8.0.10 Problem. Express the matrix $\mathbf{U} := \begin{bmatrix} -1 & 1 & -3 \\ -3 & -2 & -1 \\ -3 & 0 & 3 \end{bmatrix}$ as a product of elementary matrices.

8.1 LU-Factorizations

How do we interpret the row reduction algorithm as matrix multiplication? Any expression of a matrix as a product of two or more matrices is called a ***matrix factorization***.

We first demonstrate some advantages of having an auspicious matrix factorization.

8.1.0 Problem. Consider the matrix factorization

$$\mathbf{A} := \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \mathbf{L}\,\mathbf{U}.$$

Use this factorization of $\mathbf{A}$ to solve $\mathbf{A}\,\vec{x} = \vec{b}$ where $\vec{b} = [-9 \;\; 5 \;\; 7 \;\; 11]^{\mathsf{T}}$.

Proof. To solve the non-homogeneous linear system $\mathbf{L}\,\vec{y} = \vec{b}$, we need only 6 multiplications and 6 additions:

$$[\mathbf{L} \;\; \vec{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_2 \mapsto \vec{r}_2 - (-1)\,\vec{r}_1 \\ \vec{r}_3 \mapsto \vec{r}_3 - 2\,\vec{r}_1 \\ \vec{r}_4 \mapsto \vec{r}_4 - (-3)\,\vec{r}_1}} \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & -5 & 1 & 0 & 25 \\ 0 & 8 & 3 & 1 & -16 \end{bmatrix}$$
$$\xrightarrow[\sim]{\substack{\vec{r}_3 \mapsto \vec{r}_3 - (-5)\,\vec{r}_2 \\ \vec{r}_4 \mapsto \vec{r}_4 - 8\,\vec{r}_2}} \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 3 & 1 & 16 \end{bmatrix} \xrightarrow[\sim]{\vec{r}_4 \mapsto \vec{r}_4 - 3\,\vec{r}_3} \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Solving $\mathbf{U}\,\vec{x} = \vec{y}$ requires 9 multiplications and 6 additions:

$$[\mathbf{U} \;\; \vec{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto \vec{r}_1 - (-2)\,\vec{r}_4 \\ \vec{r}_2 \mapsto \vec{r}_2 - (-2)\,\vec{r}_4 \\ \vec{r}_3 \mapsto \vec{r}_3 - (-1)\,\vec{r}_4}} \begin{bmatrix} 3 & -7 & -2 & 0 & -7 \\ 0 & -2 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto \vec{r}_1 - 2\,\vec{r}_3 \\ \vec{r}_2 \mapsto \vec{r}_2 - \vec{r}_3}} \begin{bmatrix} 3 & -7 & 0 & 0 & -19 \\ 0 & -2 & 0 & 0 & -8 \\ 0 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$
$$\xrightarrow[\sim]{\vec{r}_1 \mapsto \vec{r}_1 - (7/2)\,\vec{r}_2} \begin{bmatrix} 3 & 0 & 0 & 0 & 9 \\ 0 & -2 & 0 & 0 & -8 \\ 0 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto (1/3)\,\vec{r}_1 \\ \vec{r}_2 \mapsto -(1/2)\,\vec{r}_2 \\ \vec{r}_3 \mapsto -\vec{r}_3}} \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Finding the solution $\vec{x} = [3 \;\; 4 \;\; -6 \;\; -1]^{\mathsf{T}}$ requires 27 arithmetic operations, excluding the cost of finding $\mathbf{L}$ and $\mathbf{U}$. In contrast, finding the reduced row echelon form of the augmented matrix $[\mathbf{A} \;\; \vec{b}]$ takes at least 62 arithmetic operations. □

8.1.1 Definition. A matrix $\mathbf{U} := [u_{j,k}]$ is ***upper triangular*** if $u_{j,k} = 0$ for all $j > k$ and a matrix $\mathbf{L} := [\ell_{j,k}]$ is ***unit lower triangular*** if $\ell_{j,k} = 0$ for all $j < k$ and $\ell_{j,j} = 1$ for all j.

$$\mathbf{U} = \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & 1 \end{bmatrix}.$$

8.1.2 Remark. From the definition of matrix multiplication [7.0.1], we see that the product of two upper triangular matrices is upper triangular and the product of two unit lower triangular matrices is also a unit lower triangular matrix.

