Exercises

9.1.6 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The identity operator is an isometry.

ii. For any complex inner product space V and any scalar c in $\mathbb{C}$, the operator $c\,\mathrm{id}_V$ is an isometry.

iii. Over a real vector space, every transformation can be triangularized by an orthogonal matrix.

9.1.7 Problem. Let $S \colon V \to V$ be an isometry on a finite-dimensional inner product space. For any eigenvalue λ of S, show that $|\lambda| = 1$.

9.1.8 Problem. Prove that the product of two isometries is also an isometry.

9.1.9 Problem. Let $\mathbf{Q}$ be an $(n \times n)$-matrix that defines, via left multiplication, an isometry on the standard inner product space $\mathbb{C}^n$. Prove that $|\det(\mathbf{Q})| = 1$.

9.1.10 Problem. Let $\mathbf{M}$ and $\mathbf{N}$ be $(n \times n)$-complex matrices. Suppose that $\mathbf{M}$ and $\mathbf{N}$ are simultaneously unitarily similar to upper-triangular matrices. In other words, there exists a unitary matrix $\mathbf{Q}$ such that $\mathbf{Q}^\star \mathbf{M}\,\mathbf{Q}$ and $\mathbf{Q}^\star \mathbf{N}\,\mathbf{Q}$ are both upper-triangular matrices. Show that every eigenvalue of the difference $\mathbf{M}\,\mathbf{N} - \mathbf{N}\,\mathbf{M}$ must be zero.

9.2 Spectral Theorem

WHICH LINEAR OPERATORS HAVE AN ORTHONORMAL EIGENBASIS? Linear operators that are compatible with their adjoint are intriguing.

9.2.0 Definition. A linear operator T is ***normal*** if $T^\star T = T\,T^\star$.

9.2.1 Problem. Confirm that every isometry on a finite-dimensional inner product space is a normal operator.

Solution. Suppose that the linear operator $S \colon V \to V$ is an isometry on a finite-dimensional inner product space. The characterization of isometries [9.1.1, 9.1.3] implies that $S^\star S = \mathrm{id}_V = S\,S^\star$, so the linear operator S is normal. □

The naive condition that a linear operator commutes with its adjoint map has extraordinary consequences.

9.2.2 Lemma (Properties of normal operators)**.** *Let $T \colon V \to V$ be a normal operator on an inner product space V.*

i. *For all vectors $\boldsymbol{v}$ in V, we have $\|T[\boldsymbol{v}]\| = \|T^\star[\boldsymbol{v}]\|$.*

ii. *We have $\mathrm{Ker}(T) = \mathrm{Ker}(T^\star)$.*

iii. *For any eigenvector $\boldsymbol{v}$ of T with eigenvalue λ, we have $T^\star[\boldsymbol{v}] = \overline{\lambda}\,\boldsymbol{v}$.*

iv. *The eigenvectors of T having distinct eigenvalues are orthogonal.*

Proof.

i. The definition [9.0.3] of the adjoint map and the definiton of a normal operator combine to give

$$\begin{aligned}\|T[\boldsymbol{v}]\|^2 = \langle T[\boldsymbol{v}], T[\boldsymbol{v}]\rangle &= \langle \boldsymbol{v}, (T^\star\, T)[\boldsymbol{v}]\rangle \\ &= \langle \boldsymbol{v}, (T\, T^\star)[\boldsymbol{v}]\rangle = \langle T^\star[\boldsymbol{v}], T^\star[\boldsymbol{v}]\rangle = \|T^\star[\boldsymbol{v}]\|^2 .\end{aligned}$$

The nonnegativity [7.0.0] of inner products shows that, by taking the square root, we obtain $\|T[\boldsymbol{v}]\| = \|T^\star[\boldsymbol{v}]\|$.

ii. The definition [3.1.0] of the kernel, the nonnegativity [7.0.0] of inner products, and part *i* give the following equivalences:

$$\begin{aligned}\boldsymbol{v} \in \operatorname{Ker}(T) \;\Leftrightarrow\; T[\boldsymbol{v}] = 0 \;\Leftrightarrow\; &\|T[\boldsymbol{v}]\| = 0 \\ \Leftrightarrow\; &\|T^\star[\boldsymbol{v}]\| = 0 \;\Leftrightarrow\; T^\star[\boldsymbol{v}] = 0 \;\Leftrightarrow\; \boldsymbol{v} \in \operatorname{Ker}(T^\star)\end{aligned}$$

which proves that $\operatorname{Ker}(T) = \operatorname{Ker}(T^\star)$.

