Problem Set #2

Due: Friday, 31 January 2020

1. Find a transversal for the subgroup $Y := \langle (2\ \ 1), (3\ \ 2), (5\ \ 4) \rangle$ in $\mathfrak{S}_5$.

2. Fix positive integer n. Let $\rho_{\text{def}} \colon \mathfrak{S}_n \to \mathrm{GL}(\mathbb{C}^n)$ be the defining permutation representation for $\mathfrak{S}_n$ and let $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \dots, \vec{\mathbf{e}}_n$ denote the standard basis for $\mathbb{C}^n$. Consider the $(n-1)$-dimensional linear subspace $W := \mathrm{Span}(\vec{\mathbf{e}}_1 - \vec{\mathbf{e}}_n, \vec{\mathbf{e}}_2 - \vec{\mathbf{e}}_n, \dots, \vec{\mathbf{e}}_{n-1} - \vec{\mathbf{e}}_n) \subset \mathbb{C}^n$.

(a) Show that W is a subrepresentation of the defining permutation representation.

(b) Calculate the matrices, with respect to the defining basis of W, corresponding to the permutations $(n\ \ 1\ \ 2\ \ \cdots\ \ n-1)$, and $(i+1\ \ i)$ for all $1 \leqslant i < n$.

3. The symmetric group $\mathfrak{S}_n$ has a natural action on each of its conjugacy classes. To be more explicit, let $C(\sigma) \subset \mathfrak{S}_n$ denote conjugacy class containing the permutation $\sigma \in \mathfrak{S}_n$. Setting $m := |C(\sigma)|$, we identify the standard basis of the vector space $\mathbb{C}^m$ with the elements of $C(\sigma)$. If $\vec{\mathbf{e}}_\tau$ denotes a basis vector for $\mathbb{C}^m$, then the induced action, given by $\tau \vec{\mathbf{e}}_\omega = \vec{\mathbf{e}}_{\tau \omega \tau^{-1}}$ for all $\tau \in \mathfrak{S}_n$ and all $\omega \in C(\sigma)$, determines a permutation representation $\rho \colon \mathfrak{S}_n \to \mathrm{GL}(\mathbb{C}^m)$.

(a) For $n = 4$ and $C\big((2\ \ 1)(4\ \ 3)\big)$, compute the matrices $\rho\big((i+1\ \ i)\big)$ for all $1 \leqslant i < n$.

(b) Prove that the representation $\rho \colon \mathfrak{S}_4 \to \mathrm{GL}(\mathbb{C}^3)$ from part (a) is not irreducible.

4. Consider the regular tetrahedron in $\mathbb{R}^3$, centered the origin, defined by the four vertices

$$\left(1, 0, -\tfrac{1}{\sqrt{2}}\right) \qquad \left(-1, 0, -\tfrac{1}{\sqrt{2}}\right), \qquad \left(0, 1, \tfrac{1}{\sqrt{2}}\right), \qquad \left(0, -1, \tfrac{1}{\sqrt{2}}\right).$$

(a) With the vertices indexed in the given order, compute the matrices corresponding to the adjacent transpositions $(i+1\ \ i)$ for all $1 \leqslant i < 4$.

(b) Show that the symmetries of this tetrahedron determine a representation of $\mathfrak{S}_4$ via the induced action on the vertices.

(c) Prove that this representation is irreducible.

Hint. Relative to the standard basis for $\mathbb{R}^3$, the reflection in the plane $ax + by + cz = 0$ is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{a^2 + b^2 + c^2} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix}.$$

5. Let U be the submodule of the group algebra $\mathbb{C}[\mathfrak{S}_3]$ generated by $\vec{\mathbf{u}}_1 := \mathbf{1} - (2\ \ 1) + (3\ \ 2) - (3\ \ 1\ \ 2)$.

(a) Show that U is isomorphic to the two-dimensional irreducible subrepresentation of the defining permutation representation of $\mathfrak{S}_3$ (also known as the standard representation).

(b) For all $\tau \in \mathfrak{S}_3$, let $U_\tau := \{\tau \vec{\mathbf{u}} \tau^{-1} \in \mathbb{C}[\mathfrak{S}_n] \mid \vec{\mathbf{u}} \in U\}$. Prove that $U \oplus U_{(2\ 1)} = U \oplus U_{(3\ 1)}$ but $U_{(2\ 1)} \neq U_{(3\ 1)}$.