8 *Coordinate Rings*

In contemporary mathematics, maps between objects are at least as important as the objects themselves. However, our dictionary between geometry and algebra does not yet include entries related to maps. To rectify this shortcoming, we associated a quotient ring to each affine subvariety.

8.0 Coordinate Rings

How do we extend our algebro-geometric dictionary to maps?

8.0.0 Definition. A *morphism* $\varphi\colon \mathbb{A}^n(\mathbb{K}) \to \mathbb{A}^m(\mathbb{K})$ of affine spaces is a polynomial map

$$(x_1, x_2, \dots, x_n) \mapsto \big(\varphi_1(x_1, x_2, \dots, x_n), \varphi_2(x_1, x_2, \dots, x_n), \dots, \varphi_m(x_1, x_2, \dots, x_n)\big)$$

where each φ_j belongs to the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$.

8.0.1 Lemma. *Any morphism of affine spaces is continuous in the Zariski topology.*

Proof. Let $\varphi\colon \mathbb{A}^n(\mathbb{K}) \to \mathbb{A}^m(\mathbb{K})$ be a morphism. For any Zariski closed subset X in $\mathbb{A}^m(\mathbb{K})$, there exists polynomials $g_1, g_2, \dots, g_r$ in the ring $\mathbb{K}[y_1, y_2, \dots, y_m]$ such that

$$X = \mathrm{V}(g_1, g_2, \dots, g_r) = \{ b \in \mathbb{A}^m(\mathbb{K}) \mid g_j(b) = 0 \text{ for all } 1 \leqslant j \leqslant r \} \, .$$

It follow that

$$\begin{aligned} \varphi^{-1}(X) &= \mathrm{V}(g_1 \circ \varphi, g_2 \circ \varphi, \dots, g_r \circ \varphi) \\ &= \big\{ a \in \mathbb{A}^n(\mathbb{K}) \;\big|\; (g_j \circ \varphi)(a) = g_j\big(\varphi(a)\big) = 0 \text{ for all } 1 \leqslant j \leqslant r \big\} \, . \end{aligned}$$

The inverse image of any closed set is closed, so φ is continuous. $\square$

8.0.2 Definition. The *pull-back* of a polynomial f in $\mathbb{K}[y_1, y_2, \dots, y_m]$ along the morphism $\varphi\colon \mathbb{A}^n(\mathbb{K}) \to \mathbb{A}^m(\mathbb{K})$ is the polynomial

$$\varphi^*(f) := f \circ \varphi = f\big(\varphi_1(x_1, \dots, x_n), \dots, \varphi_m(x_1, \dots, x_n)\big)$$

in $\mathbb{K}[x_1, x_2, \dots, x_n]$. This pull-back operation corresponds to the ring homomorphism $\varphi^*\colon \mathbb{K}[y_1, y_2, \dots, y_m] \to \mathbb{K}[x_1, x_2, \dots, x_n]$ defined for all $1 \leqslant j \leqslant m$, by $y_j \mapsto \varphi_j(x_1, x_2, \dots, x_n)$. For any element c in the field $\mathbb{K}$, we have $\varphi^*(c) = c$, so the map φ^* is a $\mathbb{K}$-algebra homomorphism.

A ring homomorphism is a map $\varphi\colon R \to S$ between commutative rings R and S such that, for all $f, g \in R$, we have $\varphi(f+g) = \varphi(f) + \varphi(g)$, $\varphi(f\,g) = \varphi(f)\,\varphi(g)$, and $\varphi(1_R) = 1_S$.

8.0.3 Proposition. *The pull-back operation gives a bijection between morphisms from $\mathbb{A}^n(\mathbb{K})$ to $\mathbb{A}^m(\mathbb{K})$ and $\mathbb{K}$-algebra homomorphisms from $\mathbb{K}[y_1, y_2, \dots, y_m]$ and $\mathbb{K}[x_1, x_2, \dots, x_n]$.*

