## 10.3 Polar Decomposition

Can we generalize the polar form for complex numbers? Since complex $(1 \times 1)$-matrices are linear operators on the vector space $\mathbb{C}^1$, we can ambitiously attempt to lift ideas from the complex numbers to linear maps. We have the following analogy in mind.

| numbers | maps |
|---|---|
| complex number: $z \in \mathbb{C}$ | linear map: $T\colon V \to W$ |
| conjugate: $\bar{z} \in \mathbb{C}$ | adjoint: $T^\star\colon W \to V$ |
| points on the unit circle: $\bar{z}\, z = 1$ | isometries: $T^\star\, T = \mathrm{id}_V$ |
| real numbers: $z = \bar{z}$ | self-adjoint operators: $T = T^\star$ |
| nonnegative real numbers: $z \geqslant 0$ | positive-semidefinite operators |
| polar form: $z = re^{\mathrm{i}\theta}$ | ?? |

Table 10.1: Analogy between complex numbers and linear operators

Given the success of the first parts in this analogy, one wonders if every linear operator can be expressed as a product of a positive-semidefinite operator and an isometry.

**10.3.0 Lemma** (Positive part)**.** *Let $V$ and $W$ be finite-dimensional inner product spaces. For any linear map $T\colon V \to W$, let $\sqrt{T^\star\, T}\colon V \to V$ denote the unique positive-semidefinite square root of the positive-semidefinite operator $T^\star\, T\colon V \to V$.*

Theorem 10.2.2 and Corollary 10.2.3 already establish that $T^\star\, T$ has a unique positive-semidefinite square root.

i. *For any vector $\boldsymbol{v}$ in $V$, we have* $\|T[\boldsymbol{v}]\|_W = \big\|\big(\sqrt{T^\star\, T}\big)[\boldsymbol{v}]\big\|_V$.
ii. *We have* $\operatorname{Ker}(T) = \operatorname{Ker}(\sqrt{T^\star\, T})$.

*Proof.*

*i.* For any vector $\boldsymbol{v}$ in $V$, the definition [7.1.0] of the norm, properties [9.0.4] of adjoint maps, and the self-adjointness of the linear map $\sqrt{T^\star\, T}$ give

$$\begin{aligned}\|T[\boldsymbol{v}]\|_W^2 &= \langle T[\boldsymbol{v}], T[\boldsymbol{v}]\rangle_W = \langle (T^\star\, T)[\boldsymbol{v}], \boldsymbol{v}\rangle_V = \big\langle \big(\sqrt{T^\star\, T}\big)^2[\boldsymbol{v}], \boldsymbol{v}\big\rangle_V = \big\langle \big(\sqrt{T^\star\, T}\big)\big[\big(\sqrt{T^\star\, T}\big)[\boldsymbol{v}]\big], \boldsymbol{v}\big\rangle_V \\ &= \langle (\sqrt{T^\star\, T})[\boldsymbol{v}], (\sqrt{T^\star\, T})^\star[\boldsymbol{v}]\rangle_V = \big\langle \big(\sqrt{T^\star\, T}\big)[\boldsymbol{v}], \big(\sqrt{T^\star\, T}\big)[\boldsymbol{v}]\big\rangle_V = \|(\sqrt{T^\star\, T})[\boldsymbol{v}]\|_V^2 .\end{aligned}$$

The nonnegative [7.0.0] of inner products shows that, by taking the square root, we have $\|T[\boldsymbol{v}]\|_W = \|(\sqrt{T^\star T})[\boldsymbol{v}]\|_V$.

*ii.* The definition of the kernel [3.1.0], the positivity [7.1.1] of norms, and part *i* yields the following equivalences:

$$\boldsymbol{v} \in \operatorname{Ker}(T) \Leftrightarrow T[\boldsymbol{v}] = \mathbf{0}_W \Leftrightarrow \|T[\boldsymbol{v}]\|_W = 0 \Leftrightarrow \|(\sqrt{T^\star T})[\boldsymbol{v}]\|_V = 0 \Leftrightarrow (\sqrt{T^\star T})[\boldsymbol{v}] = \mathbf{0}_V \Leftrightarrow \boldsymbol{v} \in \operatorname{Ker}(\sqrt{T^\star T}) \, ,$$

which proves that $\operatorname{Ker}(T) = \operatorname{Ker}(\sqrt{T^\star T})$. □