8.1.3 Definition. An **LU**-***factorization*** of a matrix $\mathbf{A}$ is an expression $\mathbf{A} = \mathbf{L}\,\mathbf{U}$ where $\mathbf{L}$ is unit lower triangular matrix and $\mathbf{U}$ is upper triangular matrix.

8.1.4 Proposition. *When a square matrix can be transformed using only elementary row add operations into a upper triangular matrix, the square matrix has an* **LU**-*factorization.*

Proof. By hypothesis, there is a sequence of elementary matrices $\mathbf{R}_i$, for all $1 \leqslant i \leqslant \ell$, such that $\mathbf{R}_\ell\,\mathbf{R}_{\ell-1}\cdots\mathbf{R}_1\,\mathbf{A} = \mathbf{U}$ is an upper triangular matrix. Each $\mathbf{R}_i$ corresponds to an elementary row add operation of the form $\vec{r}_k \mapsto \vec{r}_k + c\,\vec{r}_j$ for some scalar $c \in \mathbb{K}$ and indices satisfying $j < k$. Since each $\mathbf{R}_i$ is a unit lower triangular matrix, we obtain $\mathbf{A} = (\mathbf{R}_\ell\,\mathbf{R}_{\ell-1}\cdots\mathbf{R}_1)^{-1}\,\mathbf{U} = \mathbf{L}\,\mathbf{U}$ where $\mathbf{L} := \mathbf{R}_1^{-1}\,\mathbf{R}_2^{-1}\cdots\mathbf{R}_\ell^{-1}$. □

8.1.5 Problem. Find an **LU**-factorization of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{bmatrix}$.

Solution. Elementary row operations give

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_2 \mapsto \vec{r}_2 - 3\vec{r}_1 \\ \vec{r}_3 \mapsto \vec{r}_3 - 2\vec{r}_1}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow[\sim]{\vec{r}_3 \mapsto \vec{r}_3 - (-1)\vec{r}_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

so we deduce that $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. □

For any scalar $c \in \mathbb{K}$ and indices satisfying $j > k$, the elementary row operation $\vec{r}_j \mapsto \vec{r}_j - c\,\vec{r}_k$ means that the (j,k)-entry of the unit lower triangular matrix $\mathbf{L}$ is c.

Verification. We have

$$\begin{aligned} \mathbf{L}\,\mathbf{U} &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} (1)(1)+(0)(0)+(0)(0) & (1)(-1)+(0)(2)+(0)(0) & (1)(2)+(0)(1)+(0)(2) \\ (3)(1)+(0)(0)+(1)(0) & (3)(-1)+(1)(2)+(0)(0) & (3)(2)+(1)(1)+(0)(2) \\ (2)(1)+(-1)(0)+(1)(0) & (2)(-1)+(-1)(2)+(1)(0) & (2)(2)+(-1)(1)+(1)(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{bmatrix} = \mathbf{A}. \end{aligned}$$ □

8.1.6 Problem. Find an **LU**-factorization of $\mathbf{A} := \begin{bmatrix} 2 & 4 & 5 & -2 \\ -4 & -8 & -10 & 1 \\ 2 & 4 & 7 & 8 \\ -6 & -12 & -11 & 1 \end{bmatrix}$.

Solution. Elementary row operations yield

$$\begin{bmatrix} 2 & 4 & 5 & -2 \\ -4 & -8 & -10 & 1 \\ 2 & 4 & 7 & 8 \\ -6 & -12 & -11 & 1 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_2 \mapsto \vec{r}_2 - (-2)\,\vec{r}_1 \\ \vec{r}_3 \mapsto \vec{r}_3 - \vec{r}_1 \\ \vec{r}_4 \mapsto \vec{r}_4 - (-3)\,\vec{r}_1}} \begin{bmatrix} 2 & 4 & 5 & -2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 4 & -5 \end{bmatrix} \xrightarrow[\sim]{\vec{r}_4 \mapsto \vec{r}_4 - 2\,\vec{r}_3} \begin{bmatrix} 2 & 4 & 5 & -2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & -25 \end{bmatrix}.$$