Proof. For any morphism of affine spaces, the pull-back operation produces a $\mathbb{K}$-algebra homomorphism. Conversely, any $\mathbb{K}$-algebra homomorphism $\psi \colon \mathbb{K}[y_1, y_2, \dots, y_m] \to \mathbb{K}[x_1, x_2, \dots, x_n]$ is determined by its values on the variables $y_1, y_2, \dots, y_m$. For all $1 \leqslant j \leqslant m$, set $\psi_j(x_1, x_2, \dots, x_n) := \psi(y_j)$. The polynomials $\psi_1, \psi_2, \dots, \psi_m$ define a polynomial map $\mathbb{A}^n(\mathbb{K}) \to \mathbb{A}^m(\mathbb{K})$ given by

$$(x_1, x_2, \dots, x_n) \mapsto \big(\psi_1(x_1, x_2, \dots, x_n), \psi_1(x_1, x_2, \dots, x_n), \dots, \psi_m(x_1, x_2, \dots, x_n)\big) \,.$$

This operation and the pull-back operation are obviously mutual inverses. $\square$

How do we extend these ideas to affine varieties? Restricting polynomial functions on $\mathbb{A}^n(\mathbb{K})$ to an affine subvariety X gives

$$\begin{tikzcd}
\mathrm{I}(X) \arrow[r, hook] \arrow[d] & S := \mathbb{K}[x_1, x_2, \dots, x_n] \arrow[r, two heads] \arrow[d] & S \,/\, \mathrm{I}(X) \arrow[dl] \\
0 \arrow[r, hook] & \{\text{functions on } X\} &
\end{tikzcd}$$

When $X = \mathrm{V}(x^2 + y^2 - 1) \subseteq \mathbb{A}^2(\mathbb{R})$, the polynomials x^2 and $1 - y^2$ define the same function on the affine subvariety X because $x^2 \equiv 1 - y^2 \mod \mathrm{I}(X)$.

Since functions in the ideal $\mathrm{I}(X)$ restrict to zero, the polynomial functions restricted to X can be identified with the quotient $S \,/\, \mathrm{I}(X)$.

8.0.4 Definition. The *coordinate ring* of an affine subvariety X in $\mathbb{A}^n(\mathbb{K})$ is the quotient ring $\mathbb{K}[X] := \mathbb{K}[x_1, x_2, \dots, x_n] \,/\, \mathrm{I}(X)$.

Observe that $\mathbb{K}[\mathbb{A}^n] = \mathbb{K}[x_1, x_2, \dots, x_n]$.

8.0.5 Theorem. *The following are equivalent.*
(a) *The affine subvariety X in $\mathbb{A}^n(\mathbb{K})$ is irreducible.*
(b) *The ideal $\mathrm{I}(X)$ in $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ is prime.*
(c) *The coordinate ring $\mathbb{K}[X] := \mathbb{K}[x_1, x_2, \dots, x_n] \,/\, \mathrm{I}(X)$ is a domain.* $\square$

8.0.6 Definition. Consider an affine subvariety X in $\mathbb{A}^n(\mathbb{K})$. The morphisms $\varphi \colon \mathbb{A}^n(\mathbb{K}) \to \mathbb{A}^m(\mathbb{K})$ and $\psi \colon \mathbb{A}^n(\mathbb{K}) \to \mathbb{A}^m(\mathbb{K})$ are *equivalent on* X if the pull-back homomorphisms $\varphi^* \colon \mathbb{K}[\mathbb{A}^m] \to \mathbb{K}[X]$ and $\psi^* \colon \mathbb{K}[\mathbb{A}^m] \to \mathbb{K}[X]$ are equal. The resulting equivalence classes are morphisms $\varphi \colon X \to \mathbb{A}^m(\mathbb{K})$.

When $X = \mathrm{V}(x^2 + y^2 - 1) \subset \mathbb{A}^2(\mathbb{R})$, the morphisms $\varphi \colon \mathbb{A}^2(\mathbb{R}) \to \mathbb{A}^1(\mathbb{R})$ defined by $\varphi(x, y) = x^2$ and $\psi \colon \mathbb{A}^2(\mathbb{R}) \to \mathbb{A}^1(\mathbb{R})$ defined by $\psi(x, y) = 1 - y^2$ are equivalent.

Equivalent morphisms are equivalent as functions.

