Problems 2

Due: Friday, 3 February 2023 before 17:00 EST

Students registered in MATH 413 should submit solutions to any three problems, whereas students in MATH 813 should submit solutions to all five.

P2.1. Let $J := \langle x^{v_1}, x^{v_2}, \dots, x^{v_m} \rangle$ and $I := \langle x^{u_1}, x^{u_2}, \dots, x^{u_l} \rangle$ be two monomial ideals in the polynomial ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$.

- **(i)** For any monomial x^w in S, prove that the ideal $(J : x^w) := \{ f \in S \mid f\, x^w \in J \}$ is generated by the monomials of $x^{v_j} / \gcd(x^{v_j}, x^w)$ for all $1 \leqslant j \leqslant m$.
- **(ii)** Prove that intersection $J \cap I$ is generated by monomials $\operatorname{lcm}(x^{v_j}, x^{u_i})$ for all $1 \leqslant j \leqslant m$ and all $1 \leqslant i \leqslant l$.

P2.2. Demonstrate that the following properties characterize the monomial orders $>_{\text{lex}}$ and $>_{\text{grevlex}}$ among all monomial orders $>$ on the polynomial ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ satisfying $x_1 > x_2 > \cdots > x_n$.

- **(i)** For any polynomial $f \in S$ such that $\operatorname{LT}_{\text{lex}}(f) \in \mathbb{K}[x_i, x_{i+1}, \dots, x_n]$ for some $1 \leqslant i \leqslant n$, we have $f \in \mathbb{K}[x_i, x_{i+1}, \dots, x_n]$.
- **(ii)** The monomial order $>_{\text{grevlex}}$ refines the partial order given by total degree and, for any homogeneous $f \in S$ such that $\operatorname{LT}_{\text{grevlex}}(f) \in \langle x_i, x_{i+1}, \dots, x_n \rangle$ for some $1 \leqslant i \leqslant n$, we have $f \in \langle x_i, x_{i+1}, \dots, x_n \rangle$.

P2.3. Let $\mathbf{M}$ be an $(m \times n)$-matrix with nonnegative real entries and let $r_1, r_2, \dots, r_m$ denote the rows of $\mathbf{M}$. Assume that $\ker(\mathbf{M}) \cap \mathbb{Z}^n = \{0\}$. Define a binary relation $>_M$ on the monomials in the polynomial ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ as follows:

$x^u >_{\mathbf{M}} x^v$ if there is an positive integer i (at most m) such that $u \cdot r_j = v \cdot r_j$ for all $1 \leqslant j \leqslant i-1$ and $u \cdot r_i > v \cdot r_i$.

- **(i)** Show that $>_{\mathbf{M}}$ is a monomial order on the polynomial ring S.
- **(ii)** When $\mathbf{M} := \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$, show that $>_{\mathbf{M}}$ equals $>_{\text{grevlex}}$ on $\mathbb{K}[x, y, z]$.
- **(iii)** For the $(n \times n)$-identity matrix $\mathbf{I}$, show that $>_{\text{lex}}$ equals $>_{\mathbf{I}}$.

P2.4. Let $\mathbb{F}_2$ be a finite field with 2 elements and let I be the ideal in $\mathbb{F}_2[x, y, z]$ consisting of polynomials that vanish at every point in $\mathbb{A}^3(\mathbb{F}_2)$.

- **(i)** Show that $\langle x^2 - x, y^2 - y, z^2 - z \rangle \subseteq I$.
- **(ii)** For any $a_0, a_1, \dots, a_7 \in \mathbb{F}_2$, show that the polynomial

$$f := a_0\, xyz + a_1\, xy + a_2 x\, z + a_3\, yz + a_4\, x + a_5\, y + a_6\, z + a_7$$

  belongs to the ideal I if and only if we have $a_0 = a_1 = \cdots = a_7 = 0$.
- **(iii)** Show that $I = \langle x^2 - x, y^2 - y, z^2 - z \rangle$.

P2.5. A ring R satisfies the ***artinian*** if any descending sequence of ideals in R stabilizes. In other words, for any descending sequence $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of ideals in R, there exists a nonnegative integer m such that $I_m = I_{m+1} = I_{m+2} = \cdots$.

- **(i)** For any positive integer n, show that the quotient rings $\mathbb{Z}/\langle n \rangle$ and $\mathbb{K}[x]/\langle x^n \rangle$ are artinian.
- **(ii)** Show that rings $\mathbb{Z}$ and $\mathbb{K}[x]$ are not artinian.
- **(iii)** Show that every prime ideal in an artinian ring is maximal.