# Problems 06

Due: Friday, 17 February 2023 before 17:00 EST

**P6.1.** **(i)** Let $\mathbb{F}_3 := \mathbb{Z}/\langle 3 \rangle$ be the field with 3 elements. Consider the commutative ring

$$\mathbb{F}_3[\mathrm{i}] := \{a + b\,\mathrm{i} \mid a, b \in \mathbb{F}_3 \text{ and } \mathrm{i}^2 \equiv -1 \equiv 2 \mod 3\}.$$

Verify that $\mathbb{F}_3[\mathrm{i}]$ is a field.

**(ii)** Let $\mathbb{F}_5 := \mathbb{Z}/\langle 5 \rangle$ be the field with 5 elements. Consider the commutative ring

$$\mathbb{F}_5[\mathrm{i}] := \{a + b\,\mathrm{i} \mid a, b \in \mathbb{F}_5 \text{ and } \mathrm{i}^2 \equiv -1 \equiv 4 \mod 5\}.$$

Confirm that $\mathbb{F}_5[\mathrm{i}]$ is not a domain.

**P6.2.** **(i)** Let $R := \mathbb{Z}/\langle 6 \rangle$. For the polynomials

$$g = x^5 + 3\,x^3 + 5\,x^2 + 2\,x + 1 \qquad \text{and} \qquad f = 2\,x^2 + 4\,x + 1$$

in $R[x]$, find a quotient and remainder for division of $g$ by $f$.

**(ii)** Let $K$ be a field. Consider two polynomials $f$ and $g$ in the ring $K[x]$ such that $\deg(g) > 0$. Confirm that there exist unique polynomials $h_0, h_1, \dots, h_d$ in the ring $K[x]$ such that $f = h_0 + h_1\,g + h_2\,g^2 + h_3\,g^3 + \cdots + h_d\,g^d$ and $\deg(h_j) < \deg(g)$ or $h_j = 0$ for all $0 \leqslant j \leqslant d$.

**P6.3.** Let $R$ be a commutative ring. The *derivative operator* $D \colon R[x] \to R[x]$ is defined, for any polynomial $f = a_m\,x^m + a_{m-1}\,x^{m-1} + \cdots + a_1\,x + a_0$ in $R[x]$, by

$$D(f) = (m\,a_m)\,x^{m-1} + \big((m-1)\,a_{m-1}\big)\,x^{m-2} + \cdots + a_1\,.$$

**(i)** Prove that the operator $D$ is an $R$-linear map: for any two ring elements $r$ and $s$ in the coefficient ring $R$ and any two polynomials $f$ and $g$ in the ring $R[x]$, we have $D(r\,f + s\,g) = r\,D(f) + s\,D(g)$.

**(ii)** Prove that the operator $D$ satisfies the Leibniz product rule: for any two polynomials $f$ and $g$ in the ring $R[x]$, we have $D(f\,g) = D(f)\,g + f\,D(g)$.

**(iii)** Let $f$ be a polynomial in $R[x]$ and let $b \in R$ be root of $f$ having multiplicity $k$ with $k \geqslant 1$. Prove that $b$ is also a root of the derivative $D(f)$ having multiplicity at least $k-1$. Moreover, when the product $k\,1_R$ is invertible in $R$, prove that $b$ is a root of the derivative $D(f)$ having multiplicity $k-1$.