**10.3.1 Theorem** (Polar decomposition)**.** *Let $V$ and $W$ be two finite-dimensional inner product spaces such that* $\dim W \geqslant \dim V$. *For any linear map $T\colon V \to W$, there exists an isometry $S\colon V \to W$ such that $T = S\sqrt{T^\star\, T}$.*

*Proof.* Since the linear operator $\sqrt{T^\star T}$ is self-adjoint [10.2.3], the self-adjoint spectral theorem [10.1.4] implies that there exists an orthonormal basis of $V$ consisting of eigenvectors for $\sqrt{T^\star T}$. The basis vectors lying in the 0-eigenspace span $\operatorname{Ker}(\sqrt{T^\star T})$ and the basis vectors with nonzero eigenvalues span $\operatorname{Im}(\sqrt{T^\star T})$. Hence, for all vectors $\boldsymbol{v}$ in $V$, there exists unique vectors $\boldsymbol{v}'$ in $\operatorname{Ker}(\sqrt{T^\star T})$ and $\boldsymbol{v}''$ in $\operatorname{Im}(\sqrt{T^\star T})$ such that $\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v}''$. The properties [10.1.3] of self-adjointness prove that the linear subspaces $\operatorname{Ker}(\sqrt{T^\star T})$ and $\operatorname{Im}(\sqrt{T^\star T})$ are orthogonal. To exhibit the isometry $S$, we construct linear maps on $\operatorname{Ker}(\sqrt{T^\star T})$ and $\operatorname{Im}(\sqrt{T^\star T})$ separately.

Set $n := \dim V$ and $r := \dim \operatorname{Im}(T)$. The dimension formula [3.1.6] shows that $\dim \operatorname{Ker}(T) = n - r$ and part *ii* of the positive part lemma shows that $\operatorname{Ker}(T) = \operatorname{Ker}(\sqrt{T^\star T})$. Choose an orthonormal basis $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_{n-r}$ for the linear subspace $\operatorname{Ker}(\sqrt{T^\star T}) \subseteq V$. Similarly, set $m := \dim W$, choose an orthonormal basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_r$ for the linear subspace $\operatorname{Im}(T) \subseteq W$, and extend it to an orthonormal basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m$ of $W$. Let $W' := \operatorname{Span}(\boldsymbol{w}_{r+1}, \boldsymbol{w}_{r+2}, \dots, \boldsymbol{w}_m)$. By construction, the linear subspaces $W'$ and $\operatorname{Im}(T)$ are orthogonal and, by hypothesis, we have $\dim W' = m - r \geqslant n - r = \dim \operatorname{Ker}(\sqrt{T^\star T})$. The linear map $S_1 \colon \operatorname{Ker}(\sqrt{T^\star T}) \to W'$ is defined, for all $1 \leqslant j \leqslant n - r$, by $S_1[\boldsymbol{u}_j] = \boldsymbol{w}_{r+j}$. Using the Parseval identity [7.1.5] twice gives

$$\begin{aligned}\|S_1[c_1\,\boldsymbol{u}_1 + c_2\,\boldsymbol{u}_2 + \cdots + c_{n-r}\,\boldsymbol{u}_{n-r}]\|_W^2 &= \|c_1\,\boldsymbol{w}_{r+1} + c_2\,\boldsymbol{w}_{r+2} + \cdots + c_{n-r}\,\boldsymbol{w}_n\|_W^2 \\ &= |c_1|^2 + |c_2|^2 + \cdots + |c_{n-r}|^2 = \|c_1\,\boldsymbol{u}_1 + c_2\,\boldsymbol{u}_2 + \cdots + c_{n-r}\,\boldsymbol{u}_{n-r}\|_V^2 .\end{aligned}$$

The nonnegativity [7.0.0] shows that, by taking the square root, we obtain $\|S_1[\boldsymbol{u}]\|_W = \|\boldsymbol{u}\|_V$ for all vectors $\boldsymbol{u}$ in $\operatorname{Ker}(\sqrt{T^\star T})$.

