Fall Review Problems

1. Find all numbers x for which

(a) $x^2+x+1<0$ **(b)** $\dfrac{1}{x}+\dfrac{1}{1-x}>0$ **(c)** $|x-1|+|x-2|>1$ **(d)** $|x-1|\cdot|x+2|=3$

2. Prove the following

(a) $|x-y|\leqslant|x|+|y|$ **(b)** $|x|-|y|\leqslant|x-y|$ **(c)** $\big||x|-|y|\big|\leqslant|x-y|$

3. Evaluate the following limits.

(a) $\displaystyle\lim_{\theta\to1}\ \sin\left(\frac{\pi\theta}{2}\right)$ **(b)** $\displaystyle\lim_{x\to4}\frac{x-4}{x^2-8x+16}$ **(c)** $\displaystyle\lim_{y\to-\infty}\frac{3y^5-y^2+11}{1-y^5}$

(d) $\displaystyle\lim_{w\to\infty}\left(w-\sqrt{w^2+2w}\right)$ **(e)** $\displaystyle\lim_{t\to-2}\frac{\sqrt{t^2-3}}{1+t+t^2+t^3}$ **(f)** $\displaystyle\lim_{z\to1^-}\frac{\sqrt{2z}(z-1)}{|z-1|}$

(g) $\displaystyle\lim_{\ell\to1}\frac{\ell^2+\ell-2}{\ell^2-3\ell+2}$ **(h)** $\displaystyle\lim_{j\to0^+}\frac{\ln(j^2+2j)}{\ln(j)}$ **(i)** $\displaystyle\lim_{k\to1}\frac{2k^2-(3k+1)\sqrt{k}+2}{k-1}$

(j) $\displaystyle\lim_{x\to\infty}\int_x^{2x}\frac{1}{t}\,dt$ **(k)** $\displaystyle\lim_{y\to\infty}(1+2y)^{1/(2\ln y)}$

4. Prove that $\displaystyle\lim_{x\to0^+}f\left(\frac{1}{x}\right)=\lim_{x\to\infty}f(x)$.

5. Using the definitions, prove that each of the following limits exists.

(a) $\displaystyle\lim_{x\to2}x^2-x+1=3$ **(b)** $\displaystyle\lim_{t\to1}\frac{t^2-1}{t-1}=2$ **(c)** $\displaystyle\lim_{y\to1}y^3+y+1=3$

(d) $\displaystyle\lim_{z\to0^+}\frac{|z|}{z}=1$ **(e)** $\displaystyle\lim_{w\to-2^+}\frac{w-1}{w^2+w-2}=\infty$

6. Answer the following questions by either providing an appropriate example or proof.

(a) If $\displaystyle\lim_{x\to a}f(x)$ and $\displaystyle\lim_{x\to a}g(x)$ do not exist, can $\displaystyle\lim_{x\to a}(f+g)(x)$ or $\displaystyle\lim_{x\to a}(f\cdot g)(x)$?

(b) If $\displaystyle\lim_{x\to a}f(x)$ exists and $\displaystyle\lim_{x\to a}(f+g)(x)$ exists, must $\displaystyle\lim_{x\to a}g(x)$ exist?

(c) If $\displaystyle\lim_{x\to a}f(x)$ exists and $\displaystyle\lim_{x\to a}(f\cdot g)(x)$ exists, does it follow that $\displaystyle\lim_{x\to a}g(x)$ exists?

7. Use the definition of the limit to establish the following assertions.

(a) Prove that $\displaystyle\lim_{x\to a}f(x)=\ell$ if and only if $\displaystyle\lim_{x\to a}\big[f(x)-\ell\big]=0$.

(b) Prove that $\displaystyle\lim_{x\to0}f(x)=\lim_{x\to a}f(x-a)$.

(c) Prove that $\displaystyle\lim_{x\to0}f(x)=\lim_{x\to0}f(x^3)$.

(d) Give an example where $\displaystyle\lim_{x\to0}f(x^2)$ exists, but $\displaystyle\lim_{x\to0}f(x)$ does not.

8. Suppose that f satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and that f is continuous at 0. Prove that f is continuous at a for all $a \in \mathbb{R}$.

9. Suppose that f is continuous at a and $f(a) > 0$. Prove that there is a number $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x-a| < \delta$.

10. Consider the piecewise function

$$f(x) = \begin{cases} 2 & \text{if } x \leqslant 0 \\ 3-x & \text{if } 0 < x \leqslant 1 \\ x^2+1 & \text{if } x > 1. \end{cases}$$

Is f continuous at $x = 0$? Is f continuous at $x = 1$? Justify your answers.

11. Consider the function

$$f(x) := \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f is differentiable at 0.

12. For the following functions f, use the definition of the derivative to determine whether f is differentiable at 0. If it is differentiable, find $f'(0)$.

