Problems 4

Due: Friday, 10 March 2023 before 17:00 EST

Students registered in MATH 413 should submit solutions to any three problems, whereas students in MATH 813 should submit solutions to all five.

P4.1. Use elimination theory to solve the system

$$0 = x^2 + 2y^2 - y - 2z\,, \qquad 0 = x^2 - 8y^2 + 10z - 1\,, \qquad 0 = x^2 - 7yz\,.$$

How many solutions are in $\mathbb{A}^3(\mathbb{R})$? How many are in $\mathbb{A}^3(\mathbb{C})$?

P4.2. Assume that $\mathbb{K}$ is an algebraically closed field. Identify affine space $\mathbb{A}^9(\mathbb{K})$ with the space of (3×3)-matrices $\mathbf{A} = [a_{j,k}]$. Let $\rho \colon \mathbb{A}^9(\mathbb{K}) \dashrightarrow \mathbb{A}^9(\mathbb{K})$ be the rational map defined by

$$\mathbf{A} \mapsto \mathbf{A} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{A}^{-1}\,.$$

- **(i)** Find equations for the smallest affine subvariety X containing the image of ρ.
- **(ii)** Show that X is the set of all nilpotent (3×3)-matrices.

P4.3. For any polynomial $f = a_\ell\, x^\ell + a_{\ell-1}\, x^{\ell-1} + \cdots + a_0 \in \mathbb{C}[x]$ where $a_\ell \neq 0$ and $\ell > 0$, the ***discriminant*** of f is defined to be

$$\operatorname{disc}(f) = \frac{(-1)^{\ell(\ell-1)/2}}{a_\ell} \operatorname{Res}(f, f'; x)\,.$$

- **(i)** The polynomial $f \in \mathbb{C}[x]$ is ***separable*** if its has only simple roots. Show that f is separable if and only if f is relatively prime to its derivative f'.
- **(ii)** Prove that f has a multiple factor if and only if $\operatorname{disc}(f) = 0$.
- **(iii)** Does $6x^4 - 23x^3 + 32x^2 - 19x + 4$ have a multiple root in $\mathbb{C}$?
- **(iv)** Compute the discriminant of the quadratic polynomial $f = ax^2 + bx + c$. Explain how your answer relates to the quadratic formula.

P4.4. Suppose that $f = a_m\, x^m + a_{m-1}\, x^{m-1} + \cdots + a_0$ and $g = b_m\, x^m + b_{m-1}\, x^{m-1} + \cdots + b_0$. Consider the polynomial in two variables

$$\varphi(x, y) = \frac{f(x)g(y) - g(x)f(y)}{x - y} = \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} c_{j,k}\, x^j y^k\,.$$

- **(i)** When $m = 2$, show that $\operatorname{Res}(f, g; x) = (-1) \det \left[c_{j,k}\right]$.
- **(ii)** For any positive integer m, prove that $\operatorname{Res}(f, g; x) = (-1)^{m(m-1)/2} \det \left[c_{j,k}\right]$.

P4.5. For two polynomials f and g in $\mathbb{Q}[x, y]$, let $J := \langle f, g \rangle \cap \mathbb{Q}[y]$ be the elimination ideal.

- **(i)** When $f = xy - 1$ and $g = x^2 + y^2 - 4$, prove that $\operatorname{Res}(f, g; x)$ generates J.
- **(ii)** When $f = xy - 1$ and $g = yx^2 + y^2 - 4$, prove that $\operatorname{Res}(f, g; x)$ does not generate the ideal J.