1
Vector Spaces

The modern, more versatile, and abstract treatment of linear algebra encompasses much more than the coordinate spaces $\mathbb{K}^n$ and their linear subspaces. This chapter introduces these fundamental ideas and the most important examples.

1.0 Vector Spaces

What are the underlying objects in linear algebra? Fix a field $\mathbb{K}$ of scalars such as $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Linear algebra is the study of the following structures.

The axiomatic definition of a vector space was first given by Giuseppe Peano in 1888.

1.0.0 Definition (Vector space)**.** A $\mathbb{K}$-***vector space*** V is a set equipped with two operations

$$\begin{array}{lll}\text{(addition)} & V \times V \to V & \text{written } (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} + \boldsymbol{w} \\ \text{(scalar multiplication)} & \mathbb{K} \times V \to V & \text{written } (c, \boldsymbol{w}) \mapsto c\,\boldsymbol{v}\end{array}$$

that satisfy the following eight axioms.

$$\begin{array}{lcl}\text{(commutativity)} & \boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v} & \text{for all } \boldsymbol{v}, \boldsymbol{w} \in V. \\ \text{(associativity)} & (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) & \text{for all } \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \\ & (bc)\,\boldsymbol{v} = b(c\,\boldsymbol{v}) & \text{for all } \boldsymbol{v} \in V \text{ and all } b, c \in \mathbb{K}. \\ \text{(additive identity)} & \multicolumn{2}{l}{\text{There exists } \mathbf{0} \in V \text{ such that } \boldsymbol{v} + \mathbf{0} = \boldsymbol{v} \text{ for all } \boldsymbol{v} \in V.} \\ \text{(additive inverse)} & \multicolumn{2}{l}{\text{For all } \boldsymbol{v} \in V, \text{ there exists } \boldsymbol{w} \in V \text{ such that } \boldsymbol{v} + \boldsymbol{w} = \mathbf{0}.} \\ \text{(multiplicative identity)} & 1\,\boldsymbol{v} = \boldsymbol{v} & \text{for all } \boldsymbol{v} \in V. \\ \text{(distributivity)} & c\,(\boldsymbol{v} + \boldsymbol{w}) = c\,\boldsymbol{v} + c\,\boldsymbol{w} & \text{for all } \boldsymbol{v}, \boldsymbol{w} \in V \text{ and all } c \in \mathbb{K}. \\ & (b + c)\,\boldsymbol{v} = b\,\boldsymbol{v} + c\,\boldsymbol{v} & \text{for all } \boldsymbol{v} \in V \text{ and all } b, c \in \mathbb{K}.\end{array}$$

The elements of V are called ***vectors***.

1.0.1 Remark. The field $\mathbb{K}$ of scalars, equipped with its usual addition and multiplication, is itself a $\mathbb{K}$-vector space: the set $\mathbb{K}$ of scalars has addition and multiplication operations that are commutative, associative, distributive, and include identities and inverses.

1.0.2 Remark. The additive identity axiom establishes that the empty set $\varnothing$ cannot be a vector space.

1.0.3 Problem. Prove that the set $\mathbb{Z}$ of integers, equipped with the addition and multiplication inherited from $\mathbb{Q}$, is not a $\mathbb{Q}$-vector space.

Solution. Since $\frac{1}{3}(2) = \frac{2}{3}$ is not an integer, multiplication does not define an operation from $\mathbb{Q} \times \mathbb{Z}$ to $\mathbb{Z}$. $\square$

1.0.4 Proposition (Properties of vector spaces)**.** *Any $\mathbb{K}$-vector space V has the following properties.*

i. *The vector space V has a unique additive identity.*
ii. *Each vector $\boldsymbol{v}$ in V has a unique additive inverse denoted by $-v$.*
iii. *For all vectors $\boldsymbol{v}$ in V, we have $0\,\boldsymbol{v} = \mathbf{0}$.*
iv. *For all scalars c in $\mathbb{K}$, we have $c\,\mathbf{0} = \mathbf{0}$.*
v. *For all vectors v in V, we have $(-1)\,\boldsymbol{v} = -\boldsymbol{v}$.*

Proof.

i. Given additive identities $\mathbf{0}$ and $\mathbf{0}'$ in V, the additive identity and commutativity axioms show that $\mathbf{0}' = \mathbf{0}' + \mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}$.