iii. Consider the linear operator $S \colon V \to V$ defined by $S := \lambda \operatorname{id}_V - T$. The properties [9.0.4] of adjoint maps give $S^\star = \overline{\lambda} \operatorname{id}_V - T^\star$. Using the normality of T, we have

$$\begin{aligned}S\, S^\star = (\lambda \operatorname{id}_V - T)(\overline{\lambda} \operatorname{id}_V - T^\star) &= |\lambda| \operatorname{id}_V - \lambda\, T^\star - \overline{\lambda}\, T + T\, T^\star \\ &= |\lambda| \operatorname{id}_V - \lambda\, T^\star - \overline{\lambda}\, T + T^\star\, T = (\overline{\lambda} \operatorname{id}_V - T^\star)(\lambda \operatorname{id}_V - T) = S^\star\, S ,\end{aligned}$$

so the linear operator S is also normal. Since $T[\boldsymbol{v}] = \lambda\, \boldsymbol{v}$, we see that the vector $\boldsymbol{v}$ is in $\operatorname{Ker}(S)$ and part *ii* shows that $\boldsymbol{v}$ is also in $\operatorname{Ker}(S^\star)$. Since $\mathbf{0} = S^\star[\boldsymbol{v}] = (\overline{\lambda} \operatorname{id}_V - T^\star)[\boldsymbol{v}] = \overline{\lambda}\, \boldsymbol{v} - T^\star[\boldsymbol{v}]$, we conclude that $T^\star[\boldsymbol{v}] = \overline{\lambda}\, \boldsymbol{v}$.

iv. Suppose that $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of the linear operator T having the distinct eigenvalues λ and μ respectively. Since $T[\boldsymbol{v}] = \lambda\, \boldsymbol{v}$ and $T[\boldsymbol{w}] = \mu\, \boldsymbol{w}$, the definition [9.0.3] of the adjoint map, the properties [7.0.0] of inner products, and part *iii* give

$$\lambda\, \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \langle \lambda\, \boldsymbol{v}, \boldsymbol{w}\rangle = \langle T[\boldsymbol{v}], \boldsymbol{w}\rangle = \langle \boldsymbol{v}, T^\star[\boldsymbol{w}]\rangle = \langle \boldsymbol{v}, \overline{\mu}\, \boldsymbol{w}\rangle = \mu\, \langle \boldsymbol{v}, \boldsymbol{w}\rangle ,$$

so we obtain $(\lambda - \mu)\, \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0$. Since $\lambda \neq \mu$, we conclude that $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0$ and the eigenvectors are orthogonal. □

The next result intertwines the theory of eigenvectors and the theory of inner product spaces.

9.2.3 Normal Spectral Theorem. *Let V be a finite-dimensional complex inner product space. A linear operator $T \colon V \to V$ is normal if and only if there exists an orthonormal basis of V consisting of eigenvectors for T.*

Proof. Set $n := \dim V$.

$\Rightarrow$: Suppose that the linear operator T is normal. The orthonormal triangularization theorem [9.1.5] demonstrates that there is an orthonormal basis $\mathcal{U} := (\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n)$ of V such that the matrix

$\mathbf{A} := (T)^{\mathcal{U}}_{\mathcal{U}}$ is upper-triangular. For all $1 \leqslant j \leqslant n$ and all $1 \leqslant k \leqslant n$, let the scalar $a_{j,k}$ be the (j,k)-entry in the matrix $\mathbf{A}$. Since $\mathbf{A}$ is upper-triangular, we already know that $a_{j,k} = 0$ whenever $j > k$. We prove by induction on j that $a_{j,k} = 0$ when $j < k$. In other words, the matrix $\mathbf{A}$ is diagonal and the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ are eigenvectors of the linear operator T.

We have $\|T[\boldsymbol{u}_1]\|^2 = \|\mathbf{A}\,\boldsymbol{e}_1\|^2 = \|a_{1,1}\,\boldsymbol{e}_1\|^2 = |a_{1,1}|^2$. Since the consistency [9.0.5] between adjoints and the conjugate-transpose asserts that $(T^\star)^{\mathcal{U}}_{\mathcal{U}} = \mathbf{A}^\star$, the Parseval identity [7.1.5] yields

$$\|T^\star[\boldsymbol{u}_1]\|^2 = \|\mathbf{A}^\star\,\boldsymbol{e}_1\|^2 = \|\overline{a_{1,1}}\,\boldsymbol{e}_1 + \overline{a_{1,2}}\,\boldsymbol{e}_2 + \cdots + \overline{a_{1,n}}\,\boldsymbol{e}_n\|^2 = |a_{1,1}|^2 + |a_{1,2}|^2 + \cdots + |a_{1,n}|^2 .$$

The properties of normal operators include $\|T[\boldsymbol{u}_1]\|^2 = \|T^\star[\boldsymbol{u}_1]\|^2$, so $0 = \|T^\star[\boldsymbol{u}_1]\|^2 - \|T[\boldsymbol{u}_1]\|^2 = |a_{1,2}|^2 + |a_{1,3}|^2 + \cdots + |a_{1,n}|^2$. It follows that $a_{1,2} = a_{1,3} = \cdots = a_{1,n} = 0$ proving the base case.

