Problems 1

Due: Friday, 20 January 2023 before 17:00 EST

Students registered in MATH 413 should submit solutions to any three problems, whereas students in MATH 813 should submit solutions to all five.

P1.1. Let p be a prime integer and let $\mathbb{F}_p$ be a finite field with p elements. Demonstrate that $x^p - x$ is a nonzero polynomial in $\mathbb{F}_p[x]$ that vanishes at every point in $\mathbb{A}^1(\mathbb{F}_p)$.

P1.2. Consider the map $\sigma \colon \mathbb{A}^3(\mathbb{Q}) \to \mathbb{A}^6(\mathbb{Q})$ defined by $\sigma(x_1, x_2, x_3) := (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2)$. Let $z_1, z_2, \dots, z_6$ denote the corresponding coordinates on $\mathbb{A}^6(\mathbb{Q})$.

- **(i)** Show that the image of the map σ satisfies the equations given by the 2-minors of the symmetric matrix
$$\Omega := \begin{bmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{bmatrix}.$$
- **(ii)** Compute the dimension of the rational vector space V in $S := \mathbb{Q}[z_1, z_2, \dots, z_6]$ spanned by these 2-minors.
- **(iii)** Show that every homogeneous polynomial of degree 2 in the polynomial ring S vanishing on the image of σ is contained in V.

P1.3. Consider the curve, called a ***strophoid***, with the trigonometric parametrization given by $x = a\sin(\theta)$ and $y = a\tan(\theta)\big(1 + \sin(\theta)\big)$ where a is a constant and θ is a real parameter.

- **(i)** Find the implicit polynomial equation in x and y that describes the strophoid.
- **(ii)** Find a rational parametrization of the strophoid.

P1.4. Prove that any nonempty open subset of an irreducible topological space is dense and irreducible (in the induced topology).

P1.5. Let d be a nonnegative integer.

- **(i)** Show that the polynomial $\binom{x}{d} := \frac{1}{d!}x(x-1)\cdots(x-d+1)$ in $\mathbb{Q}[x]$ takes integer values when evaluated at any integer.
- **(ii)** Show that every integer-valued polynomial in $\mathbb{Q}[x]$ of degree at most d can be written as a unique *integer* linear combination of the polynomials $\binom{x}{d}, \binom{x}{d-1}, \dots, \binom{x}{0}$.