2 Vector Products

Unlike for pairs of scalars, we did not define multiplication between two vectors. For almost all positive integers n, the coordinate space $\mathbb{K}^n$ cannot be equipped with a second binary operation that behaves like multiplication—a reasonable vector product would be nontrivial, associative, and distributive. Nevertheless, we feature those special cases for which there exists such a vector product.

2.0 Complex Numbers

How is the multiplication of vectors in $\mathbb{R}^2$ defined? The complex numbers may viewed as the real coordinate plane $\mathbb{R}^2$ with a vector product.

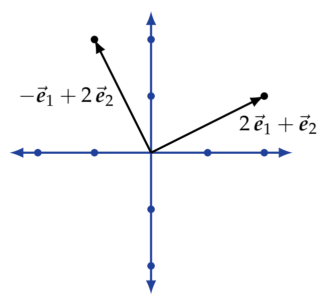

Figure 2.0: The vector product

$$\vec{e}_2(2\,\vec{e}_1 + \vec{e}_2) = -\vec{e}_1 + 2\,\vec{e}_2$$

2.0.0 Theorem. *The coordinate space $\mathbb{R}^2$ together with the vector product defined, for all $a, b, c, d \in \mathbb{R}$, by*

$$(a\,\vec{e}_1 + b\,\vec{e}_2)(c\,\vec{e}_1 + d\,\vec{e}_2) := (ac - bd)\,\vec{e}_1 + (ad + bc)\,\vec{e}_2\,,$$

forms a field of scalars called the **complex numbers**.

Proof. Since Section 1.2 already demonstrates that vector addition in $\mathbb{R}^2$ satisfies the four properties for a field of scalars that only involve addition, it suffices to verify the five properties that involve multiplication. Let $a, b, c, d, e, f \in \mathbb{R}$ denote arbitrary real numbers. The commutativity, associativity, and distributivity of addition and multiplication for real numbers give

$$\begin{aligned}
(a\,\vec{e}_1 + b\,\vec{e}_2)(c\,\vec{e}_1 + d\,\vec{e}_2) &= (ac - bd)\,\vec{e}_1 + (ad + bc)\,\vec{e}_2 \\
&= (ca - db)\,\vec{e}_1 + (da + cb)\,\vec{e}_2 \\
&= (c\,\vec{e}_1 + d\,\vec{e}_2)(a\,\vec{e}_1 + b\,\vec{e}_2)\,,
\end{aligned}$$

$$\begin{aligned}
\big((a\,\vec{e}_1 + b\,\vec{e}_2)(c\,\vec{e}_1 + d\,\vec{e}_2)\big)(e\,\vec{e}_1 + f\,\vec{e}_2) &= \big((ac - bd)e - (ad + bc)f\big)\,\vec{e}_1 + \big((ac - bd)f + (ad + bc)e\big)\,\vec{e}_2 \\
&= \big(a(ce - df) - b(cf + de)\big)\,\vec{e}_1 + \big(a(cf + de) + b(ce - df)\big)\,\vec{e}_2 \\
&= (a\,\vec{e}_1 + b\,\vec{e}_2)\big((c\,\vec{e}_1 + d\,\vec{e}_2)(e\,\vec{e}_1 + f\,\vec{e}_2)\big)\,,
\end{aligned}$$

$$\begin{aligned}
(a\,\vec{e}_1 + b\,\vec{e}_2)\big((c\,\vec{e}_1 + d\,\vec{e}_2) + (e\,\vec{e}_1 + f\,\vec{e}_2)\big) &= \big(a(c + e) - b(d + f)\big)\,\vec{e}_1 + \big(a(d + f) + b(c + e)\big)\,\vec{e}_2 \\
&= \big((ac - bd) + (ae - bf)\big)\,\vec{e}_1 + \big((ad + bc) + (af + be)\big)\,\vec{e}_2 \\
&= (a\,\vec{e}_1 + b\,\vec{e}_2)(c\,\vec{e}_1 + d\,\vec{e}_2) + (a\,\vec{e}_1 + b\,\vec{e}_2)(e\,\vec{e}_1 + f\,\vec{e}_2)\,,
\end{aligned}$$

which establishes the commutativity, associativity, and distributivity for vector multiplication on $\mathbb{R}^2$. For the existence of a multiplicative identity, we observe that

$$(1\,\vec{e}_1 + 0\,\vec{e}_2)(c\,\vec{e}_1 + d\,\vec{e}_2) = \big((1)(c) - (0)(d)\big)\,\vec{e}_1 + \big((1)(d) + (0)(c)\big)\,\vec{e}_2 = c\,\vec{e}_1 + d\,\vec{e}_2\,.$$

Lastly, if $a\,\vec{e}_1 + b\,\vec{e}_2 \neq \vec{\mathbf{0}}$, then we have $a^2 + b^2 \neq 0$ and

$$(a\,\vec{e}_1 + b\,\vec{e}_2)\left[\left(\frac{a}{a^2+b^2}\right)\vec{e}_1 - \left(\frac{b}{a^2+b^2}\right)\vec{e}_2\right] = \left(\frac{a^2+b^2}{a^2+b^2}\right)\vec{e}_1 + \left(\frac{-ab+ab}{a^2+b^2}\right)\vec{e}_2 = 1\,\vec{e}_1 + 0\,\vec{e}_2\,,$$

so every nonzero vector in $\mathbb{R}^2$ has a multiplicative inverse. □

2.0.1 Notation. The complex numbers are denoted by $\mathbb{C}$. Traditionally, one renames the standard basis vectors $\vec{e}_1, \vec{e}_2 \in \mathbb{R}^2$ as $1, \mathrm{i} \in \mathbb{C}$, so $a + b\,\mathrm{i} := a\,\vec{e}_1 + b\,\vec{e}_2$ and the multiplicative identity is $1 := 1\,\vec{e}_1 + 0\,\vec{e}_2$. With this notation, it is enough to remember the identity

$$\mathrm{i}^2 = (0+\mathrm{i})(0+\mathrm{i}) = \big((0)(0) - (1)(1)\big) + \big((0)(1) + (1)(0)\big)\mathrm{i} = -1\,.$$

When a single symbol $z := a + b\,\mathrm{i}$ represents a complex number, its ***real part*** is the real number $a := \mathrm{Re}(z)$ and its ***imaginary part*** is the real number $b := \mathrm{Im}(z)$. We identify $\mathbb{R}$ with the subset of complex numbers whose the imaginary part is zero.

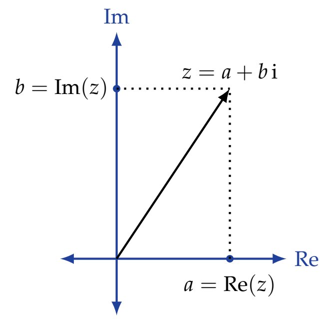

Figure 2.1: Points in the complex plane correspond to complex numbers

2.0.2 Problem. Determine the square roots of $-7 - 24\,\mathrm{i}$.

Solution. Suppose that the complex number $a + b\,\mathrm{i}$, where $a, b \in \mathbb{R}$, satisfies the equation $-7 - 24\,\mathrm{i} = (a + b\,\mathrm{i})^2 = (a^2 - b^2) + 2ab\,\mathrm{i}$. It follows that $a^2 - b^2 = -7$ and $2ab = -24$, so we have $a = -12/b$ and $\left(-\frac{12}{b}\right)^2 - b^2 = -7$. Multiplying by b^2 and gathering terms gives $0 = b^4 - 7b^2 - 144 = (b^2 - 16)(b^2 + 9) = (b-4)(b+4)(b^2+9)$. Since $b \in \mathbb{R}$, we deduce that $b = \pm 4$ and $a = \mp 3$, which means that the square roots are $3 - 4\,\mathrm{i}$ and $-3 + 4\,\mathrm{i}$. □

The methods used to find the square roots in the previous problem generalizes to all complex numbers.