(a) $f(x) = \begin{cases} x^2 & \text{if } x \leqslant 0 \\ x^3 & \text{if } x > 0 \end{cases}$ **(b)** $g(x) = \begin{cases} 2x & \text{if } x \leqslant 0 \\ x & \text{if } x > 0 \end{cases}$ **(c)** $h(x) = \begin{cases} x & \text{if } x \leqslant 0 \\ x+1 & \text{if } x > 0 \end{cases}$

13. Find derivatives for the following functions. Assume a and b are real constants.

(a) $f(x) = \sqrt{5x} + 5\sqrt{x} + \dfrac{5}{\sqrt{x}} - \sqrt{\dfrac{5}{x}} + \sqrt{5}$

(b) $g(\theta) = \big(\cos(1) + \tan(\theta)\big)^e$

(c) $h(t) = e^{e^t + e^{-t}}$

(d) $H(z) = \ln\big(\ln(2z^5)\big)$

(e) $T(y) = \dfrac{e^{ay}}{a^2 + yb^2}$

(f) $u = (v^2+\pi)^2(2-4v^3)^4$

(g) $M(r) = \sin\big((2r-\pi)^2\big)$

(h) $P(w) = \dfrac{1}{2^w + e^w}$

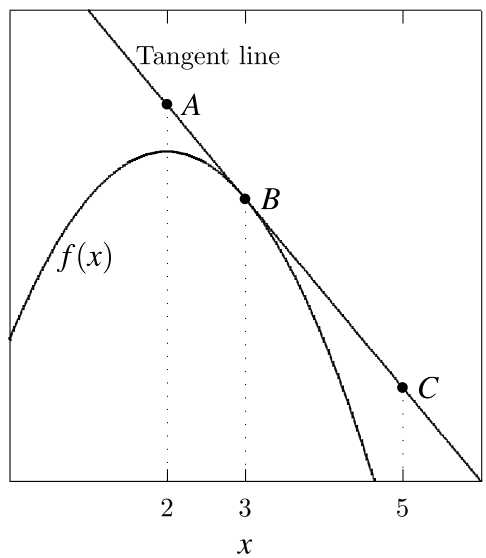
14. Let f and g be differentiable functions with values shown in the following table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	5	4	2	-2
2	3	-3	1	0

(a) If $p(x) = \dfrac{g(x)}{f(x)}$, then find $p'(1)$.

(b) Let $q(x) = x^3 f(x)$. Find the equation of the line tangent to the graph of q at $x = 2$.

15. The function graphed below has $f(3) = 6$ and $f'(3) = -2$. Find the coordinates of the points A, B, and C.



16. Assume that y is a differentiable function of x and find dy/dx. Assume a and b are constants.

(a) $e^{\cos(y)} = x^3 \arctan(y)$ **(b)** $\sin(ay) + \cos(bx) = xy$

17. Find the equations of the tangent lines to the following curves at the indicated points.

(a) $2 = \dfrac{x^2}{xy-4}$ at $(4,2)$ **(b)** $\ln(xy) = 2x$ at $(1, e^2)$

18. A spherical mothball is dissolving at a rate of $8\pi\ \mathrm{cm}\cdot\mathrm{h}^{-1}$. How fast is the radius of the mothball decreasing when the radius is 3 cm.

Hint. $V = \frac{4}{3}\pi r^3$.

19. Find the global extrema for the given functions over the specificed intervals.

(a) $f(x) = x^3 - 12x + 10$ on $[-10, 10]$ **(b)** $g(t) = (t+1)^{1/3}$ on $[-2, 7]$

(c) $h(y) = \sqrt{y} + 1/\sqrt{y}$ on $[1/2, 3/2]$ **(d)** $q(w) = |w^2 - 2w|$ on $[-2, 3]$

20. A Norman window is constructed from a rectangular sheet of glass surmounted by a semicircular sheet of glass. The light that enters through a window is proportional to the area of the window.

What are the dimensions of the Norman window having a perimeter of 30 feet that admits the most light?

21. A submarine is traveling due east and heading straight for a point P. A battleship is traveling due south and heading for the same point P. Both ships are traveling at a velocity of 30 $\text{km}\cdot\text{h}^{-1}$. Initially, their distances from P are 210 km for the submarine and 150 km for the battleship. The range of the submarine's torpedoes is 3 km. How close will the two vessels come to each other? Does the submarine have a chance to torpedo the battleship.

22. On which intervals $[a,b]$ will the following functions by injective?

(a) $f(x) = x^3 - 3x^2$ **(b)** $h(z) = \dfrac{z+1}{z^2+1}$

23. Assume that the polynomial $p(x)$ has exactly two local maxima and one local minimum and that these are the only critical points of $p(x)$.
(a) Sketch a possible graph of $p(x)$.
(b) What is the largest number of zeros $p(x)$ could have?
(c) What is the least number of zeros $p(x)$ could have?
(d) What is the least number of inflection points $p(x)$ could have?
(e) What is the smallest degree $p(x)$ could have?

