Problem Set #6

Due: Friday, 27 March 2020

1. (a) Using the Robinson–Schensted–Knuth algorithm, determine the permutation that corresponds the tableaux pair

$$\left(\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & 8 \\ \hline 4 & 9 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|c|} \hline 1 & 2 & 8 \\ \hline 3 & 5 & 9 \\ \hline 4 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \right) .$$

(b) Using Viennot's construction, to find the tableaux pairs corresponding to the permutations

$$\sigma := 2\ 3\ 7\ 9\ 1\ 6\ 5\ 4\ 8 \in \mathfrak{S}_9 \qquad \text{and} \qquad \tau := 8\ 4\ 5\ 6\ 1\ 9\ 7\ 3\ 2 \in \mathfrak{S}_9 .$$

2. Consider a permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma = \sigma^{-1}$, and let (P,Q) denote the corresponding tableaux pair. A fixed point in the permutation σ is an index i such that $\sigma_i = i$.

(a) If the standard tableau P has shape λ, then demonstrate that the number of columns having odd length equals the alternating sum of the parts: $\sum_{i \geqslant 1} (-1)^{i+1} \lambda_i$.

(b) For any positive integer i and all $1 \leqslant j \leqslant \lambda_i$, let $L_j^{(i)}$ denote the shadow lines in the Viennot construction for the permutation σ corresponding to the i-th rows in the tableau P. Prove that the number of shadow lines $L_j^{(i)}$ meeting the diagonal line $x = y$ in a northeast corner is equal to the number of shadow lines $L_j^{(i+1)}$ meeting the diagonal line $x = y$ is a southwest corner.

FIGURE 1. A southwest corner (left) and northeast corner (right)

(c) Prove that the number of fixed points in σ is equal to the number of columns having odd length in P.

3. Fix $n \in \mathbb{N}$.

(a) Prove that any permutation of more than n^2 elements has a monotonic subsequence of length greater than n.

(b) Find a formula for the number of permutations in $\mathfrak{S}_{n^2}$ that have no monotonic subsequences of length greater than n.

4. (a) If λ is an integer partition having no hook of length 2, then prove that there exists $k \in \mathbb{N}$ such that $\lambda = (k, k-1, k-2, \dots, 2, 1)$.

(b) For any integer partition, prove that there exists $k \in \mathbb{N}$ such that the number of odd hook lengths minus the number of even hook lengths equals $\binom{k+1}{2}$.

5. Fix $n \in \mathbb{N}$.

(a) Show that the number of standard tableau of shape (n^2) is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

(b) For all $0 \leqslant k < n$, show that the number of standard tableau of shape $(n-k, 1^k)$ is the binomial coefficient $\binom{n-1}{k}$.