Problems 16

Due: Friday, 4 February 2022 before 17:00 EST

P16.1. Let n be a nonnegative integer. For any nonnegative integer k such that $0 \leqslant k \leqslant n$, the ***Bernstein polynomial*** is defined to be

$$\mathrm{b}_{k,n}(t) := \binom{n}{k} t^k (1-t)^{n-k}.$$

(i) Show that the polynomials $\mathrm{b}_{0,n}(t), \mathrm{b}_{1,n}(t), \dots, \mathrm{b}_{n,n}(t)$ form a basis for $\mathbb{Q}[t]_{\leqslant n}$.

(ii) Prove that $\sum_{j=0}^{n} \mathrm{b}_{j,n}(t) = 1$.

P16.2. The set of all traceless $(n \times n)$-matrices, $\mathfrak{sl}(n,\mathbb{C}) := \{\mathbf{A} \in \mathbb{C}^{n\times n} \mid \mathrm{tr}(\mathbf{A}) = 0\}$, is a linear subspace of $\mathbb{C}^{n\times n}$. Find a basis for $\mathfrak{sl}(n,\mathbb{C})$. What is the dimension of $\mathfrak{sl}(n,\mathbb{C})$?

P16.3. Let $T \colon V \to W$ be a linear map.

(i) Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ be linearly independent vectors in the $\mathbb{K}$-vector space V. When T is injective, prove that the vectors $T[\boldsymbol{v}_1], T[\boldsymbol{v}_2], \dots, T[\boldsymbol{v}_n]$ are linearly independent in the $\mathbb{K}$-vector space W.

(ii) Consider vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ that span the $\mathbb{K}$-vector space V. When the map T is surjective, prove that the vectors $T[\boldsymbol{v}_1], T[\boldsymbol{v}_2], \dots, T[\boldsymbol{v}_n]$ span the $\mathbb{K}$-vector space W.

(iii) Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ be a basis for the $\mathbb{K}$-vector space V. When the map T is bijective, prove that the vectors $T[\boldsymbol{v}_1], T[\boldsymbol{v}_2], \dots, T[\boldsymbol{v}_n]$ are basis for the $\mathbb{K}$-vector space W.