For the induction step, assume that $0 < j < n$ and, for all $i < j$, that $a_{i,k} = 0$ when $i < k$. As in the base case, we have

$$\begin{aligned}
\big\|T[\boldsymbol{u}_j]\big\|^2 &= \big\|\mathbf{A}\,\boldsymbol{e}_j\big\|^2 = \big\|a_{1,j}\,\boldsymbol{e}_1 + a_{2,j}\,\boldsymbol{e}_2 + \cdots + a_{j,j}\,\boldsymbol{e}_j\big\|^2 = \big\|a_{j,j}\,\boldsymbol{e}_j\big\|^2 = \big|a_{j,j}\big|^2 \\
\big\|T^\star[\boldsymbol{u}_j]\big\|^2 &= \big\|\mathbf{A}^\star\,\boldsymbol{e}_j\big\|^2 = \big\|\overline{a_{j,j}}\,\boldsymbol{e}_j + \overline{a_{j,j+1}}\,\boldsymbol{e}_{j+1} + \cdots + \overline{a_{j,n}}\,\boldsymbol{e}_n\big\|^2 = \big|a_{j,j}\big|^2 + \big|a_{j,j+1}\big|^2 + \cdots + \big|a_{j,n}\big|^2 ,
\end{aligned}$$

which yields $a_{j,j+1} = a_{j,j+2} = \cdots = a_{j,n} = 0$.

$\Leftarrow$: Suppose that the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ form an orthonormal eigenbasis for the linear operator T. For each $1 \leqslant k \leqslant n$, there exists scalar λ_k in $\mathbb{C}$ such that $T[\boldsymbol{u}_k] = \lambda_k\,\boldsymbol{u}_k$. The orthonormal coordinates [7.1.7] and the definition [9.0.3] of the adjoint map establish that, for any $1 \leqslant k \leqslant n$, we have

$$\begin{aligned}
T^\star[\boldsymbol{u}_k] &= \langle T^\star[\boldsymbol{u}_k], \boldsymbol{u}_1\rangle\,\boldsymbol{u}_1 + \langle T^\star[\boldsymbol{u}_k], \boldsymbol{u}_2\rangle\,\boldsymbol{u}_2 + \cdots + \langle T^\star[\boldsymbol{u}_k], \boldsymbol{u}_n\rangle\,\boldsymbol{u}_n \\
&= \langle \boldsymbol{u}_k, T[\boldsymbol{u}_1]\rangle\,\boldsymbol{u}_1 + \langle \boldsymbol{u}_k, T[\boldsymbol{u}_2]\rangle\,\boldsymbol{u}_2 + \cdots + \langle \boldsymbol{u}_k, T[\boldsymbol{u}_n]\rangle\,\boldsymbol{u}_n \\
&= \langle \boldsymbol{u}_k, \lambda_1\,\boldsymbol{u}_1\rangle\,\boldsymbol{u}_1 + \langle \boldsymbol{u}_k, \lambda_2\,\boldsymbol{u}_2\rangle\,\boldsymbol{u}_2 + \cdots + \langle \boldsymbol{u}_k, \lambda_n\,\boldsymbol{u}_n\rangle\,\boldsymbol{u}_n \\
&= \overline{\lambda_1}\,\langle \boldsymbol{u}_k, \boldsymbol{u}_1\rangle\,\boldsymbol{u}_1 + \overline{\lambda_2}\,\langle \boldsymbol{u}_k, \boldsymbol{u}_2\rangle\,\boldsymbol{u}_2 + \cdots + \overline{\lambda_n}\,\langle \boldsymbol{u}_k, \boldsymbol{u}_n\rangle\,\boldsymbol{u}_n \\
&= \overline{\lambda_k}\,\boldsymbol{u}_k .
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
(T\,T^\star)[\boldsymbol{u}_k] &= T\big[T^\star[\boldsymbol{u}_k]\big] = T[\overline{\lambda}_k\,\boldsymbol{u}_k] = \overline{\lambda}_k T[\boldsymbol{u}_k] = \overline{\lambda}_k\lambda_k\,\boldsymbol{u}_k = |\lambda_k|^2\,\boldsymbol{u}_k , \\
(T^\star\,T)[\boldsymbol{u}_k] &= T^\star\big[T[\boldsymbol{u}_k]\big] = T^\star[\lambda_k\,\boldsymbol{u}_k] = \lambda_k\,T^\star[\boldsymbol{u}_k] = \lambda_k\overline{\lambda}_k\,\boldsymbol{u}_k = |\lambda_k|^2\,\boldsymbol{u}_k .
\end{aligned}$$

Since the linear operators $T\,T^\star$ and $T^\star\,T$ agree on a basis, we conclude that $T\,T^\star = T^\star\,T$ and T is a normal operator. □

Exercises

9.2.4 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. For any finite-dimensional inner product space V and any complex scalar c, the linear operator $c \, \mathrm{id}_V$ is normal.
ii. Every linear operator is normal.
iii. A linear operator T is normal if and only if its adjoint $T^\star$ is normal.
iv. The eigenvalues of a normal operator are always real.