ii. Given additive inverses $\boldsymbol{w}$ and $\boldsymbol{w}'$ of a vector $\boldsymbol{v}$ in V, the additive identity, additive inverse, and associativity axioms establish that $\boldsymbol{w} = \boldsymbol{w} + \mathbf{0} = \boldsymbol{w} + (\boldsymbol{v} + \boldsymbol{w}') = (\boldsymbol{w} + \boldsymbol{v}) + \boldsymbol{w}' = \mathbf{0} + \boldsymbol{w}' = \boldsymbol{w}'$.

iii. For any vector $\boldsymbol{v}$ in V, the additive identity axiom in $\mathbb{K}$ and the distributivity axiom in V give $0\,\boldsymbol{v} = (0+0)\,\boldsymbol{v} = 0\,\boldsymbol{v} + 0\,\boldsymbol{v}$. Let $\boldsymbol{w}$ denote the additive inverse of the vector $0\,\boldsymbol{v}$. The additive identity, associativity, and additive inverse axioms in V give $\mathbf{0} = 0\,\boldsymbol{v} + \boldsymbol{w} = (0\,\boldsymbol{v} + 0\,\boldsymbol{v}) + \boldsymbol{w} = 0\,\boldsymbol{v} + (0\,\boldsymbol{v} + \boldsymbol{w}) = 0\,\boldsymbol{v} + \mathbf{0} = 0\,\boldsymbol{v}$.

iv. For any scalar c in $\mathbb{K}$, the additive identity and distributivity axioms in V give $c\,\mathbf{0} = c\,(\mathbf{0} + \mathbf{0}) = c\,\mathbf{0} + c\,\mathbf{0}$. Let $\boldsymbol{w}$ be the additive inverse of the vector $c\,\mathbf{0}$. The additive identity, associativity, and additive inverse axioms in the vector space V give

$$\mathbf{0} = c\,\mathbf{0} + \boldsymbol{w} = (c\,\mathbf{0} + c\,\mathbf{0}) + \boldsymbol{w} = c\,\mathbf{0} + (c\,\mathbf{0} + \boldsymbol{w}) = c\,\mathbf{0} + \mathbf{0} = c\,\mathbf{0}\,.$$

v. For any vector $\boldsymbol{v}$ in V, the multiplicative identity and distributivity axioms in the vector space V together with property *iii* give $\boldsymbol{v} + (-1)\,\boldsymbol{v} = 1\,\boldsymbol{v} + (-1)\,\boldsymbol{v} = \big(1 + (-1)\big)\,\boldsymbol{v} = 0\,\boldsymbol{v} = \mathbf{0}$ which means $(-1)\,\boldsymbol{v}$ is the additive inverse of $\boldsymbol{v}$. □

1.0.5 Problem (Polynomial spaces)**.** Demonstrate that the set $\mathbb{K}[t]$ of all polynomials in the variable t with coefficients in $\mathbb{K}$, equipped with the usual arithmetic operations, forms a $\mathbb{K}$-vector space.

Proof. A polynomial is an expression $p := a_0 + a_1\,t + \cdots + a_n\,t^n$ where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are scalars in $\mathbb{K}$. For any $q := b_0 + b_1\,t + \cdots + b_n\,t^n$ in $\mathbb{K}[t]$ and any scalar c in $\mathbb{K}$, addition and scalar multiplication are defined by

$$\begin{aligned} p + q &:= (a_0 + b_0) + (a_1 + b_1)\,t + \cdots + (a_n + b_n)\,t^n \\ c\,p &:= (ca_0) + (ca_1)\,t + \cdots + (ca_n)\,t^n\,. \end{aligned}$$

By identifying p in $\mathbb{K}[t]$ with its sequence $a_0, a_1, a_2, \ldots$ of coefficients, we see that polynomials correspond to sequences of scalars $\mathbb{K}$ with only a finite number of nonzero terms.

These operations are defined coefficientwise, so the commutativity, associativity, identities, additive inverse, and distributivity properties are inherited from the corresponding properties in the field $\mathbb{K}$. □

Exercises

1.0.6 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The rational numbers $\mathbb{Q}$ form a $\mathbb{Q}$-vector space.
ii. Every vector space contains at least two vectors.
iii. A vector space is closed under taking linear combinations.

1.0.7 Problem. Determine if the set $\mathbb{T} := \mathbb{R} \cup \{\infty\}$, with addition and scalar multiplication defined for all $v, w \in \mathbb{T}$ and all $c \in \mathbb{R}$ by $v \oplus w := \min(v, w)$ and $c \otimes v := c + v$, is a real vector space. If it is not, then list all of the defining axioms that fail to hold.

