A Taste of Enumerative Geometry

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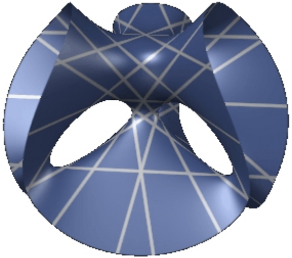
Enumerative Geometry

Basic Question: How many projective subvarieties satisfy specific geometric conditions?



Examples:

- The number of lines on a smooth cubic surface in $\mathbb{P}^3$: 27.
- The number of quadrics in $\mathbb{P}^2$ tangent to five general quadrics: $3\,264$.
- The number of rational cubic curves on a general quintic hypersurface in $\mathbb{P}^4$: $371\,206\,375$.

Goal: Describe two basic enumerative examples.

Plane Quadrics

Parameter Space: Every curve $\mathrm{V}(q) \subset \mathbb{P}^2$ having degree 2 is given by $q := a\,x^2 + b\,y^2 + c\,z^2 + d\,xy + e\,xz + f\,yz$ and corresponds to the point $[a:b:c:d:e:f] \in \mathbb{P}^5$.

Note: Setting $\mathrm{Q} := \begin{bmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{bmatrix}$ gives $[\,x\;y\;z\,]\,\mathrm{Q}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [2\,q]$.

Distinguished subvarieties in the parameter space $\mathbb{P}^5$:

- The surface defined by $\operatorname{rank} \mathrm{Q} \leqslant 1$ parametrizes the set of double lines $q = (\alpha x + \beta y + \gamma z)^2$.
- The hypersurface $\operatorname{rank} \mathrm{Q} \leqslant 2$ parametrizes the singular quadrics $q = (\alpha x + \beta y + \gamma z)(\alpha' x + \beta' y + \gamma' z)$.

Points on a Plane Quadric

Theorem: Five points, no four collinear, determine a unique quadric curve in $\mathbb{P}^2$.

Idea: For all $1 \leqslant i \leqslant 5$, $[p_i : q_i : r_i] \in \mathbb{P}^2$ corresponds to $H_i := \mathrm{V}(a\, p_i^2 + b\, q_i^2 + c\, r_i^2 + d\, p_i q_i + e\, p_i r_i + f\, q_i r_i) \subset \mathbb{P}^5$. If the hyperplanes H_i are linearly independent then $H_1 \cap H_2 \cap \cdots \cap H_5 \subset \mathbb{P}^5$ contains exactly one point.

Focus on $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, $[s:t:1]$, $[u:v:1]$. Show that these 5 points impose independent conditions on hyperplanes when no 4 are collinear.

Quadric Surfaces

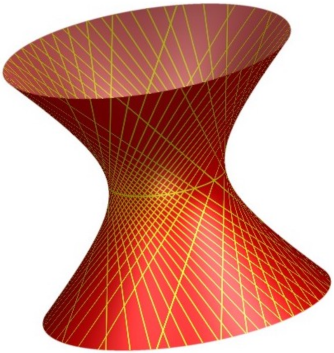
Lemma: Every smooth quadric in $\mathbb{P}^3$ is projectively equivalent to the Segre surface $\sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$.

Idea: Diagonalizing the symmetric matrix Q and rescaling the variables shows that two quadrics are projectively equivalent if and only if they have the same rank.

$\sigma(\mathbb{P}^1 \times \mathbb{P}^1) = \mathrm{V}(wz - xy)$ corresponds to $\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Consequence: There are two families of lines on a smooth quadric in $\mathbb{P}^3$ and each member of one family meets every member of the other family.

Schubert Calculus



Theorem: For any 4 pairwise-skew lines in $\mathbb{P}^3$, there exists 2 lines meeting (intersecting) them.

Idea: Choosing 3 skew lines uniquely determines a quadric surface in $\mathbb{P}^3$. All 3 belong to the same family of lines on the surface. The 4th skew line meets the quadric surface in 2 points. Each of these points lies on a line in the other family. These 2 lines meet the 4 given lines.