We next focus on $\operatorname{Im}(\sqrt{T^\star T})$. Consider vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in $V$ such that $(\sqrt{T^\star T})[\boldsymbol{v}_1] = (\sqrt{T^\star T})[\boldsymbol{v}_2]$. Part *i* of the positive part lemma and the linearity of the maps give

$$\|T[\boldsymbol{v}_1] - T[\boldsymbol{v}_2]\|_W = \|T[\boldsymbol{v}_1 - \boldsymbol{v}_2]\|_W = \|(\sqrt{T^\star T})[\boldsymbol{v}_1 - \boldsymbol{v}_2]\|_V = \|(\sqrt{T^\star T})[\boldsymbol{v}_1] - (\sqrt{T^\star T})[\boldsymbol{v}_1]\|_V = 0 ,$$

so the properties [7.1.1] of norms show that $T[\boldsymbol{v}_1] = T[\boldsymbol{v}_2]$. Hence, the linear map $S_2 \colon \operatorname{Im}(\sqrt{T^\star T}) \to \operatorname{Im}(T)$ defined, for all vectors $\boldsymbol{v}$ in $V$, by $S_2\big[(\sqrt{T^\star T})[\boldsymbol{v}]\big] = T[\boldsymbol{v}]$ is well-defined. Part *i* of the lemma also implies that, for all vectors $\boldsymbol{v}$ in $\operatorname{Im}(\sqrt{T^\star T})$, we have $\|S_2[\boldsymbol{v}]\|_W = \|\boldsymbol{v}\|_V$.

Combining $S_1$ and $S_2$ gives the linear map $S \colon V \to W$ defined by $S[\boldsymbol{v}] = S_1[\boldsymbol{v}'] + S_2[\boldsymbol{v}'']$ where $\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v}''$, $\boldsymbol{v}' \in \operatorname{Ker}(\sqrt{T^\star T})$, and $\boldsymbol{v}'' \in \operatorname{Im}(\sqrt{T^\star T})$. For all vectors $\boldsymbol{v}$ in $V$, we have

$$(S\sqrt{T^\star T})[\boldsymbol{v}] = S\big[(\sqrt{T^\star T})[\boldsymbol{v}]\big] = S_2\big[(\sqrt{T^\star T})[\boldsymbol{v}]\big] = T[\boldsymbol{v}] ,$$

so $T = S\sqrt{T^\star T}$. Moreover, the Pythagorean theorem [7.1.2] gives

$$\|S[\boldsymbol{v}]\|^2 = \big\|S_1[\boldsymbol{v}'] + S_2[\boldsymbol{v}'']\big\|^2 = \big\|S_1[\boldsymbol{v}']\big\|^2 + \big\|S_2[\boldsymbol{v}'']\big\|^2 = \big\|\boldsymbol{v}'\big\|^2 + \big\|\boldsymbol{v}''\big\|^2 = \|\boldsymbol{v}\|^2 ,$$

which proves that $S$ is an isometry. □

**10.3.2 Problem.** Find the polar decomposition of $\mathbf{A} := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix}$.

*Solution.* Since $\mathbf{A}^\star \mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, it follows that $\sqrt{\mathbf{A}^\star \mathbf{A}} = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$ and

$$\begin{aligned} \mathbf{S} &= \mathbf{A}(\sqrt{\mathbf{A}^\star \mathbf{A}})^{-1} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ s-2/\sqrt{6} & 1/\sqrt{3} & 0 \end{bmatrix}. \end{aligned}$$ □

## 10.4 Singular-Value Decomposition

CAN WE EXTEND THE SPECTRAL THEOREMS TO ALL LINEAR MAPS? To associate a diagonal matrix to every linear map, we need a pair of ordered bases: one for the source and another for the target.

**10.4.0 Definition.** Let $V$ and $W$ be finite-dimensional inner product spaces. The ***singular values*** of the linear map $T \colon V \to W$ are the eigenvalues of the linear operator $\sqrt{T^\star T} \colon V \to V$. Since $\sqrt{T^\star T}$ is the unique positive-semidefinite square root of $T^\star T \colon V \to V$, the singular values of $T$ are nonnegative real numbers and they are typically listed in increasing order.