Thus, we have $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 0 & 2 & 1 \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} 2 & 4 & 5 & -2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & -25 \end{bmatrix}$. □

Verification. We have

$$\begin{aligned} \mathbf{L}\,\mathbf{U} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 & -2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & -25 \end{bmatrix} \\ &= \begin{bmatrix} 1(2)+0(0)+0(0)+0(0) & 1(4)+0(0)+0(0)+0(0) & 1(5)+0(0)+0(-2)+0(0) & 1(-2)+0(-3)+0(10)+0(-25) \\ -2(2)+1(0)+0(0)+(0)(0) & -2(4)+1(0)+0(0)+0(0) & -2(5)+1(0)+0(-2)+0(0) & -2(-2)+1(-3)+0(10)+0(-25) \\ 1(2)+0(0)+1(0)+0(0) & 1(4)+0(0)+1(0)+0(0) & 1(5)+0(0)+1(-2)+0(0) & 1(-2)+0(-3)+1(10)+0(-25) \\ -3(2)+0(0)+2(0)+1(0) & -3(4)+0(0)+2(0)+1(0) & -3(5)+0(0)+2(-2)+1(0) & -3(-2)+0(-3)+2(10)+1(-25) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 5 & -2 \\ -4 & -8 & -10 & 1 \\ 2 & 4 & 7 & 8 \\ -6 & -12 & -11 & 1 \end{bmatrix} = \mathbf{A}\,. \end{aligned}$$ □

8.1.7 Remark. There may be more than one possible **LU**-factorization of a matrix. For example, we have

$$\begin{aligned} \mathbf{L}\,\mathbf{U} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 & -2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1(2)+0(0)+0(0)+0(0) & 1(4)+0(0)+0(0)+0(0) & 1(5)+0(0)+0(-2)+0(0) & 1(-2)+0(-3)+0(1)+0(5) \\ -2(2)+1(0)+0(0)+0(0) & -2(4)+1(0)+0(0)+0(0) & -2(5)+1(0)+0(-2)+0(0) & -2(-2)+1(-3)+0(1)+0(5) \\ 1(2)-3(0)+1(0)+0(0) & 1(4)-3(0)+1(0)+0(0) & 1(5)-3(0)+1(-2)+0(0) & 1(-2)-3(-3)+1(1)+0(5) \\ -3(2)+4(0)+2(0)+1(0) & -3(4)+4(0)+2(0)+1(0) & -3(5)+4(0)+2(-2)+1(0) & -3(-2)+4(-3)+2(1)+1(5) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 5 & -2 \\ -4 & -8 & -10 & 1 \\ 2 & 4 & 7 & 8 \\ -6 & -12 & -11 & 1 \end{bmatrix} = \mathbf{A}\,. \end{aligned}$$

8.1.8 Problem. Show that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ does not have an **LU**-factorization.

Solution. If there were an **LU**-factorization

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell_{2,1} & 1 \end{bmatrix} \begin{bmatrix} u_{1,1} & u_{1,2} \\ 0 & u_{2,2} \end{bmatrix} = \begin{bmatrix} u_{1,1} & u_{1,2} \\ \ell_{2,1}u_{1,1} & \ell_{2,1}u_{1,2} + u_{2,2} \end{bmatrix},$$

then we would have $u_{1,1} = 0$ and $\ell_{2,1}u_{1,1} = 1$ which is impossible. □

Exercises

8.1.9 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. A unit lower triangular $(n \times n)$-matrix has at least $\frac{n(n-1)}{2}$ entries that are zero.
ii. Every unit lower triangular matrix is invertible.
iii. Every upper triangular matrix is invertible.

8.1.10 Problem. Prove that the inverse of a unit lower triangular matrix is also a unit lower triangular matrix.

8.1.11 Problem. Prove that an upper triangular matrix is invertible if and only if all of its diagonal entries are nonzero. Moreover, prove that the inverse of an invertible upper triangular matrix is also an upper triangular matrix.