24. To sketch the graphs of the functions $y = f(x)$ below, follow these steps:

- Determine the domain of f;
- Determine (if possible) the x and y-intercepts of f;
- Calculate the derivative $dy/dx = f'(x)$. Find all points at which $f'(x) = 0$ or $f'(x)$ is undefined and determine where the curve is increasing and decreasing.
- Determine local maxima and minima.
- Calculate the derivative $d^2y/dx^2 = f''(x)$. Find all points at which $f''(x) = 0$ or $f''(x)$ is undefined and determine where the curve is concave-up and concave-down.
- Find all inflection points.
- Compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If either of these is finite, then there are horizontal asymptotes.
- If f is not defined at a, determine $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$. If either limit equals $\pm\infty$ then, the line $x = a$ is a vertical asymptote.

(a) $f(x) = \sqrt[3]{x^2-1}$ **(b)** $f(x) = \dfrac{x^2-4}{x^2-1}$

25. Prove that the function $f(x) = x^2 - \cos(x)$ satisfies $f(x) = 0$ for precisely two numbers x.

26. Coal gas is produced at a gasworks. Pollutants in the gas are removed by scrubbers, which become less and less efficient as time goes on. The following measurements, made at the start of each month, show the rate at which pollutants are escaping (in tons per month) in the gas:

Time (months)	0	1	2	3	4	5	6
Rate pollutants escape (in tons per month)	5	7	8	10	13	16	20

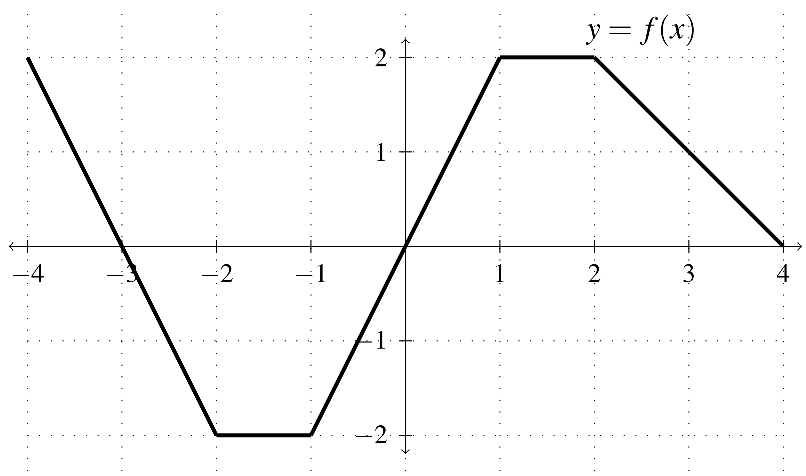
(a) Make an overestimate and an underestimate of the total quantity of pollutants the escaped during the six months.

(b) How often would measurements have to be made to find overestimates and underestimates which differ by less that one ton from the exact quantity of pollutants that escaped during the first six months?

27. Use the definition of the integral to prove that $\displaystyle\int_0^b x^3\,dx = \frac{b^4}{4}$.

28. Use the graph below to find the values of

(a) $\displaystyle\int_0^4 f(x)\,dx$ **(b)** $\displaystyle\int_{-3}^0 f(x)\,dx$ **(c)** $\displaystyle\int_{-4}^4 f(x)\,dx$ **(d)** $\displaystyle\int_{-4}^4 |f(x)|\,dx$



29. Prove that if f is continuous on $[a,b]$ then $\int_a^b f(x)\,dx = f(c)(b-a)$ for some $c \in [a,b]$.

30. Compute the following definite integrals:

(a) $\displaystyle\int_0^{\pi} \cos(x+\pi)\,dx$ **(b)** $\displaystyle\int_{-1}^3 (w^3+5w)\,dw$

(c) $\displaystyle\int_1^4 \frac{\sqrt{x}-1}{\sqrt{x}}\,dx$ **(d)** $\displaystyle\int_{-3}^3 \left|z^2+z-2\right|\,dz$

31. Find the derivatives of the following functions:

(a) $G(y) = \int_3^{\sqrt{y}} (1+u)^{200}\, du$

(b) $F(t) = \int_3^{\int_1^t \sin^3(x)\, dx} \frac{1}{1+\sin^6(y)+y^2}\, dy$

(c) $H(z) = \int_{e^z}^{\cos(z)} \ln(q^3)\, dq$

(d) $E(x) = \sin\left(\int_0^x \sin\left(\int_0^y \sin^3 t\, dt\right) dy\right)$

32. Let $F(x) = \int_2^x \frac{1}{\ln(t)}\, dt$ for $x \geqslant 2$.

(a) Find $F'(x)$.

(b) Is F increasing or decreasing? What can you say abour the concavity of its graph?

(c) Sketch a graph of $F(x)$.

33. Suppose that f is integrable on $[a,b]$. Prove that there is a number $x \in [a,b]$ such that $\int_a^x f = \int_x^b f$. Show by example that it not always possible to choose $x \in (a,b)$.

34. The area under $\frac{1}{\sqrt{x}}$ on the interval $1 \leqslant x \leqslant b$ is equal to 6. Find the value of b.

When you have completed all the problems (and not before), you may view my solutions at

http://www.mast.queensu.ca/~ggsmith/Math120/reviewFallSolutions.pdf