**10.4.1 Theorem** (Singular-value decomposition)**.** *Let $V$ and $W$ be finite-dimensional inner product spaces such that $m := \dim W \geqslant \dim V =: n$. For any linear map $T \colon V \to W$ with singular values $\sigma_1, \sigma_2, \dots, \sigma_n$, there exists an orthonormal basis $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ of $V$ and an orthonormal basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m$ of $W$ such that, for all vectors $\boldsymbol{v}$ in $V$, we have*

$$T[\boldsymbol{v}] = \sigma_1 \langle \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{w}_1 + \sigma_2 \langle \boldsymbol{v}, \boldsymbol{u}_2 \rangle \boldsymbol{w}_2 + \cdots + \sigma_n \langle \boldsymbol{v}, \boldsymbol{u}_n \rangle \boldsymbol{w}_n \,.$$

*Proof.* The self-adjoint spectral theorem [10.1.4] establishes that there exists an orthonormal basis $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ of $V$ consisting of eigenvectors for the self-adjoint linear operator $\sqrt{T^\star T}$. The polar decomposition [10.3.1] shows that there exists an isometry $S \colon V \to W$ such that $T = S\sqrt{T^\star T}$. Expressing the vector $\boldsymbol{v}$ in $V$ in terms of its orthonormal coordinate [7.1.7] and applying linear operator

$T = S\sqrt{T^\star T}$, we obtain

$$\begin{aligned} T[\boldsymbol{v}] &= \big(S\sqrt{T^\star T}\big)[\boldsymbol{v}] \\ &= S\big[(\sqrt{T^\star T})[\langle \boldsymbol{v}, \boldsymbol{u}_1\rangle\ \boldsymbol{u}_1 + \langle \boldsymbol{v}, \boldsymbol{u}_2\rangle\ \boldsymbol{u}_2 + \cdots + \langle \boldsymbol{v}, \boldsymbol{u}_n\rangle\ \boldsymbol{u}_n]\big] \\ &= S\big[\langle \boldsymbol{v}, \boldsymbol{u}_1\rangle\ (\sqrt{T^\star T})[\boldsymbol{u}_1] + \langle \boldsymbol{v}, \boldsymbol{u}_2\rangle\ (\sqrt{T^\star T})[\boldsymbol{u}_2] + \cdots + \langle \boldsymbol{v}, \boldsymbol{u}_n\rangle\ (\sqrt{T^\star T})[\boldsymbol{u}_n]\big] \\ &= S[\sigma_1\ \langle \boldsymbol{v}, \boldsymbol{u}_1\rangle\ \boldsymbol{u}_1 + \sigma_2\ \langle \boldsymbol{v}, \boldsymbol{u}_2\rangle\ \boldsymbol{u}_2 + \cdots + \sigma_n\ \langle \boldsymbol{v}, \boldsymbol{u}_n\rangle\ \boldsymbol{u}_n] \\ &= \sigma_1\ \langle \boldsymbol{v}, \boldsymbol{u}_1\rangle\ S[\boldsymbol{u}_1] + \sigma_2\ \langle \boldsymbol{v}, \boldsymbol{u}_2\rangle\ S[\boldsymbol{u}_2] + \cdots + \sigma_n\ \langle \boldsymbol{v}, \boldsymbol{u}_n\rangle\ S[\boldsymbol{u}_n]\,. \end{aligned}$$

The characterizations [9.1.3] of surjective isometries demonstrate that the vectors $S[\boldsymbol{u}_1], S[\boldsymbol{u}_2], \dots, S[\boldsymbol{u}_n]$ form an orthonormal list for $W$. For all $1 \leqslant j \leqslant n$, set $\boldsymbol{w}_j := S[\boldsymbol{u}_j]$. Extending the orthonormal list $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n$ to an orthonormal basis of $W$ completes the proof. □

**10.4.2 Corollary.** *Let $m$ and $n$ be positive integers such that $m \geqslant n$. For any complex $(m \times n)$-matrix $\mathbf{A}$, there is a factorization $\mathbf{A} = \mathbf{P}\,\boldsymbol{\Sigma}\,\mathbf{Q}^\star$ where $\mathbf{P}$ is a unitary $(m \times m)$-matrix, $\mathbf{Q}$ is a unitary $(n \times n)$-matrix, and $\boldsymbol{\Sigma}$ is a diagonal $(m \times n)$-matrix whose diagonal entries are the singular values of $\mathbf{A}$.*

*Proof.* Combining the singular-value decomposition theorem and the changes of basis theorem [4.0.2] proves the claim. □

**10.4.3 Remark.** The singular-value decomposition of an $(m \times n)$-matrix $\mathbf{A}$, where $m \geqslant n$, can be computed using the following steps.