9.2.5 Problem. Let V be a finite-dimensional inner product space. Show that a linear operator $T \colon V \to V$ is normal if and only if $\|T[\boldsymbol{v}]\| = \|T^\star[\boldsymbol{v}]\|$ for all vectors $\boldsymbol{v}$ in V.

9.2.6 Problem. Let V be a finite-dimensional inner product space. Show that a linear operator $T \colon V \to V$ is normal if and only if $\langle T[\boldsymbol{v}_1], T[\boldsymbol{v}_2] \rangle = \langle T^\star[\boldsymbol{v}_1], T^\star[\boldsymbol{v}_2] \rangle$ for all vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in V.

9.2.7 Problem. Let V be a finite-dimensional inner product space. Show that a linear operator $T \colon V \to V$ is normal if and only if the linear operators $\frac{1}{2}(T + T^\star)$ and $\frac{1}{2}(T - T^\star)$ commute.

9.2.8 Problem. Let V be a finite-dimensional complex inner product space and fix complex scalar c. Show that a linear operator $T \colon V \to V$ is normal if and only if the linear operator $c \, \mathrm{id}_V + T$ is normal.

9.2.9 Problem. Exhibit two normal operators $S \colon \mathbb{C}^2 \to \mathbb{C}^2$ and $T \colon \mathbb{C}^2 \to \mathbb{C}^2$ such that the product $S\,T$ is not normal.

9.2.10 Problem. Let V be a finite-dimensional complex inner product space. Show that a normal operator $T \colon V \to V$ is an isometry if and only if all its eigenvalues have absolute value 1.

9.2.11 Problem. Consider the complex matrix $\mathbf{C} := \frac{1}{2}\begin{bmatrix} 1+\mathrm{i} & 1-\mathrm{i} \\ 1-\mathrm{i} & 1+\mathrm{i} \end{bmatrix}$.
i. Show that $\mathbf{C}$ is normal.
ii. Find an orthonormal eigenbasis for $\mathbf{C}$.

10
Positivity of Operators

When the eigenvalues of a linear operator are all real, one may ask if they are nonnegative or positive. This innocuous requirement has deep ramifications. This chapter launches an exploration of this idea.

10.1 Self-Adjoint Operators

Which linear operators on a real inner product space have an orthonormal eigenbasis? In this situation, linear operators that equal their adjoint are especially important.

10.1.0 Definition. A linear operator $T \colon V \to V$ on an inner product space V is ***self-adjoint*** if $T = T^{\star}$.

10.1.1 Lemma. *Verify that every self-adjoint operator is normal.*

Proof. Suppose that the linear operator T is self-adjoint. It follows that $T^{\star}\, T = T^2 = T\, T^{\star}$, so T is normal. □

10.1.2 Problem. Let n be a positive integer. Consider $\mathbb{R}^n$ equipped with the standard inner product. Show that left multiplication by a real $(n \times n)$-matrix determines a self-adjoint operator on $\mathbb{R}^n$ if and only if the matrix is symmetric.

Solution. The matrix of the operator defined to left multiplication relative to the standard basis equals the original matrix. Since the standard basis is an orthonormal basis for $\mathbb{R}^n$ (relative to the canonical inner product), the adjoint of this operator is the conjugate-transpose of the matrix [9.0.5]. Thus, the matrix is self-adjoint if and only if it equals its transpose, which is equivalent to being symmetric. □

Being equal to one's adjoint has an impact on a linear operator's eigenvalues and eigenspaces.

10.1.3 Lemma (Properties of self-adjoint operators)**.** *Let $T \colon V \to V$ be a self-adjoint linear operator on an inner product space V.*

1. *For any vector $\boldsymbol{v}$ in V, the inner product $\langle T[\boldsymbol{v}], \boldsymbol{v} \rangle$ is a real number.*
2. *All of the eigenvalues of the linear operator T are real.*
3. *The linear subspaces $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$ are orthogonal.*

Proof.

i. The definition [9.0.3] of the adjoint map, the definition of a self-adjoint operator, and the conjugate-symmetry [7.0.0] of an inner product give $\langle T[\boldsymbol{v}], \boldsymbol{v}\rangle = \langle \boldsymbol{v}, T^\star[\boldsymbol{v}]\rangle = \langle \boldsymbol{v}, T[\boldsymbol{v}]\rangle = \overline{\langle T[\boldsymbol{v}], \boldsymbol{v}\rangle}$, so we see that $\langle T[\boldsymbol{v}], \boldsymbol{v}\rangle$ is a real number.