8.1.12 Problem. Consider the matrix $\mathbf{A} := \begin{bmatrix} -1 & -3 & 1 & -2 \\ -3 & -7 & 0 & -5 \\ 2 & 4 & -2 & 0 \\ -3 & -9 & 12 & 4 \end{bmatrix}$.

i. Find an **LU**-factorization of **A**.
ii. Using the **LU**-factorization, solve $\mathbf{A}\vec{x} = \vec{b}$ where $\vec{b} := \begin{bmatrix} 0 \\ -1 \\ -2 \\ 10 \end{bmatrix}$.

8.1.13 Problem. If **A** is invertible and has an **LU**-factorization, then prove that the unit lower triangular matrix **L** and the upper triangular matrix **U** are uniquely determined.

8.2 Permutations

WHAT IS A PERMUTATION? For any nonnegative integer n, there are three equivalent perspectives on the permutations of $\{1, 2, \ldots, n\}$.

8.2.0 Definition.

- A ***permutation*** is an arrangement (also known as a linear ordering) of the elements in the set $\{1, 2, \ldots, n\}$. Expressed in ***one-line notation***, both $\sigma := 2\ 5\ 4\ 3\ 1$ and $\tau := 5\ 2\ 4\ 1\ 3$ are permutations of the set $\{1, 2, 3, 4, 5\}$.
- A permutation is map from the set $\{1, 2, \ldots, n\}$ to itself such that every element occurs exact once as an image value. From this viewpoint, the permutations σ and τ are given by

$$\begin{array}{lllll} \sigma(1) = 2\,, & \sigma(2) = 5\,, & \sigma(3) = 4\,, & \sigma(4) = 3\,, & \sigma(5) = 1\,; \\ \tau(1) = 5\,, & \tau(2) = 2\,, & \tau(3) = 4\,, & \tau(4) = 1\,, & \tau(5) = 3\,. \end{array}$$

Regarding permutations as a functions allows one to compose two

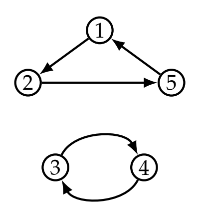


Figure 8.1: Direct graph of σ

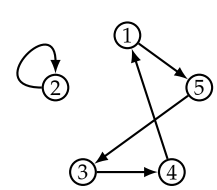


Figure 8.2: Direct graph of τ

permutation to obtain a new permutation. For example, we have

$$\begin{aligned}
(\sigma \circ \tau)(1) &= \sigma\big(\tau(1)\big) = \sigma(5) = 1 & (\tau \circ \sigma)(1) &= \tau\big(\sigma(1)\big) = \tau(2) = 2 \\
(\sigma \circ \tau)(2) &= \sigma\big(\tau(2)\big) = \sigma(2) = 5 & (\tau \circ \sigma)(2) &= \tau\big(\sigma(2)\big) = \tau(5) = 3 \\
(\sigma \circ \tau)(3) &= \sigma\big(\tau(3)\big) = \sigma(4) = 3 & (\tau \circ \sigma)(3) &= \tau\big(\sigma(3)\big) = \tau(4) = 1 \\
(\sigma \circ \tau)(4) &= \sigma\big(\tau(4)\big) = \sigma(1) = 2 & (\tau \circ \sigma)(4) &= \tau\big(\sigma(4)\big) = \tau(3) = 4 \\
(\sigma \circ \tau)(5) &= \sigma\big(\tau(5)\big) = \sigma(3) = 4 & (\tau \circ \sigma)(5) &= \tau\big(\sigma(5)\big) = \tau(1) = 5
\end{aligned}$$

so $\sigma \circ \tau = 1\ 5\ 3\ 2\ 4$ and $\tau \circ \sigma = 2\ 3\ 1\ 4\ 5$.

- A permutation is a directed graph with vertex set $\{1, 2, \dots, n\}$ such that every vertex is the head of one edge and the tail of one edge.

A ***permutation matrix*** is the adjacency matrix of some permutation.