- Compute a unitary diagonalization of the product $\mathbf{A}^\star\,\mathbf{A} = \mathbf{Q}^\star\,\boldsymbol{\Lambda}\,\mathbf{Q}$ where $\mathbf{Q}^\star\,\mathbf{Q} = \mathbf{I}$, $\boldsymbol{\Lambda} := \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$, and $\lambda_j = 0$ for all $r+1 \leqslant j \leqslant n$.
- Consider the invertible $(r \times r)$-matrix $\mathbf{D} := \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_r})$ and let $\boldsymbol{\Sigma} := \left[\begin{smallmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{smallmatrix}\right]$ be a diagonal $(m \times n)$-matrix.
- For all $1 \leqslant j \leqslant r$, set $\boldsymbol{w}_j := \frac{1}{\sqrt{\lambda_j}}\,\mathbf{A}\,\boldsymbol{u}_j$ where the vector $\boldsymbol{u}_j$ denotes the $j$-th column in the matrix $Q$. Extend the list $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_r$ to an orthonormal basis of $\mathbb{K}^m$. This orthonormal basis determines the columns of the $(m \times m)$-matrix $\mathbf{P}$.

**10.4.4 Problem.** Find a singular-value decomposition of $\mathbf{A} := \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$.

*Solution.* Since

$$\mathbf{A}^\star\,\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

we have

$$\boldsymbol{\Sigma} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\mathbf{Q} = \mathbf{I}$. Because $w_1 = \frac{1}{2}a_1$ and $w_2 = \frac{1}{2}a_2$, we obtain an orthonormal basis for $\mathbb{R}^4$ by choosing $w_3 := \frac{1}{2}\begin{bmatrix}1 & 1 & -1 & -1\end{bmatrix}^\mathsf{T}$ and $w_4 := \frac{1}{2}\begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^\mathsf{T}$. Thus, a singular-value decomposition is

$$\mathbf{A} = \left(\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1\end{bmatrix}\right)\begin{bmatrix}2 & 0\\ 0 & 2\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \qquad \square$$

**10.4.5 Problem.** Find a singular value decomposition of

$$\mathbf{B} := \begin{bmatrix}1 & -1\\ -2 & 2\\ 2 & -2\end{bmatrix}.$$

*Solution.* Since

$$\mathbf{B}^\star\,\mathbf{B} = \begin{bmatrix}1 & -2 & 2\\ -1 & 2 & -2\end{bmatrix}\begin{bmatrix}1 & -1\\ -2 & 2\\ 2 & -2\end{bmatrix} = \begin{bmatrix}9 & -9\\ -9 & 9\end{bmatrix},$$

we see the eigenvalues are 18 and 0 with unit eigenvectors given by the columns of the matrix $\mathbf{Q} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$. Hence, we have

$$\mathbf{\Sigma} := \begin{bmatrix}3\sqrt{2} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}^\mathsf{T}.$$

Since we have $w_1 = \frac{1}{3\sqrt{2}}\mathbf{A}\,u_1 = \frac{1}{3}\begin{bmatrix}1 & -2 & 2\end{bmatrix}^\mathsf{T}$, we may choose $w_2 := \frac{1}{\sqrt{5}}\begin{bmatrix}2 & 1 & 0\end{bmatrix}^\mathsf{T}$ and $w_3 := \frac{1}{\sqrt{45}}\begin{bmatrix}-2 & 4 & 1\end{bmatrix}^\mathsf{T}$. Thus, a singular-value decomposition is

$$\mathbf{B} = \begin{bmatrix}1/3 & 2/\sqrt{5} & -2\sqrt{45}\\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45}\\ 2/3 & 0 & 5/\sqrt{45}\end{bmatrix}\begin{bmatrix}3\sqrt{2} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}^\star\right). \qquad \square$$

*Exercises*

**10.4.6 Problem.** Determine which of the following statements are true. If a statement is false, then provide a counterexample.

*i.* The singular values of any linear operator on a finite-dimensional vector space are also eigenvalues of the operator.
*ii.* The singular values of any matrix $\mathbf{A}$ are the eigenvalues of $\mathbf{A}^\star\,\mathbf{A}$.
*iii.* The singular values of any linear operator are nonnegative.
*iv.* Every eigenvalue of a self-adjoint matrix $\mathbf{A}$ is a singular value of $\mathbf{A}$.