2.0.3 Proposition. *For any $z \in \mathbb{C}$, there exists $w \in \mathbb{C}$ such that $w^2 = z$. Moreover, the additive inverse $-w$ also satisfies this quadratic equation.*

Proof. Suppose that $z = a + b\,\mathrm{i}$ where $a, b \in \mathbb{R}$. Consider $w = x + y\,\mathrm{i}$ such that $x, y \in \mathbb{R}$ and $(x^2 - y^2) + 2xy\,\mathrm{i} = (x + y\,\mathrm{i})^2 = w^2 = z = a + b\,\mathrm{i}$. It follows that $x^2 - y^2 = a$ and $2xy = b$, so we have $x = \frac{b}{2y}$ and $\left(\frac{b}{2y}\right)^2 - y^2 = a$. Multiplying by y^2 and gathering terms gives

$$\begin{aligned} 0 = y^4 + ay^2 - \left(\tfrac{b}{2}\right)^2 &= \left(y^2 - \left(\tfrac{-a+\sqrt{a^2+b^2}}{2}\right)\right)\left(y^2 - \left(\tfrac{-a-\sqrt{a^2+b^2}}{2}\right)\right) \\ &= \left(y - \sqrt{\tfrac{-a+\sqrt{a^2+b^2}}{2}}\right)\left(y + \sqrt{\tfrac{-a+\sqrt{a^2+b^2}}{2}}\right)\left(y^2 + \tfrac{a+\sqrt{a^2+b^2}}{2}\right), \end{aligned}$$

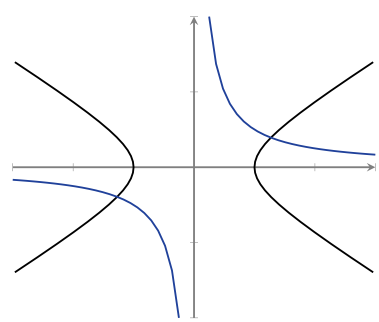

Figure 2.2: Intersection of the hyperbolas $x^2 - y^2 = a^2$ and $xy = b$ when $b > 0$

because $\sqrt{a^2+b^2} \geqslant \sqrt{a^2} = |a| \geqslant a$. Since $y \in \mathbb{R}$, we conclude that

$$y = \pm\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}\,,$$

$$x = \frac{b}{2y} = \pm\frac{\operatorname{sgn}(b)|b|}{2}\sqrt{\frac{2}{-a+\sqrt{a^2+b^2}}} = \pm\operatorname{sgn}(b)\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$$

and $w = \pm\left(\operatorname{sgn}(b)\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} + \sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}\,\mathrm{i}\right)$. □

The ***signum*** function of a real number x is defined to be

$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geqslant 0. \end{cases}$$

For all $x \in \mathbb{R}$, we have $x = \operatorname{sgn}(x)\ |x|$.

Finding all of the roots of any quadratic polynomial with complex coefficients requires only one more ingredient.

2.0.4 Problem. Solve the equation $t^2 + (-1+2\,\mathrm{i})t + (1+5\,\mathrm{i}) = 0$.

Solution. Completing the square yields

$$0 = t^2 + (-1+2\,\mathrm{i})t + (1+5\,\mathrm{i}) = \big(t^2 + 2(\mathrm{i}-\tfrac{1}{2})t + (\mathrm{i}-\tfrac{1}{2})^2\big) - \big(\mathrm{i}-\tfrac{1}{2}\big)^2 + (1+5\,\mathrm{i})\,,$$

so we have $\big(t + (\mathrm{i}-\frac{1}{2})\big)^2 = \big(-1-\mathrm{i}+\frac{1}{4}\big) - 5\mathrm{i} - 1 = \frac{1}{4}(-7-24\,\mathrm{i})$. Problem 2.0.2 shows that the square roots of $-7-24\,\mathrm{i}$ are $\pm(3-4\,\mathrm{i})$, so $t + \big(\mathrm{i}-\frac{1}{2}\big) = \frac{1}{2}(\pm 3 \mp 4\,\mathrm{i})$ and t equals $2-3\,\mathrm{i}$ or $-1+\mathrm{i}$. □

The existence of roots for polynomials over the complex numbers turns out to be the quintessential trait for this field of scalars.

2.0.5 Theorem (Fundamental theorem of algebra)**.** *Every non-constant polynomial in one variable with coefficients in the complex numbers has a complex root. More explicitly, for any positive integer n and any complex numbers $z_0, z_1, \dots, z_{n-1} \in \mathbb{C}$, there exists $w \in \mathbb{C}$ such that*

$$w^n + z_{n-1}\,w^{n-1} + z_{n-2}\,w^{n-2} + \cdots + z_1\,w + z_0 = 0\,.$$

Comment on the proof. Despite its name, there is no algebraic proof. Any proof must use the "completeness" of the real numbers. □

Exercises

2.0.6 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. Each complex number corresponds to exactly one vector in $\mathbb{R}^2$.
ii. Real numbers are not complex numbers.
iii. The imaginary part of a complex number is a real number.
iv. The real number -1 has a unique square root in $\mathbb{C}$.

2.0.7 Problem. Show that complex multiplication is compatible with scalar multiplication; for all $a, b, c \in \mathbb{R}$, we have $c(a+b\,\mathrm{i}) = ac + bc\,\mathrm{i}$.

2.0.8 Problem. Determine every complex number such that its multiplicative inverse equals its additive inverse.

2.0.9 Problem (Impossibility of order)**.** Show that the following two conditions cannot both be satisfied.

- For all $z \in \mathbb{C}$, one and only one of the relations $z > 0$, $z = 0$, and $-z > 0$ is valid.
- If $w > 0$ and $z > 0$, then $w + z > 0$ and $wz > 0$.

2.0.10 Problem (Properties of the complex conjugate)**.** For any number $z = x + y\,\mathrm{i} \in \mathbb{C}$ with $x, y \in \mathbb{R}$, its ***complex conjugate*** is the number $\overline{z} := a - b\,\mathrm{i} \in \mathbb{C}$. Geometrically, the complex conjugate is the reflection of the corresponding vector in the real axis. For all $z, w \in \mathbb{C}$, establish the following properties of the complex conjugate.

i. $\overline{z + w} = \overline{z} + \overline{w}$

ii. $\overline{zw} = \overline{z}\,\overline{w}$

iii. $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $\mathrm{Im}(z) = \frac{1}{2\mathrm{i}}(z - \overline{z})$

iv. $z\,\overline{z} = |z|^2$

v. $z = \overline{z}$ if and only if z is a real number.

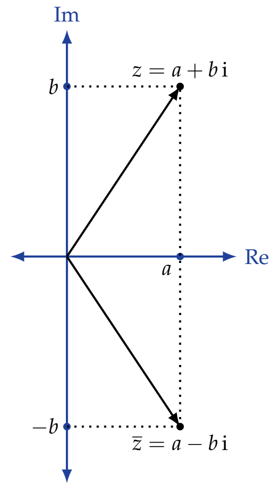

Figure 2.3: Complex conjugate

2.0.11 Problem. Prove that the three numbers $z_1, z_2, z_3 \in \mathbb{C}$ with $z_1 \neq z_2$ are collinear if and only if $\dfrac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}$.