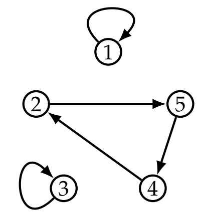


Figure 8.3: Direct graph of $\sigma \circ \tau$

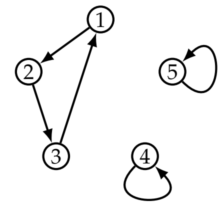


Figure 8.4: Direct graph of $\tau \circ \sigma$

To count the number of permutations of the set $\{1, 2, \dots, n\}$, we introduce a function

8.2.1 Definition. For any nonnegative integer n, the ***factorial function*** $n \mapsto n!$ is defined by

$$n! := n(n-1)(n-2)\cdots(3)(2)(1) = \prod_{j=1}^{n} j\,.$$

Since the empty product is the multiplicative identity, we have $0! = 1$. The first few values of the factorial function are $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, and $6! = 720$.

The next lemma gives the most important interpretation of the factorial function.

8.2.2 Lemma. *For any nonnegative integer n, there are $n!$ permutations of the set $\{1, 2, \dots, n\}$.*

Inductive proof. There is a unique permutation of the empty set $\varnothing$, so the base case holds. As the induction hypothesis, suppose that claim holds for some nonnegative integer n. To construct a permutation of the set $\{1, 2, \dots, n+1\}$, any element can appear in the first position, so there are $n+1$ choices. By the induction hypothesis, there are $n!$ ways to arrange the remaining n elements. Hence, the total number of permutations of the set $\{1, 2, \dots, n+1\}$ is $(n+1)(n!) = (n+1)!$. □

The ***empty set***, denoted by $\varnothing$, is the unique set having no elements.

8.2.3 Proposition (Characterization of permutation matrices)**.** *A matrix corresponds to a permutation if and only if it is square and has unique nonzero entry in each row and each column equal to 1.*

The 6 permutations of the set $\{1, 2, 3\}$ correspond to the matrices:

$$1\ 2\ 3 \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 1\ 3\ 2 \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$2\ 1\ 3 \leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 2\ 3\ 1 \leftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$3\ 1\ 3 \leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad 3\ 2\ 1 \leftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Proof. Every adjacency matrix is square and its entries are either 0 or 1 when there is at most one arrow between each pair of vertices. Conversely, every square matrix with entries 0 or 1 is the adjacency matrix for some directed graph. In an adjacency matrix, the 1s in the

j-th row corresponds to the edges with head at the j-th vertex, and the 1s in the k-th column correspond to the edges with tail at the k-th vertex. Therefore, every vertex is the head of one edge and the tail of one edge if and only if there is exactly one entry of 1 in each row and each column and 0s elsewhere. □

8.2.4 Proposition (Properties of permutation matrices). *Let n be a positive integer and let $\mathbf{P}$ be the matrix corresponding to a permutation σ of the set $\{1, 2, \dots, n\}$.*

i. *The k-th column of the matrix $\mathbf{P}$ is the vector $\vec{e}_{\sigma(k)} \in \mathbb{K}^n$, so*
$$\mathbf{P} = \mathbf{E}_{\sigma(1),1} + \mathbf{E}_{\sigma(2),2} + \cdots + \mathbf{E}_{\sigma(n),n} = \sum_{k=1}^{n} \mathbf{E}_{\sigma(k),k} \,.$$
ii. *The permutation matrix $\mathbf{P}$ is invertible and $\mathbf{P}^{-1} = \mathbf{P}^{\mathsf{T}}$.*
iii. *The product of two permutation matrices is the matrix corresponding to the composition of the permutations.*

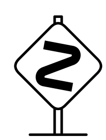
Proof.

i. In the direct graph representing the permutation σ, the edge with tail k has head $\sigma(k)$, so the k-th column is $\vec{e}_{\sigma(k)}$.

