Problem Set #8

Due: 5 November 2010

1. **(a)** Find the volume of an ice cream cone bounded by the cone $z = \sqrt{x^2 + y^2}$ and the hemisphere $z = \sqrt{8 - x^2 - y^2}$.

(b) Find the average distance to the origin for points in the ice cream cone region bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

2. **(a)** A bead is made by drilling a cylindrical hole of radius 1 mm through a sphere of radius 5 mm. Set up a triple integral in cylindrical coordinates representing the volume of the bead. Evaluate the integral.

(b) Use the change of variables $x = u - uv$, $y = uv$, to calculate $\int_R \frac{1}{x+y}\, dy\, dx$ where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$ and $x + y = 4$.

3. Suppose L is the line segment from the origin to the point $(4, 12)$ and $\vec{\boldsymbol{F}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the vector field defined by $\vec{\boldsymbol{F}}(x, y) := xy\vec{\boldsymbol{\imath}} + x\vec{\boldsymbol{\jmath}}$.

(a) Is line integral $\int_L \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ greater than, less than, or equal to zero? Give a geometric explanation.

(b) A parameterization of L is $\vec{\boldsymbol{\gamma}} \colon [0, 4] \to \mathbb{R}^2$ where $\vec{\boldsymbol{\gamma}}(t) := t\vec{\boldsymbol{\imath}} + 3t\vec{\boldsymbol{\jmath}}$. Use this to compute $\int_L \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$.

(c) Suppose a particle leaves the point $(0, 0)$, moves along the line towards the point $(4, 12)$, stops before reaching it and backs up, stops again and reverses direction, then completes its journey to the endpoint. All travel takes place along the line segment joining the point $(0, 0)$ to the point $(4, 12)$. If we call this path L', explain why $\int_{L'} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \int_L \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$.

(d) A parameterization for a path like L' is given by $\vec{\boldsymbol{\beta}} \colon [0, 4] \to \mathbb{R}^2$ with

$$\vec{\boldsymbol{\beta}}(t) = \tfrac{1}{3}(t^3 - 6t^2 + 11t)\vec{\boldsymbol{\imath}} + (t^3 - 6t^2 + 11t)\vec{\boldsymbol{\jmath}}.$$

Check that this parameterization begins at the point $(0, 0)$ and ends at the point $(4, 12)$. Also check that all points of L' lie on the line segment connecting the point $(0, 0)$ to the point $(4, 12)$. What are the values of t at which the particle changes direction?

(e) Find $\int_{L'} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ using the parameterization in part (d).