2.0.12 Problem. Prove that the four numbers $z_1, z_2, z_3, z_4 \in \mathbb{C}$ with $z_1 \neq z_4$, $z_2 \neq z_3$, and not all on the same line, lie on a circle if and only if $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in \mathbb{R}$.

2.0.13 Problem. Find all $w \in \mathbb{C}$ such that

$$w^2 + (-1 + 5\,\mathrm{i})w + (-10 + 5\,\mathrm{i}) = 0\,.$$

Express your solution(s) in the form $w = a + b\,\mathrm{i}$ where $a, b \in \mathbb{Z}$.

2.1 Complex Geometry

HOW CAN WE VISUALIZE MULTIPLICATION OF COMPLEX NUMBERS? For any complex number $z := a + b\,\mathrm{i}$ where $a, b \in \mathbb{R}$, the ***absolute value*** $|z| := \sqrt{a^2 + b^2}$ is the magnitude of the underlying vector in $\mathbb{R}^2$. The ***argument*** of z is the angle $\arg(z)$ that the vector in $\mathbb{R}^2$ makes with the positive real axis. Because it is measured in radians, the argument $\arg(z)$ can be changed be any integer multiple of 2π without changing the angle. The absolute value and argument determine a complex number: $a = |z| \cos\big(\arg(z)\big)$ and $b = |z| \sin\big(\arg(z)\big)$.

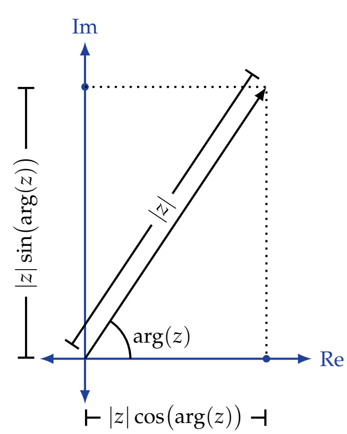

Figure 2.4: Polar form of $z \in \mathbb{C}$

2.1.0 Problem. Find the absolute value and argument of $z := -1 + \mathrm{i}$.

Solution. By definition, we have $|z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$. Since $\tan\bigl(\arg(z)\bigr) = \frac{1}{-1} = -1$ and $\operatorname{Re}(z) < 0$, it follows that

$$\arg(z) = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}\,. \qquad \square$$

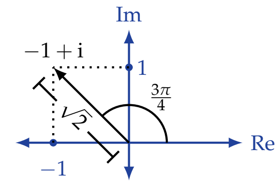

Figure 2.5: Polar form of $-1 + \mathrm{i} \in \mathbb{C}$

Multiplication of complex numbers is especially slick when using this polar representation.

2.1.1 Proposition (Geometry of complex multiplication)**.** *For any two complex numbers z and w, we have*

$$|zw| = |z|\,|w| \qquad \textit{and} \qquad \arg(zw) = \arg(z) + \arg(w)\,.$$

Thus, multiplication by a complex number is a counterclockwise rotation by the argument and a rescaling by the absolute value.

Proof. Setting $\theta := \arg(z)$ and $\phi := \arg(w)$ gives

$$z = |z|\,\bigl(\cos(\theta) + \sin(\theta)\,\mathrm{i}\bigr) \qquad \text{and} \qquad w = |w|\,\bigl(\cos(\phi) + \sin(\phi)\,\mathrm{i}\bigr)\,.$$

Hence, the addition formula for trigonometric functions gives

$$\begin{aligned} zw &= \Bigl(|z|\,\bigl(\cos(\theta) + \sin(\theta)\,\mathrm{i}\bigr)\Bigr)\Bigl(|w|\,\bigl(\cos(\phi) + \sin(\phi)\,\mathrm{i}\bigr)\Bigr) \\ &= |z|\,|w|\,\Bigl(\bigl(\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)\bigr) + \bigl(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)\bigr)\,\mathrm{i}\Bigr) \\ &= |z|\,|w|\,\bigl(\cos(\theta+\phi) + \sin(\theta+\phi)\,\mathrm{i}\bigr)\,. \qquad \square \end{aligned}$$

Examining the m-fold product of a complex number, as a special case, is peculiarly worthwhile.

2.1.2 Corollary (De Moivre formula)**.** *For any nonnegative integer m and any complex number $z := r\bigl(\cos(\theta) + \sin(\theta)\,\mathrm{i}\bigr)$, it follows that $|z^m| = |z|^m$, $\arg(z^m) = m\arg(z)$, and $z^m = r^m\bigl(\cos(m\theta) + \sin(m\theta)\,\mathrm{i}\bigr)$.*

The lowercase "theta" θ is the eighth letter in the Greek alphabet and is often used to denote an angle.

Inductive proof. When $m = 0$, we have $z^0 = 1$, so $\bigl|z^0\bigr| = 1 = |z|^0$ and $\arg(z^0) = 0 = 0\arg(z)$. Thus, the base case of the induction holds. Assume that $z^{m-1} = r^{m-1}\bigl(\cos\bigl((m-1)\theta\bigr) + \sin\bigl((m-1)\theta\bigr)\mathrm{i}\bigr)$ for any positive integer m. The geometry of complex multiplication and the induction hypothesis imply that

$$\begin{aligned} |z^m| = |zz^{m-1}| = |z|\,|z^{m-1}| &= |z|\,|z|^{m-1} = |z|^m\,, \\ \arg(z^m) = \arg(zz^{m-1}) &= \arg(z) + \arg(z^{m-1}) \\ &= \arg(z) + (m-1)\arg(z) = m\arg(z)\,. \qquad \square \end{aligned}$$

Next two problems hint at some of the beautiful applications of the De Moirve formula.

2.1.3 Problem. Express $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Solution. De Moivre's Formula for $r = 1$ and $m = 3$ gives

$$\cos(3\theta) + \sin(3\theta)\,\mathrm{i} = \big(\cos(\theta) + \sin(\theta)\,\mathrm{i}\big)^3 = \cos^3(\theta) + 3\cos^2(\theta)\sin(\theta)\,\mathrm{i} - 3\cos(\theta)\sin^2(\theta) - \sin^3(\theta)\,\mathrm{i}\,,$$

so taking real parts gives $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$. □

2.1.4 Problem. Find three solutions to the equation $z^3 = 1$.

Solution. Since $1 = 1\big(\cos(2\pi) + \sin(2\pi)\,\mathrm{i}\big)$, consider the complex number $z = \cos\big(\frac{2\pi k}{3}\big) + \sin\big(\frac{2\pi k}{3}\big)\,\mathrm{i}$ where $0 \leqslant k \leqslant 2$. The De Moivre formula shows that $|z^3| = 1$ and $\arg(z^3) = 2\pi k$, so the solutions are 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i}$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i}$. □

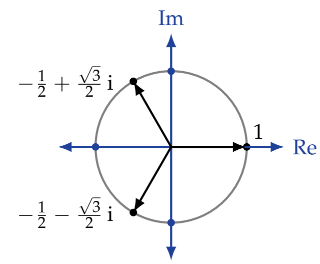

Figure 2.6: Third roots of unit

WHAT IS THE EXPONENTIAL OF A COMPLEX NUMBER? For all $x \in \mathbb{R}$, the exponential function $\exp(x) = e^x$ equals the power series

$$\sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

To extend this function to $\mathbb{C}$, it would be natural to define $e^{y\,\mathrm{i}}$, for all $y \in \mathbb{R}$, to be $1 + \frac{(y\,\mathrm{i})}{1!} + \frac{(y\,\mathrm{i})^2}{2!} + \frac{(y\,\mathrm{i})^3}{3!} + \cdots$. To make this rigorous, we would need to discuss convergence in $\mathbb{C}$ which we will not do. Nevertheless, a slight rearrangement and recognizing the power series for $\cos(y)$ and $\sin(y)$ yield

$$\exp(y\,\mathrm{i}) = e^{y\,\mathrm{i}} = \big(1 - \tfrac{y^2}{2!} + \tfrac{y^4}{4!} - \cdots\big) + \big(y - \tfrac{y^3}{3!} + \tfrac{y^5}{5!} - \cdots\big)\,\mathrm{i} = \cos(y) + \sin(y)\,\mathrm{i}\,.$$

This discussion can be strengthened to prove the next theorem.