ii. Since $\vec{e}_j \cdot \vec{e}_k = \delta_{j,k}$, we have
$$\mathbf{P}^{\mathsf{T}}\,\mathbf{P} = \begin{bmatrix} \vec{e}_{\sigma(1)} & \vec{e}_{\sigma(2)} & \cdots & \vec{e}_{\sigma(n)} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \vec{e}_{\sigma(1)} & \vec{e}_{\sigma(2)} & \cdots & \vec{e}_{\sigma(n)} \end{bmatrix} = \mathbf{I}$$
which shows $\mathbf{P}^{-1} = \mathbf{P}^{\mathsf{T}}$.

iii. Suppose that the matrix $\mathbf{Q}$ corresponds to the permutation τ. From part *i*, we see that the j-th row of the matrix $\mathbf{Q}$ is $\vec{e}^{\,\mathsf{T}}_{\tau^{-1}(j)}$. As $\vec{e}_j \cdot \vec{e}_k = \delta_{j,k}$, we deduce that $\vec{e}_{\tau^{-1}(j)} \cdot \vec{e}_{\sigma(k)} = 1$ if and only $(\tau \circ \sigma)(k) = j$. Hence, the k-th column in the product $\mathbf{Q}\,\mathbf{P}$ is the vector $\vec{e}_{(\tau\circ\sigma)(k)}$. □

8.2.5 Warning. Left multiplication by a permutation matrix permutes the entries of the vector, but the indices (or subscripts) are permuted in the opposite way. For the matrix corresponding to the permutation 3 1 2, we have



$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ v_1 \end{bmatrix} .$$

The first entry is sent to the third entry, the second entry is sent to the first entry, and the third entry is sent to the second entry, so the permutation of the entries is 3 1 2. On the other hand, by reading the subscripts in order, we see that the permutation of the indices corresponds to 2 3 1 (which is the inverse permutation).

Exercises

8.2.6 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The identity matrix is a permutation matrix.
ii. The columns of a permutation matrix are simply a permutation of the standard basis vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$.
iii. There is a unique permutation of the empty set.

8.2.7 Problem. An elementary matrix that differs from the identity matrix by interchanging a successive pair of rows is called an ***adjacent transposition***. Equivalently, an adjacent transposition is a matrix of the form $\mathbf{I} + \mathbf{E}_{j,j+1} + \mathbf{E}_{j+1,j} - \mathbf{E}_{j,j} - \mathbf{E}_{j+1,j+1}$ for some row index j.

i. Express the permutation matrix

$$\mathbf{P} := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

as a product of adjacent transpositions.
ii. Prove that every permutation matrix is a product of adjacent transpositions.

8.3 More Matrix Factorizations

HOW CAN WE EXTEND **LU**-FACTORIZATIONS TO ALL SQUARE MATRICES? An **LU**-factorization of a matrix is asymmetric—the upper triangular matrix **U** has arbitrary scalars along its diagonal whereas the unit lower triangular matrix **L** always has 1s on its diagonal. When the diagonal entries in **U** are all nonzero, this matrix factors as the product of a diagonal matrix and a unit upper triangular matrix;

$$\mathbf{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & u_{3,3} & \cdots & u_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & u_{n,n} \end{bmatrix} = \begin{bmatrix} u_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & u_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & u_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{n,n} \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{1,2}}{u_{1,1}} & \frac{u_{1,3}}{u_{1,1}} & \cdots & \frac{u_{1,n}}{u_{1,1}} \\ 0 & 1 & \frac{u_{2,3}}{u_{2,2}} & \cdots & \frac{u_{2,n}}{u_{2,2}} \\ 0 & 0 & 1 & \cdots & \frac{u_{3,n}}{u_{3,3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

In this situation, we obtain $\mathbf{A} = \mathbf{L}\,\mathbf{D}\,\mathbf{U}$ where **L** is a unit lower triangular matrix, **D** is a diagonal matrix, and **U** is a unit upper triangular matrix. For example, we have

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 1.0 & 2.0 \\ 0 & 0.0 & 1.0 \end{bmatrix} .$$

8.3.0 Problem. For any four scalars $a, b, c, d \in \mathbb{K}$ with $a \neq 0$, determine the **L D U** factorization of the (2×2)-matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

Solution. If

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & 0 \\ \ell x & y \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & xu \\ \ell x & \ell xu + y \end{bmatrix}$$

then we have $x = a$, $\ell = b/a$, $u = c/a$, and $y = (1/a)(ad - bc)$. □

8.3.1 Definition. A $\mathbf{P}^{\mathsf{T}}\mathbf{LU}$-***factorization*** of a square matrix $\mathbf{A}$ is a product $\mathbf{A} = \mathbf{P}^{\mathsf{T}}\,\mathbf{L}\,\mathbf{U}$ where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ is unit lower triangular, and $\mathbf{U}$ is upper triangular.

