Problems 12

Due: Friday, 3 December 2021 before 17:00 EDT

1. Show that the determinant of the skew-symmetric matrix

$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

is the square of a polynomial in its entries.

2. Fix a positive integer n. Consider four $(n \times n)$-matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ where $\mathbf{D}$ is invertible.

(i) Show that $\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}\right) = \det(\mathbf{A})\det(\mathbf{D})$. *Hint*: $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{D}^{-1}\mathbf{B} & \mathbf{I} \end{bmatrix}$

(ii) Find matrices $\mathbf{X}$ and $\mathbf{Y}$ which produce factorization

$$\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{X} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}.$$

(iii) Show that $\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}\right) = \det(\mathbf{A} - \mathbf{C}\mathbf{D}^{-1}\mathbf{B})\det(\mathbf{D})$.

(iv) When $\mathbf{B}\mathbf{D} = \mathbf{D}\mathbf{B}$, prove that $\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}\right) = \det(\mathbf{A}\mathbf{D} - \mathbf{C}\mathbf{B})$.

(v) When $\mathbf{B}\mathbf{D} \neq \mathbf{D}\mathbf{B}$, provide an example such that $\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}\right) \neq \det(\mathbf{A}\mathbf{D} - \mathbf{C}\mathbf{B})$.

3. (i) Compute the determinants of the following matrices:

$$\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 6 & 3 & 0 \\ 3 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 6 & 3 & 0 & 0 \\ 3 & 6 & 3 & 0 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 3 & 6 \end{bmatrix}.$$

(ii) For any nonnegative integer n, guess the determinant of the $(n \times n)$-matrix below using the results from part (i). Confirm your guess by using properties of determinants and induction.

$$\mathbf{K}_n := \begin{bmatrix} 6 & 3 & 0 & \cdots & 0 & 0 \\ 3 & 6 & 3 & \cdots & 0 & 0 \\ 0 & 3 & 6 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 6 & 3 \\ 0 & 0 & 0 & \cdots & 3 & 6 \end{bmatrix}.$$