2.1.5 Theorem (Euler formula). *For any complex number $z = x + y\,\mathrm{i}$ where $x, y \in \mathbb{R}$, the complex exponential function satisfies*

$$\exp(z) = e^z = e^x\big(\cos(y) + \sin(y)\,\mathrm{i}\big)\,.$$

In particular, we have $|e^z| = e^x = \exp\big(\mathrm{Re}(z)\big)$, and $\arg(e^z) = y = \mathrm{Im}(z)$.

The equation $e^{\pi\,\mathrm{i}} + 1 = 0$ unites the five most important numbers and the seven most important symbols in mathematics.

We summarize the main properties of the exponential function.

2.1.6 Proposition (Properties of complex exponential function)**.**

i. *The exponential function is multiplicative: for all $z, w \in \mathbb{C}$, we have $e^{z+w} = e^z e^w$.*
ii. *The exponential function is never zero: for all $z \in \mathbb{C}$, we have $e^z \neq 0$.*
iii. *We have $e^z = 1$ if and only if $z = 2\pi m\,\mathrm{i}$ for some $m \in \mathbb{Z}$.*

Proof.

i. The Euler formula and the multiplicative property of the real exponential function give

$$\begin{aligned} \big|e^{z+w}\big| &= \exp\big(\mathrm{Re}(z+w)\big) = \exp\big(\mathrm{Re}(z) + \mathrm{Re}(w)\big) = |e^z|\,|e^w|\,, \\ \arg(e^{z+w}) &= \mathrm{Im}(z+w) = \mathrm{Im}(z) + \mathrm{Im}(w) = \arg(e^z) + \arg(e^w)\,, \end{aligned}$$

so the geometry of multiplication shows that $e^{z+w} = e^z e^w$.

ii. Part (i) implies that $e^z e^{-z} = e^{z-z} = e^0 = 1$, so the complex number e^z has a multiplicative inverse and $e^z \neq 0$.

iii. Suppose $z = x + y\,\mathrm{i}$ where $x, y \in \mathbb{R}$. The equation $e^z = 1$, together with the Euler formula, implies that $e^x = 1$ and $x = 0$. Hence, we have $1 = e^{y\,\mathrm{i}} = \cos(y) + \sin(y)\,\mathrm{i}$, so $\cos(y) = 1$ and $\sin(y) = 0$ which means that $y = 2\pi m$ for some $m \in \mathbb{Z}$. □

Exercises

2.1.7 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The absolute value of a complex number is a real number.

ii. The argument of a complex number is a real number.

iii. The argument of a sum of complex numbers equals the sum of the arguments.

iv. The exponential function is periodic.

2.1.8 Problem. Compute $(1+\mathrm{i})^{1000}$.

2.1.9 Problem. Given two complex numbers $z := r\big(\cos(\theta) + \sin(\theta)\,\mathrm{i}\big)$ with $r, \theta \in \mathbb{R}$ and $w := s\big(\cos(\phi) + \sin(\phi)\,\mathrm{i}\big) \neq 0$ with $s, \phi \in \mathbb{R}$, show that

$$\frac{z}{w} = \frac{r}{s}\big(\cos(\theta - \phi) + \sin(\theta - \phi)\,\mathrm{i}\big)\,.$$

2.1.10 Problem. Simplify $\dfrac{(1-\mathrm{i})^{10}(\sqrt{3}+\mathrm{i})^5}{(-1-\sqrt{3}\,\mathrm{i})^{10}}$.

2.1.11 Problem. For $z, w \in \mathbb{C}$, show that

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|)^2\,.$$

2.1.12 Problem. If $z, w \in \mathbb{C}$ satisfy $|z| = |w| = 1$ and $zw \neq -1$, then demonstrate that

$$\frac{z+w}{1+zw} \in \mathbb{R}\,.$$

2.1.13 Problem. Find all $z \in \mathbb{C}$ such that $|z| = 1$ and

$$\left|\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right| = 1\,.$$

2.1.14 Problem. For $0 < \theta < 2\pi$, calculate the polar representation of $z = 1 + \cos(\theta) + \sin(\theta)\,\mathrm{i}$.

2.1.15 Problem. Prove the following related identities.

i. For all $z, w \in \mathbb{C}$, show that $|zw| = |z|\,|w|$ by using complex conjugates.

ii. For all $a, b, c, d \in \mathbb{R}$, show that

$$(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2)\,.$$

2.1.16 Problem. Consider the complex numbers $z := -3 - \sqrt{3}\,\mathrm{i}$ and $w := -1 + \sqrt{3}\,\mathrm{i}$.

i. Find zw and z/w. Give your answer in the form $x + y\,\mathrm{i}$ where $x, y \in \mathbb{R}$.

ii. Put z and w into polar form $re^{\theta\,\mathrm{i}} = r\big(\cos(\theta) + \sin(\theta)\,\mathrm{i}\big)$. Find zw and z/w using the polar form and verify that you get the same answer as in part (a).

2.1.17 Problem. For any complex number $z := r\big(\cos(\theta) + \sin(\theta)\,\mathrm{i}\big)$ where $r, \theta \in \mathbb{R}$, show that the n-th roots are

$$r^{1/n}\Big(\cos\big(\tfrac{\theta+2\pi k}{n}\big) + \sin\big(\tfrac{\theta+2\pi k}{n}\big)\,\mathrm{i}\Big)$$

for $k \in \mathbb{N}$ with $k < n$.

2.1.18 Problem. For all $z \in \mathbb{C}$, we define

$$\sin(z) := \frac{e^{z\,\mathrm{i}} - e^{-z\,\mathrm{i}}}{2\,\mathrm{i}} \qquad \text{and} \qquad \cos(z) := \frac{e^{z\,\mathrm{i}} + e^{-z\,\mathrm{i}}}{2}\,.$$

For $z, w \in \mathbb{C}$, prove the following identities.

i. $\sin^2(z) + \cos^2(z) = 1$;

ii. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$;

iii. $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$.

2.2 The Cross Product

HOW CAN WE MULTIPLY VECTORS IN $\mathbb{R}^3$? Although quite different than multiplication of scalars, there is a useful vector product on $\mathbb{R}^3$. Geometrically, we must associate a magnitude (or nonnegative real number) and a direction (or unit vector) to any pair $\vec{v}, \vec{w} \in \mathbb{R}^3$.