ii. Let $\boldsymbol{v}$ be an eigenvector of linear operator T with eigenvalue λ. Properties [7.0.0, 7.0.9] of inner products, the definition [9.0.3] of the adjoint map, and the definition of a self-adjoint operator give

$$\begin{aligned} \lambda \langle \boldsymbol{v}, \boldsymbol{v}\rangle = \langle \lambda\, \boldsymbol{v}, \boldsymbol{v}\rangle &= \langle T[\boldsymbol{v}], \boldsymbol{v}\rangle \\ &= \langle \boldsymbol{v}, T^\star[\boldsymbol{v}]\rangle = \langle \boldsymbol{v}, T[\boldsymbol{v}]\rangle = \langle \boldsymbol{v}, \lambda\, \boldsymbol{v}\rangle = \overline{\lambda}\, \langle \boldsymbol{v}, \boldsymbol{v}\rangle \,. \end{aligned}$$

Since $0 = (\lambda - \overline{\lambda})\, \langle \boldsymbol{v}, \boldsymbol{v}\rangle$ and eigenvectors are nonzero [5.0.0], the positivity [7.0.0] of inner products shows that $\lambda = \overline{\lambda}$. We conclude that λ is a real number.

iii. Consider the vector $\boldsymbol{u}$ in $\mathrm{Ker}(T)$ and the vector $\boldsymbol{w}$ in $\mathrm{Im}(T)$. It follows that $T[\boldsymbol{u}] = \boldsymbol{0}$ and there exists a vector $\boldsymbol{v}$ in V such that $T[\boldsymbol{v}] = \boldsymbol{w}$. The definition [9.0.3] of the adjoint map, the definition of self-adjoint, and the properties [7.0.9] of inner products give $\langle \boldsymbol{u}, \boldsymbol{w}\rangle = \langle \boldsymbol{u}, T[\boldsymbol{v}]\rangle = \langle T^\star[\boldsymbol{u}], \boldsymbol{v}\rangle = \langle T[\boldsymbol{v}], \boldsymbol{v}\rangle = \langle \boldsymbol{0}, \boldsymbol{v}\rangle = 0$, so the linear subspaces $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are orthogonal. □

Over the real numbers, we have the different spectral theorem.

10.1.4 Self-Adjoint Spectral Theorem. *Let V be a finite-dimensional real inner product space. A linear operator $T\colon V \to V$ is self-adjoint if and only if there exists an orthonormal basis of V consisting of eigenvectors for T.*

Proof.

$\Rightarrow$: Suppose that T is self-adjoint operator. Since every self-adjoint operator is normal [10.1.1], the normal spectral theorem [9.2.3] implies that there exists an orthonormal basis for V consisting of eigenvectors for T. The properties of self-adjoint operators show that the eigenvalues of T are all real, so taking the real part of each vector in this basis for V yields an eigenbasis for T. Applying the orthonormalization algorithm [7.2.0] to this real eigenbasis produces an orthonormal basis of V consisting of real eigenvectors for T because eigenvectors of T having distinct eigenvalues are orthogonal [9.2.2].

$\Leftarrow$: Suppose that real inner product space V has an orthonormal basis consisting of eigenvectors for T. As the matrix of T relative to this basis is a real diagonal matrix, it is equal to its conjugate transpose. Hence, we have $T = T^\star$ and T is self-adjoint. □

10.1.5 Problem. Let V be a finite-dimensional complex inner product space and let the linear operator $T\colon V \to V$ be normal operator. Prove that T is self-adjoint if and only if all its eigenvalues are real.

Solution. Since the properties of self-adjoint operators establishes that all eigenvalues are real, it suffices to prove the converse. The normal spectral theorem [9.2.3] implies that there exists an orthonormal basis $\mathcal{U}$ of V consisting of eigenvectors for T. Since the eigenvalues of T are all real, the consistency [9.0.5] between adjoints and the conjugate-transpose implies that $(T^\star)^{\mathcal{U}}_{\mathcal{U}} = \big((T)^{\mathcal{U}}_{\mathcal{U}}\big)^\star = (T)^{\mathcal{U}}_{\mathcal{U}}$. Therefore, it follows that $T^\star = T$. □

