Problems 24

Due: Friday, 8 April 2022 before 17:00 EST

P24.1. Let V be a finite-dimensional complex inner product space and let $T : V \to V$ be a linear operator.

(i) Show that T is normal if and only if the linear operators $\frac{1}{2}(T + T^\star)$ and $\frac{1}{2}(T - T^\star)$ commute.

(ii) Show that T is normal if and only if $\|T[\boldsymbol{v}]\| = \|T^\star[\boldsymbol{v}]\|$ for all vectors $\boldsymbol{v}$ in V.

P24.2. (i) Let $T : V \to V$ be a self-adjoint operator on a finite-dimensional inner product space. Assuming that 3 and 5 are the only eigenvalues of T, prove that $T^2 - 8\,T + 15\,\mathrm{id}_V = 0$.

(ii) Exhibit real (3×3)-matrix $\mathbf{A}$ such that 3 and 5 are its only eigenvalues, but $\mathbf{A}^2 - 8\,\mathbf{A} + 15\,\mathbf{I} \neq \mathbf{0}$.

P24.3. Let V be a finite-dimensional complex inner product space and let $T : V \to V$ be a linear operator satisfying $T^\star = -T$.

(i) Show that all eigenvalues of T are purely imaginary (in other words, the real part equals zero).

(ii) Show that the linear operators $\mathrm{id}_V + T$ and $\mathrm{id}_V - T$ are invertible.

(iii) Show that $(\mathrm{id}_V - T)(\mathrm{id}_V + T)^{-1}$ is an isometry.