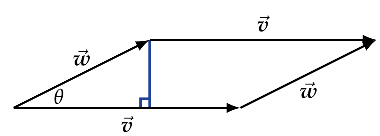

Figure 2.7: Area of parallelogram

- The associated magnitude is the area of the parallelogram formed by the two vectors. If $0 \leqslant \theta \leqslant \pi$ denotes the angle between $\vec{v}$ and $\vec{w}$, then the height of the parallelogram equals $\|\vec{w}\|\sin(\theta)$ and the area of the parallelogram equals $\|\vec{v}\|\,\|\vec{w}\|\sin(\theta)$.
- To visualize the associated direction, position the vectors $\vec{v}$ and $\vec{w}$ so that their tails coincide. Orient your right hand so that its edge and all of your finger point in the same direction as $\vec{v}$. With a flat hand, extend your thumb so that it is perpendicular to your fingers. When curling your fingers through the angle θ, from $\vec{v}$ to $\vec{w}$, your thumb points in the direction of the unit vector $\vec{n}$. By construction, the vector $\vec{n}$ is perpendicular to the plane containing $\vec{v}$ and $\vec{w}$, and this ***right-hand rule*** chooses one side of the plane.

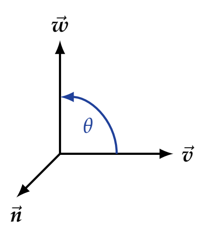

Figure 2.8: The right-hand rule: if $\vec{v}$ and $\vec{w}$ lie in the plane the page, then $\vec{n}$ points out of the page toward the reader.

2.2.0 Definition. For any two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, the following two definitions of the ***cross product*** $\vec{v} \times \vec{w} \in \mathbb{R}^3$ are equivalent.

(geometric) For any two vectors $\vec{v}$ and $\vec{w}$ that are not parallel, we set

$$\vec{v} \times \vec{w} := \big(\|\vec{v}\|\,\|\vec{w}\| \sin(\theta)\big)\,\vec{n}$$

where $0 < \theta < \pi$ is the angle between $\vec{v}$ and $\vec{w}$, and $\vec{n}$ is the unit vector determined by the right-hand rule. For parallel vectors $\vec{v}$ and $\vec{w}$, we set $\vec{v} \times \vec{w} := \vec{0}$.

(algebraic) Given the two vectors $\vec{v} := v_1\,\vec{e}_1 + v_2\,\vec{e}_2 + v_3\,\vec{e}_3$ and $\vec{w} := w_1\,\vec{e}_1 + w_2\,\vec{e}_2 + w_3\,\vec{e}_3$, we set

$$\vec{v} \times \vec{w} := (v_2 w_3 - v_3 w_2)\,\vec{e}_1 + (v_3 w_1 - v_1 w_3)\,\vec{e}_2 + (v_1 w_2 - v_2 w_1)\,\vec{e}_3\,.$$

The geometric definition was made in 1878 by W.K. Clifford. The name "cross product" and the notation were introduced in 1881 by J.W. Gibbs.

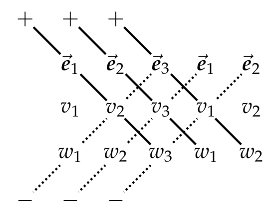

Figure 2.9: The cross product is the sum of the product along the solid diagonals minus the sum of the produces along the dotted diagonals. This mnemonic is named after P.F. Sarrus.

2.2.1 Problem. Compute $\vec{e}_j \times \vec{e}_k$ for all $1 \leqslant j < k \leqslant 3$.

Geometric solution. Since the standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are pairwise perpendicular unit vectors, the geometric definition of the cross product gives

$$\begin{aligned} \vec{e}_1 \times \vec{e}_2 &= \|\vec{e}_1\|\,\|\vec{e}_2\| \sin\big(\tfrac{\pi}{2}\big)\vec{e}_3 = \vec{e}_3\,, \\ \vec{e}_1 \times \vec{e}_3 &= \|\vec{e}_1\|\,\|\vec{e}_3\| \sin\big(\tfrac{\pi}{2}\big)(-\vec{e}_2) = -\vec{e}_2\,, \\ \vec{e}_2 \times \vec{e}_3 &= \|\vec{e}_2\|\,\|\vec{e}_3\| \sin\big(\tfrac{\pi}{2}\big)\vec{e}_1 = \vec{e}_1\,. \end{aligned}$$ □

Algebraic solution. The algebraic definition of the cross product yields

$$\begin{aligned} \vec{e}_1 \times \vec{e}_2 &= \big((0)(0) - (0)(1)\big)\,\vec{e}_1 + \big((0)(0) - (1)(0)\big)\,\vec{e}_2 + \big((1)(1) - (0)(0)\big)\,\vec{e}_3 = \vec{e}_3\,, \\ \vec{e}_1 \times \vec{e}_3 &= \big((0)(1) - (0)(0)\big)\,\vec{e}_1 + \big((0)(0) - (1)(1)\big)\,\vec{e}_2 + \big((1)(0) - (0)(0)\big)\,\vec{e}_3 = -\vec{e}_2\,, \\ \vec{e}_2 \times \vec{e}_3 &= \big((1)(1) - (0)(0)\big)\,\vec{e}_1 + \big((0)(0) - (0)(1)\big)\,\vec{e}_2 + \big((0)(0) - (1)(0)\big)\,\vec{e}_3 = \vec{e}_1\,, \end{aligned}$$

because $\vec{e}_j$ has 1 in the j-th entry and zeros elsewhere. □

2.2.2 Proposition (Properties of the cross product)**.** *For any three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ and any scalar $c \in \mathbb{R}$, we have the following:*

(anti-commutativity) $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$

(compatibility with scalar multiplication) $\vec{v} \times (c\,\vec{w}) = c(\vec{v} \times \vec{w}) = (c\,\vec{v}) \times \vec{w}$

(distributivity) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

Geometric proof. For all $\vec{v} \in \mathbb{R}^3$, we have $\vec{v} \times \vec{v} = \vec{0}$ because $\vec{v}$ is parallel to itself. The right-hand rule tells us that $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ point in opposite directions. Since the magnitudes of $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ both equal the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$, we have $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$ which proves anti-commutativity property.

By anti-commutativity, we may assume that $c \geqslant 0$ by interchanging $\vec{v}$ and $\vec{w}$ if necessary. Hence, the relation between magnitude and scalar multiplication gives

$$\|c\vec{v}\|\,\|\vec{w}\| \sin(\theta) = c\,\|\vec{v}\|\,\|\vec{w}\| \sin(\theta) = \|\vec{v}\|\,\|c\vec{w}\| \sin(\theta)\,,$$

whence the compatibility with scalar multiplication follows.