10.1.6 Problem. Find an orthonormal eigenbasis for the matrix

$$\mathbf{A} := \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Solution. The characteristic polynomial of the matrix $\mathbf{A}$ is

$$\begin{aligned}
\det(t\,\mathbf{I} - \mathbf{A}) &= \det\left(\begin{bmatrix} t-3 & 2 & -4 \\ 2 & t-6 & -2 \\ -4 & -2 & t-3 \end{bmatrix}\right) \xrightarrow[=]{\substack{r_1 \mapsto r_1 + r_3 \\ r_3 \mapsto r_3 + 2r_2}} \det\left(\begin{bmatrix} t-7 & 0 & t-7 \\ 2 & t-6 & -2 \\ 0 & 2(t-7) & t-7 \end{bmatrix}\right) \\
&= (t-7)^2 \det\left(\begin{bmatrix} 1 & 0 & 1 \\ 2 & t-6 & -2 \\ 0 & 2 & 1 \end{bmatrix}\right) \xrightarrow[=]{r_2 \mapsto r_2 - 2r_1} (t-7)^2 \det\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & t-6 & -4 \\ 0 & 2 & 1 \end{bmatrix}\right) \\
&\xrightarrow[=]{r_2 \mapsto r_2 + 4r_3} (t-7)^2 \det\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & t+2 & 0 \\ 0 & 2 & 1 \end{bmatrix}\right) = (t-7)^2(t+2),
\end{aligned}$$

so the eigenvalues are -2 and 7. Since

$$\begin{aligned}
-2\,\mathbf{I} - \mathbf{A} &= \begin{bmatrix} -5 & 2 & -4 \\ 2 & -8 & -2 \\ -4 & -2 & -5 \end{bmatrix} \xrightarrow[\sim]{\substack{r_1 \mapsto r_1 + 2.5r_2 \\ r_2 \mapsto 0.5r_2 \\ r_3 \mapsto r_3 + 2r_2}} \begin{bmatrix} 0 & -18 & -9 \\ 1 & -4 & -1 \\ 0 & -18 & -9 \end{bmatrix} \xrightarrow[\sim]{\substack{r_1 \mapsto -9^{-1} r_2 \\ r_2 \mapsto r_1 \\ r_3 \mapsto r_3 - r_1}} \begin{bmatrix} 1 & -4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\sim]{r_1 \mapsto r_1 + 2r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
7\,\mathbf{I} - \mathbf{A} &= \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \xrightarrow[\sim]{\substack{r_1 \mapsto r_1 - 2r_2 \\ r_3 \mapsto r_3 + 2r_2}} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\sim]{\substack{r_1 \mapsto r_2 \\ r_2 \mapsto r_1}} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

the eigenspaces are $\operatorname{Ker}(-2\,\mathbf{I} - \mathbf{A}) = \operatorname{Span}\big(\tfrac{1}{3}\begin{bmatrix} 2 & 1 & -2 \end{bmatrix}^\mathsf{T}\big)$ and $\operatorname{Ker}(7\,\mathbf{I} - \mathbf{A}) = \operatorname{Span}\big(\begin{bmatrix} 1 & -2 & 0 \end{bmatrix}^\mathsf{T}, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\mathsf{T}\big)$ For the 7-eigenspace, the orthonormalization algorithm [7.2.0] gives $\boldsymbol{u}_1 = (1/\sqrt{2})\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\mathsf{T}$ and

$$\boldsymbol{w}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{2}\left(\begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2-1 \\ -4-0 \\ 0-1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}.$$

Thus, the orthogonal matrix $\begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & -\frac{4}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{bmatrix}$ diagonalizes $\mathbf{A}$. □

Exercises

10.1.7 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. Every normal operator is self-adjoint.
ii. A linear operator has real eigenvalues if and only if it is self-adjoint.
iii. A complex symmetric matrix must have real eigenvalues.
iv. A real skew-symmetric matrix must have real eigenvalues.

10.1.8 Problem. Let $T \colon V \to V$ be a self-adjoint operator on a finite-dimensional inner product space. Assuming that 3 and 5 are the only eigenvalues of T, prove that $T^2 - 8\,T + 15\,\mathrm{id}_V = 0$.

10.1.9 Problem. Exhibit a linear operator $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that 3 and 5 are its only eigenvalues, but $T^2 - 8\,T + 15\,\mathrm{id}_{\mathbb{R}^3} \neq 0$.

10.1.10 Problem. When $b \neq 0$, orthogonally diagonalize the matrix

$$\mathbf{A} := \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{bmatrix}.$$

10.2 Positive-Semidefinite Operators

How are self-adjoint operators with nonnegative eigenvalues distinctive? Capitalizing on the first property of self-adjoint operators [10.1.3], we single out a special class of linear operators.

10.2.0 Definition. Let V be an inner product space. A self-adjoint operator $T \colon V \to V$ is ***positive-semidefinite*** if, for all vectors $\boldsymbol{v}$ in V, we have $\langle T[\boldsymbol{v}], \boldsymbol{v} \rangle \geqslant 0$. Similarly, a self-adjoint operator T is ***positive-definite*** if, for all nonzero vectors $\boldsymbol{v}$ in V, we have $\langle T[\boldsymbol{v}], \boldsymbol{v} \rangle > 0$.

For a matrix, being positive-definite is not the same as having nonnegative entries.