8.0.7 Proposition. *Let X in $\mathbb{A}^n(\mathbb{K})$ be an affine subvariety. For any morphisms $\varphi \colon \mathbb{A}^n(\mathbb{K}) \to \mathbb{A}^m(\mathbb{K})$ and $\psi \colon \mathbb{A}^n(\mathbb{K}) \to \mathbb{A}^m(\mathbb{K})$ that are equivalent on X, we have $\varphi(a) = \psi(a)$ for all $a \in X$.*

Proof. Suppose that $\varphi(a) \neq \psi(a)$. There exists an index i such that $\varphi^*(y_i) = \varphi(a)_i \neq \psi(a)_i = \psi^*(y_i)$ which violates the equivalence. $\square$

8.0.8 Definition. For affine subvarieties $X \subseteq \mathbb{A}^n(\mathbb{K})$ and $Y \subseteq \mathbb{A}^m(\mathbb{K})$, a *morphism of affine varieties* $\varphi\colon X \to Y$ is a morphism $\varphi\colon X \to \mathbb{A}^m(\mathbb{K})$ satisfying $\varphi^*\big(\mathrm{I}(Y)\big) \subseteq \mathrm{I}(X)$ or equivalently $\varphi(X) \subseteq Y$.

Verification of equivalence. Suppose that $\varphi^*\big(\mathrm{I}(Y)\big) \subseteq \mathrm{I}(X)$. For any $a \in X$ and any $g \in \mathrm{I}(Y)$, the membership $\varphi^*(g) \in \mathrm{I}(X)$ implies that $g\big(\varphi(a)\big) = \big(\varphi^*(g)\big)(a) = 0$ which means $\varphi(a) \in Y$. Conversely, for any $\varphi(X) \subseteq Y$, the membership $g \in \mathrm{I}(Y)$ implies that, for all $a \in X$, we have $g\big(\varphi(a)\big) = 0$ which means $\varphi^*(g) \in \mathrm{I}(X)$. □

8.0.9 Proposition. *Any morphism $\varphi\colon X \to Y$ of affine subvarieties induces a $\mathbb{K}$-algebra homomorphism $\varphi^*\colon \mathbb{K}[Y] \to \mathbb{K}[X]$.*

$$\begin{tikzcd} \mathbb{K}[\mathbb{A}^m] \arrow[r, "\varphi^*"] & \mathbb{K}[\mathbb{A}^n] \arrow[r] & \mathbb{K}[X] \\ \mathrm{I}(Y) \arrow[u, hook] \arrow[r, "\varphi^*"] & \mathrm{I}(X) \arrow[u, hook] \arrow[r] & 0 \arrow[u, hook] \end{tikzcd}$$

Proof. Since the ideal $\mathrm{I}(Y)$ in $\mathbb{K}[\mathbb{A}^m]$ maps to zero in $\mathbb{K}[X]$, there is an induced $\mathbb{K}$-algebra homomorphism $\varphi^*\colon \mathbb{K}[\mathbb{A}^m]/\,\mathrm{I}(Y) \to \mathbb{K}[X]$. □

8.1 Morphisms of Affine Subvarieties

How do morphisms of affine subvarieties encode geometry?

8.1.0 Proposition. *Let X and Y be affine subvarieties. For any $\mathbb{K}$-algebra homomorphism $\psi\colon \mathbb{K}[Y] \to \mathbb{K}[X]$, there exists a morphism $\varphi\colon X \to Y$ of affine subvarieties such that $\varphi^* = \psi$.*

Proof. The affine subvarieties X in $\mathbb{A}^n$ and Y in $\mathbb{A}^m$ correspond to the ideals $\mathrm{I}(X)$ and $\mathrm{I}(Y)$ in the rings $\mathbb{K}[\mathbb{A}^n] = \mathbb{K}[x_1, x_2 \ldots, x_n]$ and $\mathbb{K}[\mathbb{A}^m] = \mathbb{K}[y_1, y_2, \ldots, y_m]$ respectively. It suffices to find a $\mathbb{K}$-algebra homomorphism $\psi'\colon \mathbb{K}[\mathbb{A}^m] \to \mathbb{K}[\mathbb{A}^n]$ such that the diagram

$$\begin{tikzcd} \mathbb{K}[\mathbb{A}^m] \arrow[r, dotted, "\psi'"] \arrow[d] & \mathbb{K}[\mathbb{A}^n] \arrow[d] \\ \mathbb{K}[Y] \arrow[r, "\psi"] & \mathbb{K}[X] \end{tikzcd}$$

commutes. Any such homomorphism arises as the pullback $(\varphi')^*$ of a polynomial map $\varphi'\colon \mathbb{A}^n(\mathbb{K}) \to \mathbb{A}^m(\mathbb{K})$. The commutativity of the diagram guarantees that $\psi'\big(\mathrm{I}(Y)\big) \subseteq \mathrm{I}(X)$, so $\varphi'(X) \subseteq Y$ and the induced map on coordinate rings $\mathbb{K}[Y] \to \mathbb{K}[X]$ is just ψ.

