Problems 11

Due: Friday, 31 March 2023 before 17:00 EST

P11.1. Consider the subring $\mathbb{Z}[\sqrt{-5}] := \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ of field $\mathbb{C}$ of complex numbers

(i) Show that the norm function $\mathrm{N}\colon \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}$ defined by $\mathrm{N}(a + b\sqrt{-5}) = a^2 + 5\,b^2$ is compatible with multiplication, meaning that the norm of a product is equal to the product of the norms of the factors.

(ii) Confirm that $2 + \sqrt{-5}$ is an irreducible element in $\mathbb{Z}[\sqrt{-5}]$.

(iii) Verify that the ideal $\langle 2 + \sqrt{-5} \rangle$ is not prime in $\mathbb{Z}[\sqrt{-5}]$.

P11.2. Let R be a principal ideal domain. For any two distinct nonzero elements f and g with no common irreducible factor, prove that $\langle f \rangle + \langle g \rangle = \langle 1 \rangle$.

P11.3. Let R be a unique factorization domain such that the sum of two principal ideals in R is again a principal ideal. Prove that R is a principal ideal domain.

Hint. First establish that every finitely-generated ideal is principal, and then show that every ideal is finitely generated.