10.2.1 Problem. Show that $\mathbf{C} := \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ defines a positive-definite operator on $\mathbb{R}^4$.

Solution. For all vectors $\boldsymbol{v}$ in $\mathbb{R}^4$, we have

$$\begin{aligned} \langle \mathbf{C}\,\boldsymbol{v}, \boldsymbol{v} \rangle = \boldsymbol{v}^{\mathsf{T}}\,\mathbf{C}\,\boldsymbol{v} &= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \\ &= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} 2v_1 - v_2 \\ -v_1 + 2v_2 - v_3 \\ -v_2 + 2v_3 - v_4 \\ -v_3 + 2v_4 \end{bmatrix} \\ &= 2v_1^2 - 2v_1v_2 + 2v_2^2 - 2v_2v_3 + 2v_3^2 - 2v_3v_4 + 2v_4^2 \\ &= v_1^2 + (v_1 - v_2)^2 + (v_2 - v_3)^2 + (v_3 - v_4)^2 + v_4^2 \geqslant 0\,. \end{aligned}$$

It follows that $\langle \mathbf{C}\, \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$. Since $\mathbf{C} = \mathbf{C}^{*} = \mathbf{C}^{\mathsf{T}}$, we conclude that the matrix $\mathbf{C}$ is positive-definite. □

The next result imitates the various descriptions of nonnegative real numbers among all complex numbers.

10.2.2 Theorem (Characterizations of positive-semidefinite operators). *Let V and W be finite-dimensional inner product spaces over the same field of scalars. For any linear operator $T \colon V \to V$, the following are equivalent:*

a. *The linear operator T is positive-semidefinite.*
b. *The linear operator T is self-adjoint and its eigenvalues are all nonnegative real numbers.*
c. *The linear operator T has a positive-semidefinite square root.*
d. *The linear operator T has a self-adjoint square root.*
e. *There exists a linear map $S \colon V \to W$ such that $T = S^{*}\, S$.*

Proof. Set $n := \dim V$.

$a \Rightarrow b$: A positive-semidefinite linear operator is, by definition, self-adjoint. For any eigenvector $\boldsymbol{v}$ of the self-adjoint operator T with eigenvalue λ, the linearity [7.0.0] of inner products gives $0 \leqslant \langle T[\boldsymbol{v}], \boldsymbol{v} \rangle = \langle \lambda\, \boldsymbol{v}, \boldsymbol{v} \rangle = \lambda\, \langle \boldsymbol{v}, \boldsymbol{v} \rangle = \lambda\, \|\boldsymbol{v}\|^2$. As eigenvectors are nonzero [5.0.0], we deduce that $\lambda \geqslant 0$.

$b \Rightarrow c$: Since any self-adjoint linear operator is normal [10.1.1], the normal spectral theorem [9.2.3] shows that T has an orthonormal eigenbasis $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$. Let $\lambda_1, \lambda_2 \dots, \lambda_n$ be the corresponding nonnegative eigenvalues. Consider the linear map $S \colon V \to V$ defined, for all $1 \leqslant k \leqslant n$, by $S[\boldsymbol{u}_k] := \sqrt{\lambda_k}\, \boldsymbol{u}_k$; see [3.0.7]. Since the matrix of S relative to the ordered basis $(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n)$ is a real diagonal matrix, we see that S is a self-adjoint operator. For all vectors $\boldsymbol{v}$ in V, the orthonormal coordinates [7.1.7], the linearity of S, and the properties [7.0.0,7.0.9] of an inner product give

$$\begin{aligned}
\langle S[\boldsymbol{v}], \boldsymbol{v} \rangle &= \left\langle S\left[\sum_{j=1}^{n} \langle \boldsymbol{v}, \boldsymbol{u}_j \rangle\, \boldsymbol{u}_j\right], \sum_{k=0}^{n} \langle \boldsymbol{v}, \boldsymbol{u}_k \rangle\, \boldsymbol{u}_k \right\rangle = \left\langle \sum_{j=1}^{n} \langle \boldsymbol{v}, \boldsymbol{u}_j \rangle\, S[\boldsymbol{u}_j], \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{u}_k \rangle\, \boldsymbol{u}_k \right\rangle \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{u}_j \rangle\, \overline{\langle \boldsymbol{v}, \boldsymbol{u}_k \rangle}\, \langle S[\boldsymbol{u}_j], \boldsymbol{u}_k \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{u}_j \rangle\, \overline{\langle \boldsymbol{v}, \boldsymbol{u}_k \rangle} \sqrt{\lambda_j}\, \langle \boldsymbol{u}_j, \boldsymbol{u}_k \rangle \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{u}_j \rangle\, \overline{\langle \boldsymbol{v}, \boldsymbol{u}_k \rangle} \sqrt{\lambda_j}\, \delta_{j,k} = \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{u}_k \rangle\, \overline{\langle \boldsymbol{v}, \boldsymbol{u}_k \rangle} \sqrt{\lambda_k} = \sum_{k=1}^{n} |\langle \boldsymbol{v}, \boldsymbol{u}_k \rangle|^2 \sqrt{\lambda_k} \geqslant 0\,,
\end{aligned}$$