1.0.8 Problem. Determine if the set $\mathbb{P} := \{x \in \mathbb{R} \mid x > 0\}$, with addition and scalar multiplication defined for all $v, w \in \mathbb{P}$ and all $c \in \mathbb{R}$ by $v \boxplus w := v\,w$ and $c \boxtimes v := v^c$, is a real vector space. If it is not, then list all of the defining axioms that fail to hold.

1.1 Function Spaces

WHAT IS THE UNIVERSAL EXAMPLE OF A VECTOR SPACE? The set of all functions from an arbitrary set to a vector space has a canonical vector space structure.

In mathematics, the word "canonical" means either 'conventionally agreed to be the most useful' or sometimes 'the most elegant option known'.

1.1.0 Theorem (Function spaces)**.** *Let $\mathcal{X}$ be a set and let V be a $\mathbb{K}$-vector space. The set $V^{\mathcal{X}}$ of all functions from $\mathcal{X}$ to V is a $\mathbb{K}$-vector space under pointwise addition and scalar multiplication.*

Proof. Consider three functions f, g, h in $V^{\mathcal{X}}$ and two scalars b, c in $\mathbb{K}$. The linear combination $b\,f + c\,g$ is the function from $\mathcal{X}$ to V defined, for all $x \in \mathcal{X}$, by $(b\,f + c\,g)(x) := b\,f(x) + c\,g(x)$. Let $0 \in V^{\mathcal{X}}$ denote the zero function that sends each element in $\mathcal{X}$ to the additive identity in vector space V. From the defining axioms of the vector space V, the pointwise operations on the set $V^{\mathcal{X}}$ satisfy the following:

$$
\begin{aligned}
(f+g)(x) &= f(x) + g(x) = g(x) + f(x) = (g+f)(x) \\
\big((f+g)+h\big)(x) &= \big(f(x)+g(x)\big) + h(x) = f(x) + \big(g(x)+h(x)\big) = \big(f+(g+h)\big)(x) \\
\big((bc)f\big)(x) &= (bc)\big(f(x)\big) = (b)\big(c\big(f(x)\big)\big) = \big(b(cf)\big)(x) \\
(f+0)(x) &= f(x) + 0(x) = f(x) + 0 = f(x) \\
\big(f+(-1)f\big)(x) &= f(x) + (-1)\big(f(x)\big) = f(x) - f(x) = 0 = 0(x) \\
\big((1)f\big)(x) &= 1\big(f(x)\big) = f(x) \\
\big(c(f+g)\big)(x) &= c\big(f(x)+g(x)\big) = c\big(f(x)\big) + c\big(g(x)\big) = (cf+cg)(x) \\
\big((b+c)f\big)(x) &= (b+c)\big(f(x)\big) = b\big(f(x)\big) + c\big(f(x)\big) = (bf+cf)(x)\,.
\end{aligned}
$$

Therefore, the set $V^{\mathcal{X}}$ is also a $\mathbb{K}$-vector space. $\square$

1.1.1 Corollary (Coordinate spaces)**.** *For all nonnegative integers n, the set $\mathbb{K}^n$, equipped with entrywise operations, is a $\mathbb{K}$-vector space.*

Proof. The field $\mathbb{K}$ of scalars is a $\mathbb{K}$-vector space; see Remark 1.0.1. For the finite set $[n] := \{1, 2, \dots, n\}$, Theorem 1.1.0 shows that $\mathbb{K}^{[n]}$ is $\mathbb{K}$-vector space. Since functions from $[n]$ to $\mathbb{K}$ are determined by their outputs, we may identify $\mathbb{K}^{[n]}$ with $\mathbb{K}^n$. More explicitly, the function f in $\mathbb{K}^{[n]}$ corresponds to the vector $\begin{bmatrix} f(1) & f(2) & \cdots & f(n) \end{bmatrix}^{\mathsf{T}} \in \mathbb{K}^n$ and pointwise operations on $\mathbb{K}^{[n]}$ correspond to entrywise operations on $\mathbb{K}^n$. We conclude that $\mathbb{K}^n$ also a $\mathbb{K}$-vector space. □

1.1.2 Corollary (Space of matrices)**.** *The set $\mathbb{K}^{m \times n}$ of all $(m \times n)$-matrices, with the entrywise addition and scalar multiplication, is a $\mathbb{K}$-vector space.*

The space of matrices has features, such as matrix multiplication, that are not captured by its vector space structure. This is typical. The power of our abstract approach is that consequences derived from only the axioms can be applied to a wide range of examples.