8.3.2 Problem. Find a $\mathbf{P}^{\mathsf{T}}\mathbf{LU}$-factorization of $\mathbf{A} = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 6 & 7 & 5 \end{bmatrix}$.

Solution. Elementary operations give

$$\begin{bmatrix} 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 6 & 7 & 5 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto \vec{r}_4 \\ \vec{r}_3 \mapsto \vec{r}_1 \\ \vec{r}_4 \mapsto \vec{r}_3}} \begin{bmatrix} \underline{1} & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 \\ 2 & 6 & 7 & 5 \\ 0 & 2 & 2 & 4 \end{bmatrix} \xrightarrow[\sim]{\vec{r}_3 \mapsto \vec{r}_3 - 2\vec{r}_1} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & \underline{2} & 2 & 2 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 2 & 4 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_3 \mapsto \vec{r}_3 - \vec{r}_2 \\ \vec{r}_4 \mapsto \vec{r}_4 - \vec{r}_2}} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Hence, we have

$$\begin{bmatrix} 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 6 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$ □

The theory of row-reduction is encapsulated by the next result.

8.3.3 Proposition. *Every square matrix has a* $\mathbf{P}^{\mathsf{T}}\mathbf{LU}$*-factorization. When* $\mathbf{A}$ *is invertible, the matrix* $\mathbf{A}$ *has a* $\mathbf{P}^{\mathsf{T}}\mathbf{LDU}$*-factorization.*

8.3.4 Problem. Find a $\mathbf{P}^{\mathsf{T}}\mathbf{LDU}$-factorization of $\mathbf{A} = \begin{bmatrix} 0 & 3 & -6 & 1 \\ -2 & -2 & 2 & 6 \\ 1 & 1 & -1 & -1 \\ 2 & -1 & 2 & -2 \end{bmatrix}$.

Solution. Elementary operations yield

$$\begin{bmatrix} 0 & 3 & -6 & 1 \\ -2 & -2 & 2 & 6 \\ 1 & 1 & -1 & -1 \\ 2 & -1 & 2 & -2 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 \mapsto \vec{r}_3 \\ \vec{r}_2 \mapsto \vec{r}_4 \\ \vec{r}_3 \mapsto \vec{r}_1 \\ \vec{r}_4 \mapsto \vec{r}_2}} \begin{bmatrix} \underline{1} & 1 & -1 & -1 \\ 2 & -1 & 2 & -2 \\ 0 & 3 & -6 & 1 \\ -2 & -2 & 2 & 6 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_2 \mapsto \vec{r}_2 - 2\vec{r}_1 \\ \vec{r}_4 \mapsto \vec{r}_4 + 2\vec{r}_1}} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & \underline{-3} & 4 & 0 \\ 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \xrightarrow[\sim]{\vec{r}_3 \mapsto \vec{r}_3 + \vec{r}_2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Hence, we obtain

$$\begin{bmatrix} 0 & 3 & -6 & 1 \\ -2 & -2 & 2 & 6 \\ 1 & 1 & -1 & -1 \\ 2 & -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -4/3 & 0 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

as the desired factorization. □

Exercises

8.3.5 Problem. For all $t \in \mathbb{C}$, find a $\mathbf{P}^{\mathsf{T}}\mathbf{LDU}$-factorization of the matrix

$$\mathbf{B} := \begin{bmatrix} 2t & 2t-1 & 2t^2+2 & -t-9 \\ 2 & 2 & 2t & 0 \\ -6 & -7 & -6t-2 & -t \\ -2 & -t-2 & -4t+2 & -t^2-6 \end{bmatrix}.$$