises

1.2.1 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. Zero belongs to the set $\mathbb{N}$.
ii. The integers $\mathbb{Z}$ form a field of scalars.
iii. The symbol $\mathbb{K}$ denotes a random collection of numbers.
iv. Every entry in the zero vector $\vec{\mathbf{0}} \in \mathbb{K}^n$ equals zero.
v. The coordinate space $\mathbb{K}^0$ contains the zero vector.

1.2.2 Problem. For all $a, b \in \mathbb{Z}$, the ***order relation*** on $\mathbb{Z}$ is defined by $a \leqslant b$ if and only if $b - a \in \mathbb{N}$. We write $a < b$ if $a \leqslant b$ and $a \neq b$.

i. For all $a, b, c \in \mathbb{Z}$, show that the relation $a \leqslant b$ implies that $a + c \leqslant b + c$.
ii. For all $a, b, c \in \mathbb{Z}$ with $c \geqslant 0$, show that the relation $a \leqslant b$ implies that $ac \leqslant bc$.

1.2.3 Problem. Verify that the rational numbers $\mathbb{Q}$ satisfy the nine defining properties of a field of scalars.

1.2.4 Problem. Given two scalars $a, b \in \mathbb{K}$ such that $ab = 0$, prove that $a = 0$ or $b = 0$.

1.2.5 Problem. ***Exponentiation*** maps $(n, a) \in \mathbb{N} \times \mathbb{K}$ to the product

$$a^n := \prod_{i=1}^{n} a = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}} .$$

i. Show that exponentiation is not commutative.
ii. Show that exponentiation is not associative.

1.2.6 Problem. For all $a, b, c, d \in \mathbb{Z}$ with $b > 0$ and $d > 0$, the ***order relation*** on $\mathbb{Q}$ is defined by $\frac{a}{b} \leqslant \frac{c}{d}$ if and only if $ad < bc$.

i. For all $a, b, c, d, e, f \in \mathbb{Z}$ with $b > 0$, $d > 0$, and $f > 0$, show that the relation $\frac{a}{b} \leqslant \frac{c}{d}$ implies that $\frac{a}{b} + \frac{e}{f} \leqslant \frac{c}{d} + \frac{e}{f}$.

ii. For all $a, b, c, d, e, f \in \mathbb{Z}$ with $b > 0$, $d > 0$, $e \geqslant 0$, and $f > 0$, show that the relation $\frac{a}{b} \leqslant \frac{c}{d}$ implies that $\left(\frac{a}{b}\right)\left(\frac{e}{f}\right) \leqslant \left(\frac{c}{d}\right)\left(\frac{e}{f}\right)$.

iii. For all $a, b, c, d \in \mathbb{Z}$ with $b > 0$ and $d > 0$, show that the relation $\frac{a}{b} \leqslant \frac{c}{d}$ implies that $\frac{a}{b} \leqslant \frac{ad+bc}{2bd} \leqslant \frac{c}{d}$.

1.2.7 Problem. Show that the 2-element subset $\{0, 1\} \subset \mathbb{Z}$ with the operations $a \oplus b := a + b - 2ab$ and $a \wedge b := \min(a, b)$ forms a field of scalars. In this context, the addition operation $\oplus$ is called "exclusive or" and the multiplication operation $\wedge$ is called "and".

1.2.8 Problem. Consider the set $\mathbb{R}$ of real numbers equipped with the two binary operations: $a \boxplus b := \min(a, b)$ and $a \boxtimes b := a + b$. Determine which of the nine defining properties of a field of scalars this ***tropical algebra*** satisfies.

1.2.9 Problem. Two forces, represented by the vectors

$$\vec{F}_1 := 6\,\vec{e}_1 - 2\,\vec{e}_2 \qquad \text{and} \qquad \vec{F}_2 := -3\,\vec{e}_1 + 8\,\vec{e}_2\,,$$

are acting on an object. Find a vector of the force that must be applied to the object if it is to remain stationary.

1.2.10 Problem. An airplane is flying at an airspeed of $600\ \mathrm{km} \cdot \mathrm{h}^{-1}$ in a cross-wind that is blowing from the northeast at a speed of $50\ \mathrm{km} \cdot \mathrm{h}^{-1}$. In what direction should the plane head to end up going due east?

1.2.11 Problem. Which pairs of the following vectors are parallel?

$$\vec{w} := -\vec{e}_1 - 2\vec{e}_2 + 2\vec{e}_3 \qquad \vec{x} := -2\vec{e}_1 + 4\vec{e}_2 + 4\vec{e}_3 \qquad \vec{y} := 3\vec{e}_1 + 6\vec{e}_2 - 6\vec{e}_3 \qquad \vec{z} := -4\vec{e}_1 - 8\vec{e}_2 + 8\vec{e}_3$$