Problem Set #9

Due: 7 November 2008

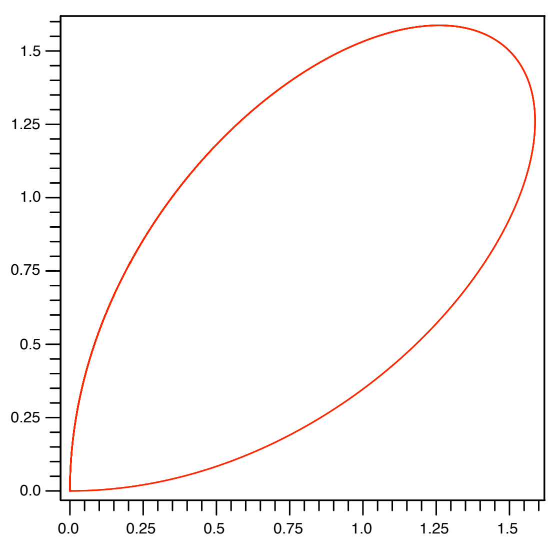
1. **(a)** Consider $\vec{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\vec{F}(x,y) := \sin(x)\vec{\imath} + (x+y)\vec{\jmath}$. Find the line integral of $\vec{F}$ around the perimeter of the rectangle with corners $(3,0)$, $(3,5)$, $(-1,5)$, and $(-1,0)$ traversed in that order.

(b) Let D be a region for which Green's theorem holds. For any two differentiable functions $P(x,y)$ and $Q(x,y)$, prove that

$$\int_{\partial D} PQ\,dx + PQ\,dy = \int_D \left[Q\left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right) + P\left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right)\right]\,dA\,.$$

2. **(a)** If $\vec{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\vec{F}(x,y) := x\vec{\jmath}$, then show that the line integral of vector field $\vec{F}$ around a closed curve in the xy-plane, oriented as in Green's Theorem, measures the area of the region enclosed by the curve.

(b) Calculate the area of the region within the folium of Descartes $x^3 + y^3 = 3xy$; it is parameterized by $\vec{\gamma}\colon [0,\infty) \to \mathbb{R}^2$ where $\vec{\gamma}(t) = \left(\frac{3t}{1+t^3}\right)\vec{\imath} + \left(\frac{3t^2}{1+t^3}\right)\vec{\jmath}$.



3. Consider the vector field $\vec{F}\colon \mathbb{R}\times(0,\infty) \to \mathbb{R}^2$ given by $\vec{F}(x,y) := \frac{x+xy^2}{y^2}\vec{\imath} - \frac{x^2+1}{y^3}\vec{\jmath}$

(a) Determine if $\vec{F}$ is path-independent.

(b) Find the work done by $\vec{F}$ in moving a particle along the curve $y = 1 + x - x^2$ from $(0,1)$ to $(1,1)$.