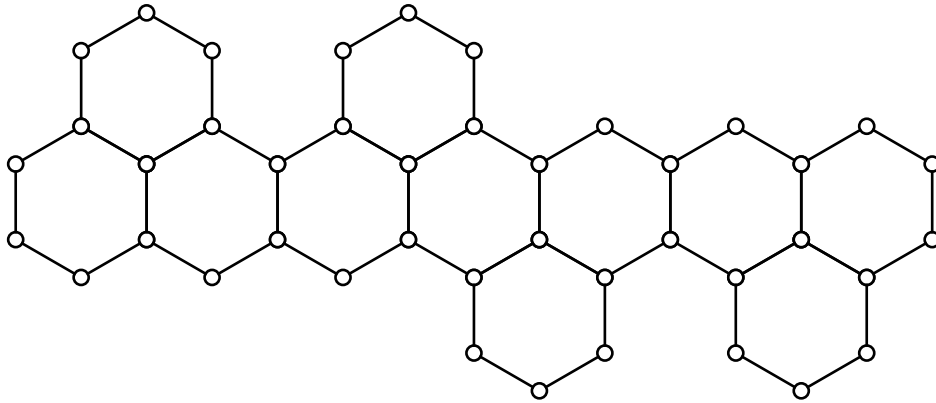# Problem Set #8
Due: Thursday, 1 November 2012

Students registered in MATH 401 should submit solutions to three of the following problems. Students in MATH 801 should submit solutions to all five.

1. Exhibit a perfect matching in the graph below or give a short proof that it has none.

2. For every graph G, prove that $\beta(G) \leq 2\alpha'(G)$. For each $k \in \mathbb{N}$, construct a graph G with $\alpha'(G) = k$ and $\beta(G) = 2k$.

3. A ***permutation matrix*** P is a matrix with entries 0 or 1 having exactly one 1 in each row and column. Prove that a square matrix of nonnegative integers can be expressed as the sum of k permutations matrices if and only if all row sums and columns sums equal k.

Hint. Every nonempty regular bipartite multigraph has a perfect matching.

4. Let M be a nonnegative real $(n \times n)$-matrix with row and column and sums all equal to m. Show that M can be written as a nonnegative linear combination of permutation matrices; in other words, $M = c_0 P_0 + \cdots + c_k P_k$ where $c_0, \ldots, c_k$ are nonnegative real numbers and $P_0, \ldots, P_k$ are permutation matrices.

Hint. Use induction on the number of nonzero entries and Hall's theorem.

5. The Turán graph $T_{n,k}$ is the complete k-partite graph with n vertices, b parts of size $a+1$, and $k-b$ parts of size a where $a = \lfloor \frac{n}{k} \rfloor$ and $b = n - ka$. Prove that
$$e(T_{n,k}) = \left(1 - \frac{1}{k} \right) \frac{n^2}{2} - \frac{b(k-b)}{2k} .$$