To construct ψ', consider $\overline{\psi}(y_j)$ for all $1 \leqslant j \leqslant m$. Choosing a representative in each coset lifts these elements in the quotient ring $\mathbb{K}[X]$ to polynomials $\varphi_j \in \mathbb{K}[x_1, x_2, \ldots, x_n]$ for all $1 \leqslant j \leqslant m$. These polynomials yield a $\mathbb{K}$-algebra homomorphism $\psi'\colon \mathbb{K}[\mathbb{A}^m] \to \mathbb{K}[\mathbb{A}^m]$ defined by $y_j \mapsto \varphi_j$ that makes the diagram commute. □

8.1.1 Corollary. *There is a bijection between morphisms $X \to Y$ of affine subvarieties and $\mathbb{K}$-algebra homomorphisms $\mathbb{K}[Y] \to \mathbb{K}[X]$.* □

8.1.2 Remark. Corollary 8.1.1 implies that any automorphism of an affine subvariety X corresponds to a $\mathbb{K}$-algebra isomorphism $\mathbb{K}[X] \to \mathbb{K}[X]$. For example, every automorphism of $\mathbb{A}^1(\mathbb{Q})$ has the form $x \mapsto ax + b$ for some $a, b \in \mathbb{Q}$.

8.1.3 Remark. Let $\varphi \colon \mathbb{A}^n(\mathbb{C}) \to \mathbb{A}^n(\mathbb{C})$ be a morphism such that the determinant of matrix $\left[{}^{\partial \varphi_i} / {}_{\partial x_j} \right]$ is nonzero. Jacobian conjecture claims that the morphism φ has an inverse. This open problem is notorious for the large number of attempted proofs that turned out to contain subtle errors.

8.1.4 Definition. A morphism $\varphi \colon X \to Y$ is *dominant* if the image $\varphi(X)$ is not contained in a proper subvariety of Y or equivalently the Zariski closure of the image is Y.

Any surjective morphism is dominant.

8.1.5 Lemma. *A morphism $\varphi \colon X \to Y$ is dominant if and only if the $\mathbb{K}$-algebra homomorphism $\varphi^* \colon \mathbb{K}[Y] \to \mathbb{K}[X]$ is injective.*

Proof. The image $\varphi(X)$ is contained in an affine subvariety $Z \subset Y$ if and only if $\varphi^*\big(\mathrm{I}(Z)\big) \subseteq \mathrm{I}(X)$. Suppose that $\varphi(X)$ is contained in a proper subvariety $Z \subset Y$. Hence, there exists a nonzero element $f \in \mathrm{I}(Z) \subseteq \mathbb{K}[Y]$ that vanishes on the image $\varphi(X)$. It follows that $\varphi^*(f) = 0 \in \mathbb{K}[X]$. Conversely, suppose that there exists a nonzero element $f \in \mathbb{K}[Y]$ such that $\varphi^*(f) = 0$. Hence, we deduce that $\varphi(X) \subset Y \cap \{b \in Y \mid f(b) = 0\} \subset Y$. □

8.1.6 Example. Consider the projection

$$\pi \colon X := \big\{(x, y) \bigm| xy = 1\big\} \to \mathbb{A}^1(k)$$

defined by $(x, y) \mapsto x$. Its image is the subset $\mathbb{A}^1(k) \setminus \{0\}$ which is not contained in a proper closed subset of $\mathbb{A}^1(k)$. ◇

8.1.7 Proposition. *Let X be an irreducible affine subvariety. For any dominant morphism $\varphi \colon X \to Y$, the affine subvariety Y is also irreducible.*

Proof. Since X is irreducible, the coordinate ring $\mathbb{K}[X]$ has no zero-divisors. The dominance assumption implies that $\mathbb{K}$-algebra homomorphism $\varphi^* \colon \mathbb{K}[Y] \to \mathbb{K}[X]$ has no kernel. Since any zero divisor in $\mathbb{K}[Y]$ would yield a zero divisor in $\mathbb{K}[X]$, we conclude that $\mathbb{K}[Y]$ is also a domain. □

8.1.8 Definition. An affine subvariety X is irreducible if and only if its coordinate ring $\mathbb{K}[X]$ is a domain. Under this assumption, the fraction field $\mathbb{K}(X)$ of $\mathbb{K}[X]$ is defined to be

$$\mathbb{K}(X) := \big\{ f/g \bigm| f, g \in \mathbb{K}[X], g \neq 0 \big\} .$$

The function field of affine space $\mathbb{A}^n$ is the field of rational functions: $\mathbb{K}(\mathbb{A}^n) = \mathbb{K}(x_1, x_2, \dots, x_n)$.