We postpone the proof of distributivity until we have introduced the triple product; see Problem 3.0.8. □

Algebraic proof. The algebraic definition of the cross product gives

$$\begin{aligned}\vec{v} \times \vec{w} &= (v_2 w_3 - v_3 w_2)\,\vec{e}_1 + (v_3 w_1 - v_1 w_3)\,\vec{e}_2 + (v_1 w_2 - v_2 w_1)\,\vec{e}_3 \\ &= -\big((v_3 w_2 - v_2 w_3)\,\vec{e}_1 + (v_1 w_3 - v_3 w_1)\,\vec{e}_2 + (v_2 w_1 - v_1 w_2)\,\vec{e}_3\big) = -(\vec{w} \times \vec{v})\,, \\ \vec{v} \times (c\,\vec{w}) &= (c v_2 w_3 - c v_3 w_2)\,\vec{e}_1 + (c v_3 w_1 - c v_1 w_3)\,\vec{e}_2 + (c v_1 w_2 - c v_2 w_1)\,\vec{e}_3 = (c\,\vec{v}) \times \vec{w} \\ &= c\big((v_3 w_2 - v_2 w_3)\,\vec{e}_1 + (v_1 w_3 - v_3 w_1)\,\vec{e}_2 + (v_2 w_1 - v_1 w_2)\,\vec{e}_3\big) = c(\vec{v} \times \vec{w})\,, \\ \vec{u} \times (\vec{v} + \vec{w}) &= \big(u_2(v_3 + w_3) - u_3(v_2 + w_2)\big)\,\vec{e}_1 + \big(u_3(v_1 + w_1) - u_1(v_3 + w_3)\big)\,\vec{e}_2 \\ &\qquad + \big(u_1(v_2 + w_2) - u_2(v_1 + w_1)\big)\,\vec{e}_3 \\ &= (u_2 v_3 - u_3 v_2)\,\vec{e}_1 + (u_3 v_1 - u_1 v_3)\,\vec{e}_2 + (u_1 v_2 - u_2 v_1)\,\vec{e}_3 \\ &\qquad + (u_2 w_3 - u_3 w_2)\,\vec{e}_1 + (u_3 w_1 - u_1 w_3)\,\vec{e}_2 + (u_1 w_2 - u_2 w_1)\,\vec{e}_3 = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}\,,\end{aligned}$$

which establishes the three properties of the cross product. □

WHY DO THE TWO DEFINITIONS OF THE CROSS PRODUCT AGREE? For any $\vec{u} \in \mathbb{R}^3$, the anti-commutativity of the cross product means that $\vec{u} \times \vec{u} = -(\vec{u} \times \vec{u})$, which implies that $2(\vec{u} \times \vec{u}) = \vec{0}$ and $\vec{u} \times \vec{u} = \vec{0}$. For all $\vec{v} := v_1\,\vec{e}_1 + v_2\,\vec{e}_2 + v_3\,\vec{e}_3$ and $\vec{w} := w_1\,\vec{e}_1 + w_2\,\vec{e}_2 + w_3\,\vec{e}_3$, the properties of cross product establish that

$$\begin{aligned}\vec{v} \times \vec{w} &= (v_1\,\vec{e}_1 + v_2\,\vec{e}_2 + v_3\,\vec{e}_3) \times (w_1\,\vec{e}_1 + w_2\,\vec{e}_2 + w_3\,\vec{e}_3) \\ &= \quad v_1 w_1(\vec{0}\qquad) + v_1 w_2(\vec{e}_1 \times \vec{e}_2) + v_1 w_3(\vec{e}_1 \times \vec{e}_3) \\ &\quad - v_2 w_1(\vec{e}_1 \times \vec{e}_2) + v_2 w_2(\vec{0}\qquad) + v_2 w_3(\vec{e}_2 \times \vec{e}_3) \\ &\quad - v_3 w_1(\vec{e}_1 \times \vec{e}_3) - v_3 w_2(\vec{e}_2 \times \vec{e}_3) + v_3 w_3(\vec{0}\qquad)\,.\end{aligned}$$

$\times$	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$
$\vec{e}_1$	$\vec{0}$	$\vec{e}_3$	$-\vec{e}_2$
$\vec{e}_2$	$-\vec{e}_3$	$\vec{0}$	$\vec{e}_1$
$\vec{e}_3$	$\vec{e}_2$	$-\vec{e}_1$	$\vec{0}$

Figure 2.10: Multiplication table for the cross product of the standard basis vectors

Hence, it suffices to know that geometric and algebraic definitions of the cross products agree on $\vec{e}_j \times \vec{e}_k$ where $1 \leqslant j < k \leqslant 3$. Fortuitously, Problem 2.2.1 already does this.

Exercises

2.2.3 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The cross product of two vectors in $\mathbb{R}^3$ is another vector in $\mathbb{R}^3$.
ii. The cross product is defined for any two vectors in $\mathbb{R}^n$.
iii. The cross product of two vectors is zero if and only if one vector is parallel to the other.
iv. The cross product is commutative.

2.2.4 Problem. For all vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, simplify the following expressions.

i. $(\vec{v} - \vec{w}) \times (\vec{v} + \vec{w})$
ii. $(\vec{u} + \vec{v} + \vec{w}) \times (\vec{v} + \vec{w})$

2.2.5 Problem. Demonstrate that the cross product is not associative by exhibiting three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ such that

$$(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})\,.$$

2.2.6 Problem. Show that the cross product is not cancellative by exhibiting three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ such that $\vec{u} \neq \vec{0}$, $\vec{v} \neq \vec{w}$, and $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$.

2.2.7 Problem. Find the area of the triangle with edges corresponding to the vectors $\vec{v} := 2\,\vec{e}_1 + \vec{e}_2 - 3\,\vec{e}_3$, $\vec{w} := \vec{e}_1 + 3\,\vec{e}_2 + 2\,\vec{e}_3$, and $\vec{w} - \vec{v}$.

2.2.8 Problem. For the position vector $\vec{r}$ locating a particle, having mass m and velocity $\vec{v}$, relative to a fixed point, the ***angular momentum*** $\vec{\ell}$ of the particle relative to the fixed is defined to be $\vec{\ell} := m(\vec{r} \times \vec{v})$. A 2 kg object has position vector $\vec{r} = (2\,\vec{e}_1 + 4\,\vec{e}_2 - 3\,\vec{e}_3)$ m and velocity vector $\vec{v} = (-6\,\vec{e}_1 + 3\,\vec{e}_2 + 3\,\vec{e}_3)$ m $\cdot$ s^{-1}. Determine the angular momentum of the object about the origin.

2.2.9 Problem. When a force $\vec{F}$ is applied to a particle and $\vec{r}$ is the position vector locating the particle relative to a fixed point, the ***torque*** on the particle relative to the fixed point is defined to be $\vec{\tau} := \vec{r} \times \vec{F}$. A particle moves in $\mathbb{R}^3$ while a force acts on it. When the particle has the position vector $\vec{r} = (2\,\vec{e}_1 - 5\,\vec{e}_2 + \vec{e}_3)$ m, the force is given by $\vec{F} = (F_1\,\vec{e}_1 + 4\,\vec{e}_2 - 3\,\vec{e}_3)$ N, and the torque about the origin is $\vec{\tau} = (11\,\vec{e}_1 + 5\,\vec{e}_2 + \tau_3\,\vec{e}_3)$ N $\cdot$ m. Find the scalars F_1 and τ_3.

2.2.10 Problem (Law of sines)**.** If a, b, and c are the lengths of the sides in a triangle and α, β, and γ are the opposite angles, then prove that

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}\,,$$

using the cross product.

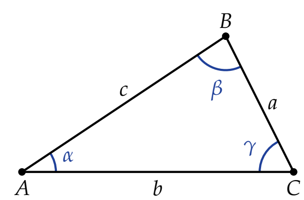

Figure 2.11: The angles α, β, γ in the triangle are opposite to the sides a, b, c.

2.3 Quaternions*

HOW IS VECTOR MULTIPLICATION ON $\mathbb{R}^4$ DEFINED? The quaternions may viewed as $\mathbb{R}^4$ with a vector product.

The quaternions were introduced in 1843 by W.R. Hamilton.

