Problem Set #1

Due: Friday, 17 January 2020

1. Use *MathSciNet* (available at http://www.ams.org.proxy.queensu.ca/mathscinet/), the *arXiv* (available at https://arxiv.org or http://front.math.ucdavis.edu), and *mathoverflow* (available at https://mathoverflow.net) to answer the following questions:

(a) Count the publications with the phrase "symmetric group" in their title.

(b) How many representation theory preprints appeared on the e-print archives in December 2019?

(c) Estimate the number of research level math questions tagged with 'rt.representation-theory'.

2. (a) How many of the following references

- Miklós Bóna, *Combinatorics of permutations*, Second edition, Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2012.
- David M. Bressoud, *Proofs and confirmations*, MAA Spectrum, Mathematical Association of America, Cambridge University Press, Cambridge, 1999.
- Peter J. Cameron, *Permutation groups*, London Mathematical Society Student Texts 45, Cambridge University Press, Cambridge, 1999.
- Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli, *Representation theory of the symmetric groups*, Cambridge Studies in Advanced Mathematics 121, Cambridge University Press, Cambridge, 2010.
- William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics 129, Readings in Mathematics, Springer-Verlag, New York, 1991.
- Gordon D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics 682, Springer, Berlin, 1978.
- Tsit Yuen Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics 67, American Mathematical Society, Providence, RI, 2005.
- Ambar N. Sengupta, *Representing finite groups*, Springer, New York, 2012.

are available through the Queen's library?

(b) Can you find another interesting reference related to the material in this course?

3. For any permutation $\sigma \in \mathfrak{S}_n$, the ***inversion number*** $N(\sigma)$ counts the pairs (k, ℓ) such that $1 \leqslant k < \ell \leqslant n$ and $\sigma(k) > \sigma(\ell)$. For example, the inversion number of $7\ 2\ 3\ 5\ 6\ 1\ 4\ 9\ 8 \in \mathfrak{S}_9$ is

$$13 = \left|\{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,6),(3,6),(4,6),(4,7),(5,6),(5,7),(8,9)\}\right| .$$

(a) For any permutation $\sigma \in \mathfrak{S}_n$ and any transposition $\tau \in \mathfrak{S}_n$, show that the inversion numbers $N(\sigma)$ and $N(\sigma\,\tau)$ have the opposite parity (in other words, one number is even and the other is odd).

(b) For any $\sigma \in \mathfrak{S}_n$, demonstrate that $\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)}$.

4. Fix $n \in \mathbb{N}$. For all $1 \leqslant i < n$, consider the ***adjacent transposition*** $\boldsymbol{s}_i \in \mathfrak{S}_n$ defined by $\boldsymbol{s}_i(i) = i+1$, $\boldsymbol{s}_i(i+1) = i$, and $\boldsymbol{s}_i(j) = j$ for all $j \in [n] \setminus \{i, i+1\}$.

(a) Verify that the adjacent transpositions satisfy the following relations:

$$\begin{aligned} \boldsymbol{s}_i^2 &= \mathbf{1} && \text{for all } 1 \leqslant i < n, \\ \boldsymbol{s}_i\, \boldsymbol{s}_{i+1}\, \boldsymbol{s}_i &= \boldsymbol{s}_{i+1}\, \boldsymbol{s}_i\, \boldsymbol{s}_{i+1} && \text{for all } 1 \leqslant i < n, \\ \boldsymbol{s}_i\, \boldsymbol{s}_j &= \boldsymbol{s}_j\, \boldsymbol{s}_i && \text{for all } |i-j| \geqslant 2. \end{aligned}$$

(b) Prove that every permutation is a product of adjacent transpositions.

(c) Express $7\ 2\ 3\ 5\ 6\ 1\ 4\ 9\ 8 \in \mathfrak{S}_9$ as a product of adjacent transpositions.

5. **(a)** Let $\sigma \in \mathfrak{S}_n$ be a cycle of length k. Prove that $\sigma^k = \mathbf{1}$, but $\sigma^j \neq \mathbf{1}$ for all $1 \leqslant j < k$.

(b) For any integer n greater than 2, show that the symmetric group $\mathfrak{S}_n$ is generated by two elements: the transposition $(2\ \ 1)$ and the cycle $(n\ \ 1\ \ 2\ \ 3\ \ \cdots\ \ n-1)$.