so S is positive-semidefinite. For all $1 \leqslant k \leqslant n$, we also have

$$S^2[\boldsymbol{u}_k] = S\big[S[\boldsymbol{u}_k]\big] = S[\sqrt{\lambda_k}\, \boldsymbol{u}_k] = \sqrt{\lambda_k}\, S[\boldsymbol{u}_k] = \big(\sqrt{\lambda_k}\big)^2\, \boldsymbol{u}_k = \lambda_k\, \boldsymbol{u}_k = T[\boldsymbol{u}_k]\,,$$

establishing that S is a positive-semidefinite square root of T.

$c \Rightarrow d$: A positive-semidefinite linear operator is, by definition, self-adjoint, so this implication is trivial.

$d \Rightarrow e$: Suppose that there exists a self-adjoint operator $S \colon V \to V$ such that $T = S^2$. It follows that $S^\star = S$ and $T = S^2 = S\,S = S^\star\,S$.

$e \Rightarrow a$: Consider a linear map $S \colon V \to W$ such that $T = S^\star\,S$. The properties [9.0.4] of adjoints give $T^\star = (S^\star\,S)^\star = S^\star\,S = T$, so the linear operator T is self-adjoint. For all vectors $\boldsymbol{v}$ in V, we also have

$$\langle T[\boldsymbol{v}], \boldsymbol{v} \rangle = \langle (S^\star\,S)[\boldsymbol{v}], \boldsymbol{v} \rangle = \big\langle S^\star\big[S[\boldsymbol{v}]\big], \boldsymbol{v} \big\rangle = \langle S[\boldsymbol{v}], S[\boldsymbol{v}] \rangle = \|S[\boldsymbol{v}]\|^2 \geqslant 0\,,$$

which proves that T is positive-semidefinite. □

10.2.3 Corollary (Unique positive-semidefinite square roots)**.** *A positive-semidefinite operator on a finite-dimensional inner product space has unique positive-semidefinite square root.*

Proof. Let V be a finite-dimensional inner product space. Suppose that $T \colon V \to V$ and $S \colon V \to V$ are a positive-semidefinite operators such that $T = S^2$. It follows that $S\,T = S\,S^2 = S^3 = S^2\,S = T\,S$. Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the distinct eigenvalues of T and, for all $1 \leqslant k \leqslant r$, let W_k denote the λ_k-eigenspace of T. For any vector $\boldsymbol{v}$ in W_k, we have $T\big[S[\boldsymbol{v}]\big] = (T\,S)[\boldsymbol{v}] = (S\,T)[\boldsymbol{v}] = S\big[T[\boldsymbol{v}]\big] = S[\lambda_k\,\boldsymbol{v}] = \lambda_k\,S[\boldsymbol{v}]$, proving that the image vector $S[\boldsymbol{v}]$ is also in W_k. Thus, the restriction $S|_{W_k}$ is a linear operator on W_k. Since S is self-adjoint, the normal spectral theorem [9.2.3] establishes that there exists an orthonormal basis $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_{m_k}$ for the linear subspace W_k consisting of eigenvectors for S. For all $1 \leqslant j \leqslant m_k$, let μ_j be the nonnegative eigenvalue corresponding to $\boldsymbol{u}_j$. Since

$$\lambda_k\,\boldsymbol{u}_j = T[\boldsymbol{u}_j] = S^2[\boldsymbol{u}_j] = S\big[S[\boldsymbol{u}_j]\big] = S[\mu_j\,\boldsymbol{u}_j] = \mu_j\,S[\boldsymbol{u}_j] = \mu_j^2\,\boldsymbol{u}_j\,,$$

we see that $\mu_j = \sqrt{\lambda_k}$. It follows that the restriction $S|_{W_k} \colon W_k \to W_k$ is the diagonal operator $\sqrt{\lambda_k}\,\mathrm{id}_{W_k}$. Therefore, the positive-semidefinite square root of T is uniquely determined. □

Exercises

10.2.4 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. A linear operator is positive-definite if and only if it is self-adjoint and its eigenvalues are all positive.
ii. Every positive-definite operator is invertible.
iii. The inverse of a positive-definite operator is also itself positive-definite.
iv. Only positive-semidefinite operators have square roots.
v. Every positive-semidefinite operator has a unique self-adjoint square root.

10.2.5 Problem. For any linear map $T \colon V \to W$ between inner product spaces, prove that $T^\star\,T$ is a positive-semidefinite operator on V and $T^\star\,T$ is a positive-semidefinite operator on W.