Problem Set #11

Due: Friday, March 30, 2007

1. Suppose that $T \in \mathrm{End}(V)$ has a singular-value decomposition given by

$$Tv = s_1 \langle v, u_1 \rangle e_1 + \cdots + s_n \langle v, u_n \rangle e_n$$

for all $v \in V$, where $s_1, \ldots, s_n$ are the singular values of T and $(u_1, \ldots, u_n)$, $(e_1, \ldots, e_n)$ are orthonormal bases of V.

(a) Prove that $T^* v = s_1 \langle v, e_1 \rangle u_1 + \cdots + s_n \langle v, e_n \rangle u_n$ for all $v \in V$.

(b) If T is invertible, then prove $T^{-1} v = \frac{1}{s_1} \langle v, e_1 \rangle u_1 + \cdots + \frac{1}{s_n} \langle v, e_n \rangle u_n$ for all $v \in V$.

2. **(a)** Let V be an inner-product space. Suppose that $S \in \mathrm{End}(V)$ is self-adjoint and nilpotent. Prove that $S = 0$.

(b) Define $N \in \mathrm{End}(\mathbb{K}^5)$ by $N(z_1, z_2, z_3, z_4, z_5) = (2z_2, 4z_3, -6z_4, 8z_5, 0)$. Find a square root of $I + N$.

3. Consider $T \in \mathrm{End}(V)$.

(a) Suppose T is invertible. Prove that there exists a polynomial $f \in \mathbb{K}[t]$ such that $T^{-1} = f(T)$.

(b) Suppose that $0 \neq v \in V$ and g is the monic polynomial of smallest degree such that $g(T)v = 0$. Prove that g divides the minimal polynomial of T.