

## Problem Set #12

Due: 2 December 2010

1. (a) Show that the path  $\vec{\gamma}: [0, 2\pi] \rightarrow \mathbb{R}^3$  defined by  $\vec{\gamma}(t) := \cos(t)\vec{i} + \sin(t)\vec{j} + \sin(2t)\vec{k}$  lies on the surface  $z = 2xy$ .
 

(b) Evaluate  $\int_C (y^3 + \cos(x))dx + (\sin(y) + z^2)dy + xdz$  where  $C$  is the closed curve parametrized by  $\vec{\gamma}$ .
  
2. (a) Evaluate the circulation of the vector field  $\vec{G}(x, y, z) := xy\vec{i} + z\vec{j} + 3y\vec{k}$  around a square of side length 6, centered at the origin lying in the  $yz$ -plane, and oriented counterclockwise viewed from the positive  $x$ -axis.
 

(b) Let  $\vec{H}(x, y, z) := (y - z)\vec{i} + (x + z)\vec{j} + xy\vec{k}$  and let  $C$  be the circle of radius 3 centered at  $(2, 1, 0)$  in the  $xy$ -plane oriented counterclockwise when viewed from above. Compute  $\int_C \vec{H} \cdot d\vec{r}$ . Is  $\vec{H}$  path-independent? Explain.

3. Water in a bathtub has velocity vector field near the drain given, for  $x, y, z$  in cm, by

$$\vec{V}(x, y, z) := \frac{-y\vec{i} + x\vec{j}}{(z^2 + 1)^2} + \frac{-z(x\vec{i} + y\vec{j})}{(z^2 + 1)^2} - \frac{\vec{k}}{z^2 + 1} = -\frac{y + xz}{(z^2 + 1)^2}\vec{i} - \frac{yz - x}{(z^2 + 1)^2}\vec{j} - \frac{1}{z^2 + 1}\vec{k} \text{ cm}\cdot\text{s}^{-1}.$$

- (a) The drain in the bathtub is a disk in the  $xy$ -plane with center at the origin and radius 1 cm. Find the rate at which the water is leaving the bathtub.
- (b) Find the divergence of  $\vec{V}$ .
- (c) Find the flux of the water through the hemisphere of radius 1, centered at the origin, lying below the  $xy$ -plane and oriented downward.
- (d) Consider the vector field

$$\vec{U}(x, y, z) := \frac{1}{2} \left( \frac{y}{z^2 + 1}\vec{i} - \frac{x}{z^2 + 1}\vec{j} - \frac{x^2 + y^2}{(z^2 + 1)^2}\vec{k} \right).$$

Compute  $\int_E \vec{U} \cdot d\vec{r}$  where  $E$  is the edge of the drain oriented clockwise when viewed from above.

- (e) Calculate  $\vec{\nabla} \times \vec{U}$ .
- (f) Explain why your answers in parts (a) and (d) are equal.