2.3.0 Theorem (Quaternions)**.** *The coordinate space $\mathbb{R}^4$, together with the vector multiplication defined by*

$$\begin{aligned}\vec{v}\vec{w} &= (v_1\,\vec{e}_1 + v_2\,\vec{e}_2 + v_3\,\vec{e}_3 + v_4\,\vec{e}_4)(w_1\,\vec{e}_1 + w_2\,\vec{e}_2 + w_3\,\vec{e}_3 + w_4\,\vec{e}_4) \\ &= (v_1w_1 - v_2w_2 - v_3w_3 - v_4w_4)\,\vec{e}_1 + (v_1w_2 + v_2w_1 + v_3w_4 - v_4w_3)\,\vec{e}_2 \\ &\qquad + (v_1w_3 - v_2w_4 + v_3w_1 + v_4w_2)\,\vec{e}_3 + (v_1w_4 + v_2w_3 - v_3w_2 + v_4w_1)\,\vec{e}_4\,,\end{aligned}$$

satisfies the defining properties a field of scalars, except that multiplication is not commutative.

Proof. Since Section 1.2 already demonstrates that vector addition in $\mathbb{R}^4$ satisfies the four properties for a field of scalars that only involve addition, it suffices to verify the four of the properties that involve vector multiplication. Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^4$ be arbitrary vectors. The associativity and distributivity of addition and multiplication on the real numbers give

$$\begin{aligned}
\vec{u}(\vec{v}+\vec{w}) &= \big(u_1(v_1+w_1) - u_2(v_2+w_2) - u_3(v_3+w_3) - u_4(v_4+w_4)\big)\,\vec{e}_1 \\
&\qquad + \big(u_1(v_2+w_2) + u_2(v_1+w_1) + u_3(v_4+w_4) - u_4(v_3+w_3)\big)\,\vec{e}_2 \\
&\qquad + \big(u_1(v_3+w_3) - u_2(v_4+w_4) + u_3(v_1+w_1) + u_4(v_2+w_2)\big)\,\vec{e}_3 \\
&\qquad + \big(u_1(v_4+w_4) + u_2(v_3+w_3) - u_3(v_2+w_2) + u_4(v_1+w_1)\big)\,\vec{e}_4 \\
&= \big((u_1v_1 - u_2v_2 - u_3v_3 - u_4v_4) + (u_1w_1 - u_2w_2 - u_3w_3 - u_4w_4)\big)\,\vec{e}_1 \\
&\qquad + \big((u_1v_2 + u_2v_1 + u_3v_4 - u_4v_3) + (u_1w_2 + u_2w_1 + u_3w_4 - u_4w_3)\big)\,\vec{e}_2 \\
&\qquad + \big((u_1v_3 - u_2v_4 + u_3v_1 + u_4v_2) + (u_1w_3 - u_2w_4 + u_3w_1 + u_4w_2)\big)\,\vec{e}_3 \\
&\qquad + \big((u_1v_4 + u_2v_3 - u_3v_2 + u_4v_1) + (u_1w_4 + u_2w_3 - u_3w_2 + u_4w_1)\big)\,\vec{e}_4 \\
&= \vec{u}\vec{v} + \vec{u}\vec{w}\,,
\end{aligned}$$

$$\begin{aligned}
\vec{e}_1\vec{v} &= (1v_1 - 0v_2 - 0v_3 - 0v_4)\,\vec{e}_1 + (1v_2 + 0v_1 + 0v_4 - 0v_3)\,\vec{e}_2 \\
&\qquad + (1v_3 - 0v_4 + 0v_1 + 0v_2)\,\vec{e}_3 + (1v_4 + 0v_3 - 0v_2 + 0v_1)\,\vec{e}_4 \\
&= v_1\,\vec{e}_1 + v_2\,\vec{e}_2 + v_3\,\vec{e}_3 + v_4\,\vec{e}_4 = \vec{v}\,,
\end{aligned}$$

$$\begin{aligned}
(\vec{u}\vec{v})\vec{w} = \;& \big((u_1v_1 - u_2v_2 - u_3v_3 - u_4v_4)w_1 - (u_1v_2 + u_2v_1 + u_3v_4 - u_4v_3)w_2 \\
&\qquad - (u_1v_3 - u_2v_4 + u_3v_1 + u_4v_2)w_3 - (u_1v_4 + u_2v_3 - u_3v_2 + u_4v_1)w_4\big)\,\vec{e}_1 \\
&+ \big((u_1v_1 - u_2v_2 - u_3v_3 - u_4v_4)w_2 + (u_1v_2 + u_2v_1 + u_3v_4 - u_4v_3)w_1 \\
&\qquad + (u_1v_3 - u_2v_4 + u_3v_1 + u_4v_2)w_4 - (u_1v_4 + u_2v_3 - u_3v_2 + u_4v_1)w_3\big)\,\vec{e}_2 \\
&+ \big((u_1v_1 - u_2v_2 - u_3v_3 - u_4v_4)w_3 - (u_1v_2 + u_2v_1 + u_3v_4 - u_4v_3)w_4 \\
&\qquad + (u_1v_3 - u_2v_4 + u_3v_1 + u_4v_2)w_1 + (u_1v_4 + u_2v_3 - u_3v_2 + u_4v_1)w_2\big)\,\vec{e}_3 \\
&+ \big((u_1v_1 - u_2v_2 - u_3v_3 - u_4v_4)w_4 + (u_1v_2 + u_2v_1 + u_3v_4 - u_4v_3)w_3 \\
&\qquad - (u_1v_3 - u_2v_4 + u_3v_1 + u_4v_2)w_2 + (u_1v_4 + u_2v_3 - u_3v_2 + u_4v_1)w_1\big)\,\vec{e}_4 \\
= \;& \big(u_1(v_1w_1 - v_2w_2 - v_3w_3 - v_4w_4) - u_2(v_1w_2 + v_2w_1 + v_3w_4 - v_4w_3) \\
&\qquad - u_3(v_1w_3 - v_2w_4 + v_3w_1 + v_4w_2) - u_4(v_1w_4 + v_2w_3 - v_3w_2 + v_4w_1)\big)\,\vec{e}_1 \\
&+ \big(u_1(v_1w_2 + v_2w_1 + v_3w_4 - v_4w_3) + u_2(v_1w_1 - v_2w_2 - v_3w_3 - v_4w_4) \\
&\qquad + u_3(v_1w_4 + v_2w_3 - v_3w_2 + v_4w_1) - u_4(v_1w_3 - v_2w_4 + v_3w_1 + v_4w_2)\big)\,\vec{e}_2 \\
&+ \big(u_1(v_1w_3 - v_2w_4 + v_3w_1 + v_4w_2) - u_2(v_1w_4 + v_2w_3 - v_3w_2 + v_4w_1) \\
&\qquad + u_3(v_1w_1 - v_2w_2 - v_3w_3 - v_4w_4) + u_4(v_1w_2 + v_2w_1 + v_3w_4 - v_4w_3)\big)\,\vec{e}_3 \\
&+ \big(u_1(v_1w_4 + v_2w_3 - v_3w_2 + v_4w_1) + u_2(v_1w_3 - v_2w_4 + v_3w_1 + v_4w_2) \\
&\qquad - u_3(v_1w_2 + v_2w_1 + v_3w_4 - v_4w_3) + u_4(v_1w_1 - v_2w_2 - v_3w_3 - v_4w_4)\big)\,\vec{e}_4 \\
= \;& \vec{u}(\vec{v}\vec{w})\,,
\end{aligned}$$

