Problems 19

Due: Friday, 4 March 2022 before 17:00 EST

P19.1. Let $\mathbb{Q}^{2\times 2}$ denote the $\mathbb{Q}$-vector space of rational (2×2)-matrices and consider the linear operator $T\colon \mathbb{Q}^{2\times 2} \to \mathbb{Q}^{2\times 2}$ defined, for all (2×2)-matrices $\mathbf{A}$, by $T(\mathbf{A}) := \mathbf{A}^{\mathsf{T}}$.

(i) Show that ± 1 are the only eigenvalues of T.

(ii) Describe the eigenvectors corresponding to each eigenvalue of T.

(iii) Find an ordered basis $\mathcal{C}$ such that $(T)_{\mathcal{C}}^{\mathcal{C}}$ is a diagonal matrix.

P19.2. Let $D\colon \mathbb{R}[t]_{\leqslant 2} \to \mathbb{R}[t]_{\leqslant 2}$ be defined by $D[f] := \frac{1}{2}\,t\,(t-1)\,f''(t) + t\,f'(t) + f(t) + t^2\,f'(0)$ where f' and f'' are the first and second derivatives of the polynomial f respectively.

(i) Let $\mathcal{M} := (1, t, t^2)$ denote the monomial basis of $\mathbb{R}[t]_{\leqslant 2}$. Compute the matrix $(D)_{\mathcal{M}}^{\mathcal{M}}$.

(ii) Find the eigenvalues of D. What is the algebraic multiplicity of each eigenvalue?

(iii) For each eigenvalue, determine linear subspace spanned by all its eigenvectors. What is the dimension of each of these linear subspace?

P19.3. The union of the zero vector and the set of all eigenvectors with an eigenvalue λ is called the ***λ-eigenspace***. The dimension of the λ-eigenspace, or equivalently the maximum number of linearly independent eigenvectors with eigenvalue λ, is the eigenvalue's ***geometric multiplicity***.

(i) Let $\mathbf{A}$ and $\mathbf{B}$ be similar matrices. Prove that the geometric multiplicities of the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are the same.

(ii) Are the following matrices similar?

$$\mathbf{A} := \begin{bmatrix} 1 & -4 & -2 \\ 0 & 1 & 0 \\ 0 & 4 & 3 \end{bmatrix} \qquad \mathbf{B} := \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$