Proof. Corollary 1.1.1 establishes that $\mathbb{K}^{mn}$ is a $\mathbb{K}$-vector space. By vertically concatenating the columns of an $(m \times n)$-matrix to obtain a $(mn \times 1)$-matrix, we may identify $\mathbb{K}^{m \times n}$ with $\mathbb{K}^{mn}$. Under this identification, entrywise operations on $\mathbb{K}^{m \times n}$ correspond to entrywise operations on $\mathbb{K}^{mn}$. We conclude that $\mathbb{K}^{m \times n}$ is a $\mathbb{K}$-vector space. □

In practice, almost all vector spaces are obtained as subsets of an appropriate function space.

1.1.3 Definition (Linear subspace)**.** A subset W of a $\mathbb{K}$-vector space V is a ***linear subspace*** if W forms a $\mathbb{K}$-vector space when equipped with the addition and scalar multiplication operations inherited from V.

A $\mathbb{K}$-vector space V itself is obviously the largest linear subspace it contains.

To be more explicit, we rephrase this definition.

1.1.4 Lemma (Naive test)**.** *Let V be a $\mathbb{K}$-vector space. A subset W of V is a linear subspace if and only if the following three properties hold.*

(additive identity)	*The additive identity $\mathbf{0} \in V$ lies in W.*
(closed under addition)	*For all $\boldsymbol{v}, \boldsymbol{w} \in W$, we have $\boldsymbol{v} + \boldsymbol{w} \in W$.*
(closed under scalar multiplication)	*For any $c \in \mathbb{K}$ and any $\boldsymbol{v} \in W$, we have $c\,\boldsymbol{v} \in W$.*

Proof.

$\Rightarrow$: Suppose that W is a linear subspace of V. The addition and scalar multiplication operations on W are obtain by restricting the operations on V. Saying that these restrictions yield well-defined operations on W is equivalent to saying that W is closed under addition and scalar multiplication. Since the empty set is never a vector space [1.0.2], there exists a vector $\boldsymbol{w}$ in W. Since W is closed under scalar multiplication, the properties of a vector space [1.0.4] yield $0\,\boldsymbol{w} = \mathbf{0} \in W$.

$\Leftarrow$: Suppose that W satisfies the three properties. Since W is closed under addition and scalar multiplication, the restrictions of the addition and scalar multiplication operations on V yield well-defined operations on W. By assumption, the additive identity of V belongs to W. As the $\mathbb{K}$-vector space V satisfies the eight defining axioms [1.0.0] and the operations on W agree with the operations on V, it follows that these properties also hold in W. We conclude that W is a $\mathbb{K}$-vector space. □

Every vector space has at least one linear subspace.

1.1.5 Problem (Zero subspace)**.** For any $\mathbb{K}$-vector space V, show that the subset $\{\mathbf{0}\} \subseteq V$, consisting of just the additive identity in V, is a linear subspace. Moreover, prove that every linear subspace contains this zero subspace.

Proof. The additive identity axiom in V shows that the singleton $\{\mathbf{0}\}$ is closed under addition. Since $c\,\mathbf{0} = \mathbf{0}$ for all scalars c in $\mathbb{K}$ [1.0.4], the subset $\{\mathbf{0}\}$ is also closed under scalar multiplication. Thus, the naive test [1.1.4] proves that $\{\mathbf{0}\}$ is a linear subspace. Finally, the additive identity axiom implies that every linear subspace of V contains $\mathbf{0} \in V$, so every linear subspace contains the zero subspace. □

Exercises

1.1.6 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The set $\mathbb{R}^{\mathbb{N}}$ of a real sequences is a $\mathbb{R}$-vector space.
ii. The set $\mathbb{R}^{\mathbb{R}}$ of a real-valued functions on the real line is a $\mathbb{R}$-vector space.
iii. The set $\mathbb{Z}^{\mathbb{N}}$ of a integer sequences is a $\mathbb{Q}$-vector space.
iv. The set $\mathbb{Q}[t]$ of rational polynomials is a linear subspace of $\mathbb{Q}^{\mathbb{C}}$.

1.1.7 Problem. Give an example of a nonempty subset U in $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a linear subspace of $\mathbb{R}^2$.