which establishes distributivity, the existence of a multiplicative identity, and associativity for vector multiplication in $\mathbb{R}^4$. Lastly, if $\vec{v} \neq \vec{\mathbf{0}}$, then we have $v_1^2 + v_2^2 + v_3^2 + v_4^2 \neq 0$. Hence, commutativity of multiplication in $\mathbb{R}$ yields

$$\begin{aligned}
\vec{v}(v_1\,\vec{e}_1 - v_2\,\vec{e}_2 - v_3\,\vec{e}_3 - v_4\,\vec{e}_4) &= (v_1^2 + v_2^2 + v_3^2 + v_4^2)\,\vec{e}_1 + (-v_1v_2 + v_2v_1 - v_3v_4 + v_4v_3)\,\vec{e}_2 \\
&\qquad + (-v_1v_3 + v_2v_4 + v_3v_1 - v_4v_2)\,\vec{e}_3(-v_1v_4 - v_2v_3 + v_3v_2 - v_4v_1)\,\vec{e}_4 \\
&= (v_1^2 + v_2^2 + v_3^2 + v_4^2)\,\vec{e}_1\,,
\end{aligned}$$

so $(v_1^2 + v_2^2 + v_3^2 + v_4^2)^{-1}(v_1\,\vec{e}_1 - v_2\,\vec{e}_2 - v_3\,\vec{e}_3 - v_4\,\vec{e}_4) \in \mathbb{R}^4$ is the multiplicative inverse of the vector $\vec{v}$. □

2.3.1 Notation. The quaternions are denoted by $\mathbb{H}$. Traditionally, one renames the standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \in \mathbb{R}^2$ as $1, \mathrm{i}, \mathrm{j}, \mathrm{k} \in \mathbb{H}$, so $a + b\,\mathrm{i} + c\,\mathrm{j} + d\,\mathrm{k} := a\,\vec{e}_1 + b\,\vec{e}_2 + c\,\vec{e}_3 + d\,\vec{e}_4$ and the multiplicative identity is 1.

	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

Figure 2.12: The multiplication table for the quaternion units

2.3.2 Problem. Show that quaternion units satisfy the relations

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{i}\,\mathrm{j}\,\mathrm{k} = -1\,.$$

Solution. The definition of the multiplication in $\mathbb{H}$ implies that

$$\begin{aligned}
\mathrm{i}^2 &= \big((0)(0) - (1)(1) - (0)(0) - (0)(0)\big) + \big((0)(1) + (1)(0) + (0)(0) - (0)(0)\big)\,\mathrm{i} \\
&\quad + \big((0)(0) - (1)(0) + (0)(0) + (0)(1)\big)\,\mathrm{j} + \big((0)(0) + (1)(0) - (0)(1) + (0)(0)\big)\,\mathrm{k} = -1\,, \\
\mathrm{j}^2 &= \big((0)(0) - (0)(0) - (1)(1) - (0)(0)\big) + \big((0)(0) + (0)(0) + (1)(0) - (0)(1)\big)\,\mathrm{i} \\
&\quad + \big((0)(1) - (0)(0) + (1)(0) + (0)(0)\big)\,\mathrm{j} + \big((0)(0) + (0)(1) - (1)(0) + (0)(0)\big)\,\mathrm{k} = -1\,, \\
\mathrm{k}^2 &= \big((0)(0) - (0)(0) - (0)(0) - (1)(1)\big) + \big((0)(0) + (0)(0) + (0)(1) - (1)(0)\big)\,\mathrm{i} \\
&\quad + \big((0)(0) - (0)(1) + (0)(0) + (1)(0)\big)\,\mathrm{j} + \big((0)(1) + (0)(0) - (0)(0) + (1)(0)\big)\,\mathrm{k} = -1\,, \\
\mathrm{i}\,\mathrm{j} &= \big((0)(0) - (1)(0) - (0)(1) - (0)(0)\big) + \big((0)(0) + (1)(0) + (0)(0) - (0)(1)\big)\,\mathrm{i} \\
&\quad + \big((0)(1) - (1)(0) + (0)(0) + (0)(0)\big)\,\mathrm{j} + \big((0)(0) + (1)(1) - (0)(0) + (0)(0)\big)\,\mathrm{k} = \mathrm{k}\,,
\end{aligned}$$

and $\mathrm{i}\,\mathrm{j}\,\mathrm{k} = \mathrm{k}^2 = -1$. □

Exercises

2.3.3 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The quaternions form a field of scalars.

ii. The quaternion units satisfy $\mathrm{j}\,\mathrm{i} = -\mathrm{i}\,\mathrm{j}$.

2.3.4 Problem. From the basic relations $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{i}\,\mathrm{j}\,\mathrm{k} = -1$ deduce the following relations:

$$\begin{aligned}
\mathrm{i}\,\mathrm{i} &= -1\,, & \mathrm{i}\,\mathrm{j} &= \mathrm{k}\,, & \mathrm{i}\,\mathrm{k} &= -\mathrm{j}\,, \\
\mathrm{j}\,\mathrm{i} &= -\mathrm{k}\,, & \mathrm{j}\,\mathrm{j} &= -1\,, & \mathrm{j}\,\mathrm{k} &= \mathrm{i}\,, \\
\mathrm{k}\,\mathrm{i} &= -\mathrm{j}\,, & \mathrm{k}\,\mathrm{j} &= -\mathrm{i}\,, & \mathrm{k}\,\mathrm{k} &= -1\,.
\end{aligned}$$

2.3.5 Problem. For any numbers $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$, consider the quaternions $p := v_1\,\mathrm{i} + v_2\,\mathrm{j} + v_3\,\mathrm{k} \in \mathbb{H}$ and $q := w_1\,\mathrm{i} + w_2\,\mathrm{j} + w_3\,\mathrm{k} \in \mathbb{H}$, and the corresponding vectors $\vec{p} := v_1\,\vec{e}_1 + v_2\,\vec{e}_2 + v_3\,\vec{e}_3 \in \mathbb{R}^3$ and $\vec{q} := w_1\,\vec{e}_1 + w_2\,\vec{e}_2 + w_3\,\vec{e}_3 \in \mathbb{R}^3$. How is the quaternion product pq related to the cross product $\vec{p} \times \vec{q}$?

2.3.6 Problem. Prove the following related identities.

i. For all $p, q \in \mathbb{H}$, show that $|pq| = |p|\,|q|$.

ii. For all $v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4 \in \mathbb{R}$, show that

$$\begin{aligned}(v_1^2 + v_2^2 + v_3^2 + v_4^2)(w_1^2 + w_2^2 + w_3^2 + w_4^2) = {} & (v_1 w_1 - v_2 w_2 - v_3 w_3 - v_4 w_4)^2 \\ & + (v_1 w_2 + v_2 w_1 + v_3 w_4 - v_4 w_3)^2 \\ & + (v_1 w_3 - v_2 w_4 + v_3 w_1 + v_4 w_2)^2 \\ & + (v_1 w_4 + v_2 w_3 - v_3 w_2 + v_4 w_1)^2 \,.\end{aligned}$$

2.3.7 Problem. For any $h \in \mathbb{H}$, show that there exists $a, b \in \mathbb{R}$ such that $h^2 = a\,h + b$.

2.3.8 Problem. For any $h := a\,\mathrm{i} + b\,\mathrm{j} + c\,\mathrm{k} \in \mathbb{H}$ such that $a, b, c \in \mathbb{R}$ and $a^2 + b^2 + c^2 = 1$, then prove that $h^2 - 1 = 0$, which show that quadratic polynomials may have infinitely many roots over $\mathbb{H}$.