The field $\mathbb{K}(X)$ is called the *function field* of X.

8.1.9 Definition. Two affine subvarieties X and Y are *birational* (over the field $\mathbb{K}$) if their associated function fields $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ are isomorphic (as $\mathbb{K}$-algebras).

8.1.10 Remark. One can show that two irreducible varieties X and Y are birational if and only if there are rational maps $\rho\colon X \dashrightarrow Y$ and $\xi\colon Y \dashrightarrow X$ which are mutual inverses. One has to correctly define rational maps between varieties and interpret their composition.

8.1.11 Remark. An affine subvariety X is rational if and only if it is birational to affine space $\mathbb{A}^n(\mathbb{K})$ for some nonnegative integer n.

8.1.12 Remark. Is there a unique simplest variety in each birational equivalence class? The *minimal model program* aims to construct a birational model of any variety which is as simple as possible. At least three Fields medalist are connected to this program: David Mumford (1974), Shigefumi Mori (1990), and Caucher Birkar (2018).

8.2 Projective Space

What is better as an ambient space than affine space? Projective space has several different useful interpretations.

8.2.0 Definition. Projective space $\mathbb{P}^n(\mathbb{K})$ is the *set* of 1-dimensional linear subspaces of the $\mathbb{K}$-vector space $\mathbb{K}^{n+1}$.

Equivalently, $\mathbb{P}^n(\mathbb{K})$ is the set of all lines through the origin in $\mathbb{A}^{n+1}(\mathbb{K})$. Each such line has the form $\lambda\,(a_0, a_1 \dots, a_n)$ for some $\lambda \in \mathbb{K}$ where $(a_0, a_1, \dots, a_n) \in \mathbb{K}^{n+1}$ is nonzero. The points $(a_0, a_1, \dots, a_n)$ and $(a'_0, a'_1, \dots, a'_n)$ in the $\mathbb{K}$-vector space $\mathbb{K}^{n+1}$ span the same line if and only if there is a nonzero scalar $\lambda \in \mathbb{K}^*$ such that $(a'_0, a'_1, \dots, a'_n) = \lambda(a_0, a_1, \dots, a_n)$. Hence, points in projective space are identified with equivalence classes $\mathbb{P}^n = \big(\mathbb{A}^{n+1} \setminus \{0\}\big) / \sim$ where the equivalent relation $(a'_0, a'_1, \dots, a'_n) \sim (a_0, a_1, \dots, a_n)$ means that, for some $\lambda \in \mathbb{K}^*$, we have $(a'_0, a'_1, \dots, a'_n) = \lambda(a_0, a_1, \dots, a_n)$. The notation $[a_0 : a_1 : \cdots : a_n]$ represents one of these equivalence classes.

8.2.1 Example. Consider $\mathbb{P}^1(\mathbb{C})$. Fixing a reference line in $\mathbb{A}^2(\mathbb{C})$ (an affine line not passing through the origin) produces representatives for points in $\mathbb{P}^1$. Namely, the unique point where the reference line meets the line through the origin. Only one point in $\mathbb{P}^1$ fails to have such a representative: the point in projective space corresponding to the line parallel to our reference line. We call this point the "point at infinity".

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{A}^1(\mathbb{C}) \cup \{\infty\} \qquad [a_0 : a_1] \mapsto \begin{cases} \frac{a_1}{a_0} & \text{for } a_0 \neq 0; \\ \infty & \text{for } a_0 = 0. \end{cases} \qquad \diamond$$