1.2 Linear Subspaces

WHICH SUBSETS INHERIT THE STRUCTURE OF A VECTOR SPACE? We develop better tools for recognizing such vector spaces.

1.2.0 Proposition (Subspace test)**.** *A nonempty subset W of a $\mathbb{K}$-vector space is a linear subspace if and only if, for all vectors $\boldsymbol{v}, \boldsymbol{w}$ in W and all scalars b, c in $\mathbb{K}$, the linear combination $c\,\boldsymbol{v} + d\,\boldsymbol{w}$ belongs to W.*

Proof.

⇒: Suppose that W is a linear subspace of a $\mathbb{K}$-vector space. The naive test [1.1.4] shows that the subset W is closed under scalar multiplication, so $b\,\boldsymbol{v}$ and $c\,\boldsymbol{w}$ belong to W. Since W is also closed under addition, we see that $b\,\boldsymbol{v} + c\,\boldsymbol{w}$ belongs to W.

$\Leftarrow$: Suppose that the subset W is nonempty and closed under linear combinations. There exists a vector $\boldsymbol{v} \in W$. Setting $\boldsymbol{w} := \boldsymbol{v}$, $b := 1$, and $c := -1$, we obtain $b\,\boldsymbol{v} + c\,\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v} = \mathbf{0} \in W$. For all $\boldsymbol{v}, \boldsymbol{w} \in W$ and $b = c = 1$, it follows that $b\,\boldsymbol{v} + c\,\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{w} \in W$, so W is closed under addition. Finally, if $\boldsymbol{v} \in W$, $\boldsymbol{w} := \mathbf{0}$, $b \in \mathbb{K}$, and $c := 1$, we have $b\,\boldsymbol{v} + c\,\boldsymbol{w} = b\,\boldsymbol{v} \in W$, so W is closed under scalar multiplication. The naive test [1.1.4] implies that W is a linear subspace of V. □

1.2.1 Problem. Verify that the subset $\mathcal{S}$ of all functions satisfying the differential equation $f'' + f = 0$ is a linear subspace of the function space $\mathbb{R}^{\mathbb{R}}$.

Solution. The zero function in $\mathbb{R}^{\mathbb{R}}$ has derivatives of all orders and trivially satisfies the differential equation $f'' + f = 0$, so we see that $\mathcal{S} \neq \varnothing$. Given functions $f, g \in \mathcal{S}$ and scalars $b, c \in \mathbb{R}$, we have

$$\begin{aligned}(b\,f + c\,g)'' + (b\,f + c\,g) &= b\,f'' + c\,g'' + b\,f + c\,g \\ &= b\,(f'' + f) + c\,(g'' + g) = c\,0 + d\,0 = 0\end{aligned}$$

which establishes that $b\,f + c\,g$ lies in $\mathcal{S}$. Thus, the subspace test [1.2.0] implies that $\mathcal{S}$ is a linear subspace of $\mathbb{R}^{\mathbb{R}}$. □

1.2.2 Problem. Prove that the subset of non-differentiable functions is not a linear subspace of $\mathbb{R}^{\mathbb{R}}$.

Calculus demonstrates that continuous functions, differentiable functions, and integrable functions each form a linear subspace of $\mathbb{R}^{\mathbb{R}}$.

Solution. The zero function in $\mathbb{R}^{\mathbb{R}}$ is differentiable. Since the additive identity does not belong to this subset, it is not a linear subspace. □

1.2.3 Problem. Show that the set $W := \{f \in \mathbb{Q}[t] \mid f(2) = 0\}$ is a linear subspace of $\mathbb{Q}[t]$.

Solution. Since the zero polynomial belongs to W, we have $W \neq \varnothing$. For any polynomials f, g in W and any scalars b, c in $\mathbb{Q}$, it follows that $(b\,f + c\,g)(2) = b\,f(2) + c\,g(2) = b\,0 + c\,0 = 0$, so $b\,f + c\,g \in W$. Thus, the subspace test [1.2.0] implies that W is a linear subspace. □

1.2.4 Problem. Let n be a positive integer. Show that the subset of symmetric matrices is a linear subspace of $\mathbb{K}^{n \times n}$.

