Problems 6

Due: Monday, 10 April 2023 before 17:00 EDT

Students registered in MATH 413 should submit solutions to any three problems, whereas students in MATH 813 should submit solutions to all four.

P6.1. We analyze how affine lines in $\mathbb{R}^n$ relate to the points at infinity in $\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^n \cup \mathbb{P}^{n-1}(\mathbb{R})$. Given a line L in $\mathbb{R}^n$, we can parametrize L by the formula $a + b\,t$ where a is a point on L and b is a nonzero vector parallel to L. In coordinates, we write this parametrization as $t \mapsto (a_1 + b_1\,t, a_2 + b_2\,t, \dots, a_n + b_n\,t)$.

- **(i)** Regard the line L as lying in $\mathbb{P}^n(\mathbb{R})$ via $t \mapsto [1 : a_1 + b_1\,t : a_2 + b_2\,t : \cdots : a_n + b_n\,t]$. What happens as $t \to \pm\infty$? What is $L \cap \mathbb{P}^{n-1}(\mathbb{R})$?
- **(ii)** Show that the point of $\mathbb{P}^{n-1}(\mathbb{R})$ given by part (i) is the same for all parameterizations of the line L.
- **(iii)** Show that two lines in $\mathbb{R}^n$ are parallel if and only if they pass through the same point at infinity in $\mathbb{P}^n(\mathbb{R})$.

P6.2. A ***homogeneous prime ideal*** in the polynomial ring $\mathbb{K}[x_0, x_1, \dots, x_n]$ is an ideal that is both homogeneous and prime.

- **(i)** Show that a homogeneous ideal I in the ring $\mathbb{K}[x_0, x_1, \dots, x_n]$ is prime if and only if $I \neq \langle 1 \rangle$ and, for all homogeneous polynomials $f, g \in I$, the relation $f\,g \in I$ implies that $f \in I$ or $g \in I$.
- **(ii)** Prove that, for all homogeneous prime ideals I in the ring $\mathbb{K}[x_0, x_1, \dots, x_n]$, the projective variety $\mathrm{V}(I)$ is irreducible if I is a prime ideal. When I is radical, prove that the converse also holds.

P6.3. A projective line in $\mathbb{P}^2(\mathbb{C})$ is an subvariety defined by the equation $a\,x + b\,y + c\,z = 0$ where $(0,0,0) \neq (a,b,c) \in \mathbb{A}^3(\mathbb{C})$.

- **(i)** Show that the triples (a,b,c) and (a',b',c') define the same projective line if and only if $(a,b,c) = \lambda(a',b',c')$ for some nonzero $\lambda \in \mathbb{C}$.
- **(ii)** Show that the map sending the projective line with equation $a\,x + b\,y + c\,z = 0$ to the vector (a,b,c) gives a bijection $\psi \colon \{\text{lines in } \mathbb{P}^2\} \to (\mathbb{A}^3 \setminus \{0\}) / \sim$, where $\sim$ is the equivalence relation in part (i). This quotient is called the *dual projective plane* and is denoted by $\check{\mathbb{P}}^2$. Geometrically, the points of $\check{\mathbb{P}}^2$ are lines in $\mathbb{P}^2$.
- **(iii)** Describe the subset of $\check{\mathbb{P}}^2$ corresponding to affine lines in the distinguished affine subset $\mathbb{A}^2 \cong U_0 = \mathbb{P}^2 \setminus \mathrm{V}(x)$.
- **(iv)** Show that the *incidence correspondence*

$$Y := \left\{ (P, L) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 \;\middle|\; P \in L \right\} \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2$$

  is a closed subvariety.
- **(v)** Given a point p in $\mathbb{P}^2$, consider the set X_p of all projective lines L containing p. We can regard X_p as a subset of $\check{\mathbb{P}}^2$. Show that X_p is a projective line in $\check{\mathbb{P}}^2$. We call the X_p the *pencil of lines* through p.

P6.4. The ***general linear group*** $\mathrm{GL}(\mathbb{K}^{n+1})$ is the set of invertible $(n+1) \times (n+1)$ matrices with entries in $\mathbb{K}$ together the binary operation of ordinary matrix multiplication. The center $Z(\mathbb{K}^{n+1})$ of this groups consists of scalar matrices, namely, matrices of the form λI for some $\lambda \in \mathbb{K}$. The ***projective linear group*** $\mathrm{PGL}(\mathbb{K}^{n+1})$ is the quotient of the general linear group by its center.

- **(i)** Show that each element of $\mathrm{PGL}(\mathbb{K}^{n+1})$ induces an automorphism of $\mathbb{P}^n$.
- **(ii)** For any two sets $\{p_0, p_1, \dots, p_{n+1}\}, \{q_0, q_1, \dots, q_{n+1}\} \subset \mathbb{P}^n$ of $n+2$ points in general position (meaning no $n+1$ lying on a hyperplane), show that there exists an element of $\mathrm{PGL}(\mathbb{K}^{n+1})$ carrying p_i to q_i for all $0 \leqslant i \leqslant n+1$.