How do we endow $\mathbb{P}^n$ with the structure of an algebraic variety? We introduce a covering by affine open subsets. For all $0 \leqslant i \leqslant n$, consider the subset $U_i := \big\{ [a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n \;\big|\; a_i \neq 0 \big\}$; the set U_i is well-defined because $(a_0, a_1, \dots, a_n) \sim (a'_0, a'_1, \dots, a'_n)$ implies that $a_i = 0$ if and only if $a'_i = 0$. We see that $\mathbb{P}^n(\mathbb{K}) = U_0 \cup U_1 \cup \cdots \cup U_n$. Each class $[a_0 : a_1 : \cdots : a_n] \in U_i$ has a distinguish representative $\big[\frac{a_0}{a_i} : \frac{a_1}{a_i} : \cdots : \frac{a_{i-1}}{a_i} : 1 : \frac{a_{i+1}}{a_i} : \cdots : \frac{a_n}{a_i} \big]$. Thus, we obtain the bijections

$$\begin{aligned} \psi_i \colon U_i &\to \mathbb{A}^n & \qquad \psi_i^{-1} \colon \mathbb{A}^n &\to U_i \\ [a_0 : a_1 : \cdots : a_n] &\mapsto \big(\tfrac{a_0}{a_i}, \cdots, \tfrac{a_{i-1}}{a_i}, \tfrac{a_{i+1}}{a_i}, \cdots, \tfrac{a_n}{a_i} \big) & (b_1, b_2, \dots, b_n) &\mapsto [b_0 : \cdots : b_{i-1} : 1 : b_i : \cdots : b_n] \end{aligned}$$

We declare ψ_i to be a homeomorphism of U_i to $\mathbb{A}^n$ with its Zariski topology, giving U_i with the structure of an affine subvariety.

How do the subsets U_i fit together or overlap? We claim that the identifications $\psi_i \colon U_i \to \mathbb{A}^n(\mathbb{K})$ transform the set $\mathbb{P}^n$ into an algebraic variety. It suffices to show that, on the intersection $U_i \cap U_j$ where $i < j$, the induced structures are compatible. We have two maps $\psi_i|_{U_i \cap U_j} \colon U_i \cap U_j \to \mathbb{A}^n$ and $\psi_j|_{U_i \cap U_j} \colon U_i \cap U_j \to \mathbb{A}^n$ inducing the composition

$$(b_1, b_2, \dots, b_n) \xrightarrow{\psi_i^{-1}} [b_1 : b_2 : \cdots : b_{i-1} : 1 : b_i : \cdots : b_n] \xrightarrow{\psi_j} \left(\frac{b_0}{b_j}, \frac{b_1}{b_j}, \dots, \frac{b_{i-1}}{b_j}, \frac{1}{b_j}, \frac{b_{i+1}}{b_j}, \dots, \frac{b_{j-1}}{b_j}, \frac{b_{j+1}}{b_j}, \dots, \frac{b_n}{b_j} \right).$$

One verifies that this is an isomorphism of affine varieties

$$\begin{array}{ccc} \psi_i(U_i \cap U_j) & \xrightarrow{\psi_j \circ \psi_i^{-1}} & \psi_j(U_i \cap U_j) \\ \| & & \| \\ \mathbb{A}^n \setminus \mathrm{V}(b_j) & \longrightarrow & \mathbb{A}^n \setminus \mathrm{V}(b_i) \end{array}$$

The open cover $\{U_i\}$ of $\mathbb{P}^n(\mathbb{C})$ defines an altas making projective space into a complex n-dimensional manifold. The change of coordinates $\psi_j \circ \psi_i^{-1}$ are holomorphic maps.

and note that $(\psi_i \circ \psi_j^{-1}) \circ (\psi_j \circ \psi_k^{-1}) = \psi_i \circ \psi_k^{-1}$.

8.2.2 Definition. A *projective subvariety* X in $\mathbb{P}^n(\mathbb{K})$ is a subset such that, for each distinguished $U_i \cong \mathbb{A}^n(\mathbb{K})$, the intersection $U_i \cap X \subseteq U_i$ is an affine subvariety. A subset $X \subseteq \mathbb{P}^n(\mathbb{K})$ is *Zariski closed* if $X \cap U_i$ is closed in each U_i. For any subset $X \subseteq \mathbb{P}^n(\mathbb{K})$, the *projective closure* $\overline{X} \subseteq \mathbb{P}^n(\mathbb{K})$ is the smallest closed subset containing X.

8.2.3 Lemma. *The union of two projective subvarieties is itself a projective subvariety. The intersection of any family of projective subvarieties is also a projective subvariety. The empty set and whole space are projective subvarieties.* □

The *Zariski topology* on projective space $\mathbb{P}^n$ is defined by taking the open sets to be complements of projective subvarieties.

8.2.4 Remark. Since there are no non-constant analytic functions on $\mathbb{P}^n(\mathbb{C})$, we cannot hope to define a projective subvariety as the common zeros of a collection of functions.