Solution. Since the identity matrix $\mathbf{I}_n$ equals its transpose, the subset of symmetric matrices is nonempty. For any symmetric matrices $\mathbf{A}, \mathbf{B}$ in $\mathbb{K}^{n\times n}$ and any scalars b, c in $\mathbb{K}$, the properties of the transpose imply that $(b\,\mathbf{A} + c\,\mathbf{B})^{\mathsf{T}} = b\,\mathbf{A}^{\mathsf{T}} + c\,\mathbf{B}^{\mathsf{T}} = b\,\mathbf{A} + c\,\mathbf{B}$, so $b\,\mathbf{A} + c\,\mathbf{B}$ is also symmetric. Hence, the subspace test [1.2.0] shows that the subset of symmetric matrices is a linear subspace. □

1.2.5 Problem. Let n be a positive integer. Demonstrate that the subset $\mathfrak{sl}(n, \mathbb{K})$, consisting of all matrices with trace equal to zero, is a linear subspace of $\mathbb{K}^{n \times n}$.

Solution. Since the zero matrix has trace equal to zero, we deduce that $\mathfrak{sl}(n, \mathbb{K}) \neq \varnothing$. For any matrices $\mathbf{A}, \mathbf{B}$ in $\mathfrak{sl}(n, \mathbb{K})$ and any scalars b, c in $\mathbb{K}$, the linearity of the trace operation gives

$$\operatorname{tr}(b\,\mathbf{A} + c\,\mathbf{B}) = b\ \operatorname{tr}(\mathbf{A}) + c\ \operatorname{tr}(\mathbf{B}) = b\,0 + c\,0 = 0\,,$$

so $b\,\mathbf{A} + c\,\mathbf{B} \in \mathfrak{sl}(n, \mathbb{K})$. Therefore, the subspace test [1.2.0] establishes that $\mathfrak{sl}(n, \mathbb{K})$ is a linear subspace of $\mathbb{K}^{n \times n}$. □

1.2.6 Problem. Let n be a positive integer. Prove that the set $\mathrm{GL}(n, \mathbb{K})$, consisting of the invertible $(n \times n)$-matrices, is not a linear subspace.

Solution. The zero matrix is not invertible. Since the additive identity in $\mathbb{K}^{n \times n}$ is not in $\mathrm{GL}(n, \mathbb{K})$, the subset of invertible matrices is not a linear subspace. □

1.2.7 Problem. Let n be an integer greater than 1. Confirm that the set of non-invertible $(n \times n)$-matrices is not a linear subspace of $\mathbb{K}^{n \times n}$.

Solution. None of the matrix units $\mathbf{E}_{1,1}, \mathbf{E}_{2,2}, \dots, \mathbf{E}_{n,n}$ is invertible because all but one row or column is zero. However, their sum

$$\mathbf{E}_{1,1} + \mathbf{E}_{2,2} + \cdots + \mathbf{E}_{n,n} = \mathbf{I}_n$$

is invertible. Since the set of non-invertible matrices is not closed under addition, this subset is not a linear subspace. □

The matrix unit $\mathbf{E}_{j,k}$ is a matrix having 1 in the (j,k)-entry as its only nonzero entry.

1.2.8 Problem. Prove that the subset c_{00} of all infinite sequences with only a finite number of nonzero terms is a linear subspace of $\mathbb{K}^{\mathbb{N}}$.

Calculus proves that the convergent sequences also form a linear subspace of $\mathbb{R}^{\mathbb{N}}$.

Proof. Since the zero sequence lies in c_{00}, we have $\mathsf{c}_{00} \neq \varnothing$. For all sequences $(v_j)_{j=0}^{\infty}, (w_j)_{j=0}^{\infty}$ in c_{00} and all scalars b, c in $\mathbb{K}$, the term $c\,v_j + d\,w_j = 0$ when $v_j = 0$ and $w_j = 0$. Since both $(v_j)_{j=0}^{\infty}$ and $(w_j)_{j=0}^{\infty}$ have only a finite number of nonzero terms, it follows that $b(v_j)_{j=0}^{\infty} + c(w_j)_{j=0}^{\infty} = (b\,v_j + c\,w_j)_{j=0}^{\infty}$ also has only a finite number of nonzero terms. In other words, the linear combination $(c\,v_j + d\,w_j)_{j=0}^{\infty}$ lies in c_{00}, so the subspace test [1.2.0] confirms that c_{00} is a linear subspace of $\mathbb{R}^{\mathbb{N}}$. □

Exercises

1.2.9 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The empty set is closed under taking linear combinations.
ii. The set of skew-symmetric matrices form a linear subspace.

1.2.10 Problem. Let V be a $\mathbb{K}$-vector space. Prove that the intersection of any collection of linear subspaces in V is also a linear subspace.