3.2 Products and Direct Sums

The product of a family of module is the "most general" module which admits a homomorphism to each of member in the family. Reversing the maps, the direct sum of a family of modules is the "least specific" module to which each member in the family admits a homomorphism. Despite this seemingly innocuous change, direct sums are quite different from products.

3.2.1 Definition. Let J be an arbitrary index set and consider $\{V_j\}_{j \in J}$ a family of R-modules. The ***product*** $V := \prod_{j \in J} V_j$ is the Cartesian product of sets with the R-module structure defined componentwise: for all $r \in R$ and all $u := (u_j) \in V, v := (v_j) \in V$, we have $r\,u + v = (r\,u_j + v_j)$. This structure are equivalent to saying that the projection maps $\varpi_j : V \to V_j$ are R-module homomorphisms.

When $J = \varnothing$, we have $\prod_{j \in J} V_j = 0$.

3.2.2 Proposition (Mapping property of products). *Let $V := \prod_{j \in J} V_J$ be the product of a family of R-modules $\{V_j\}_{j \in J}$. For any R-module W and any family of R-module homomorphisms $\varphi_j : W \to V_j$ for all $j \in J$, there exists a unique R-module homomorphism $\varphi : W \to V$ such that $\varpi_j \circ \varphi = \varphi_j$ for all $j \in J$.*

Proof. This follows directly from the definitions. □

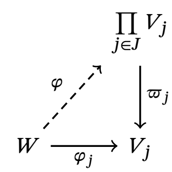

Figure 3.2: Commutative diagrams arising from Proposition 3.2.2

3.2.3 Remark. Rephrasing Proposition 3.2.2, it follows that, for any R-module W and any family of R-modules $\{V_j\}_{j \in J}$, the mapping

$$\operatorname{Hom}_R\Big(W, \prod_{j \in J} V_j\Big) \to \prod_{j \in J} \operatorname{Hom}_R(W, V_j)$$

which associates $\varphi \in \operatorname{Hom}_R\big(W, \prod_{j \in J} V_j\big)$ to the family $(\varpi_j \circ \varphi)$ is an R-module isomorphism.

3.2.4 Remark. The product is commutative and associative. For any three R-modules U, V, and W, we have the canonical isomorphisms $U \times V \cong V \times U$ and $(U \times V) \times W \cong U \times (V \times W)$.

3.2.5 Definition. Let J be an arbitrary index set and consider $\{V_j\}_{j \in J}$ a family of R-modules. The ***direct sum*** $\bigoplus_{j \in J} V_j$ is the submodule of the product $V := \prod_{j \in J} V_j$ consisting of all $v \in V$ such that $\varpi_j(v) = 0$ for all but a finite number of indices j.

When $J = \varnothing$, we have $\bigoplus_{j \in J} V_j = 0$. When the index set J is finite, we have $\bigoplus_{j \in J} V_j = \prod_{j \in J} V_j$.

For all $k \in J$, let $\gamma_k : V_k \to V$ be the map that associates to each $v_k \in V_k$ the element in V such that $\varpi_j\big(\gamma_k(v_k)\big) = v_k\,\delta_{j,k}$. Clearly, γ_k is an injective R-module homomorphism from V_k into the $\bigoplus_{j \in J} V_j$. For all $v \in \bigoplus_{j \in J} V_j$, we have $v = \sum_{j \in J} (\gamma_j \circ \varpi_j)(v)$.

3.2.6 Proposition (Mapping property of direct sums). *Let $\{V_j\}_{j \in J}$ be a family of R-modules. For any R-module W and any family of R-module homomorphisms $\psi_j : V_j \to W$ for all $j \in J$, there is a unique R-module homomorphism $\psi : \bigoplus_{j \in J} V_j \to W$ such that $\psi \circ \gamma_j = \psi_j$ for all $j \in J$.*

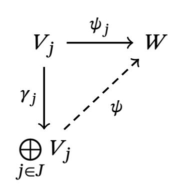

Figure 3.3: Commutative diagrams arising from Proposition 3.2.6

Proof. Suppose that ψ exists. For all $v \in \bigoplus_{j\in J} V_j$, we have

$$\psi(v) = \sum_{j\in J} (\psi \circ \gamma_j \circ \varpi_j)(v) = \sum_{j\in J} (\psi_j \circ \varpi_j)(v),$$

whence the uniqueness of ψ. Setting $\psi(v) := \sum_{j\in J} (\psi_j \circ \pi_j)(v)$, one immediately verifies that an R-module homomorphism has been defined satisfying the conditions in the statement. □

3.2.7 Remark. Rephrasing Proposition 3.2.6, it follows that, for any R-module W and any family of R-modules $\{V_j\}_{j\in J}$, the mapping

$$\mathrm{Hom}_R\Big(\bigoplus_{j\in J} V_j, W\Big) \to \prod_{j\in J} \mathrm{Hom}_R(V_j, W)$$

which associates $\varphi \in \mathrm{Hom}_R\big(W, \bigoplus_{j\in J} V_j\big)$ to the family $(\varphi \circ \gamma_j)$ is an R-module isomorphism.

3.2.8 Remark. The direct sum is commutative and associative. For any three R-modules U, V, and W, we have the canonical isomorphisms $U \oplus V \cong V \oplus U$ and $(U \oplus V) \oplus W \cong U \oplus (V \oplus W)$.

3.2.9 Definition. An R-module W is a ***direct sum*** of a family $\{V_j\}_{j\in J}$ of submodules of W if the canonical map $\bigoplus_{j\in J} V_j \to W$ is an isomorphism. This is equivalent to saying that every $w \in W$ can be written uniquely in the form $w = \sum_{j\in J} v_j$ where $v_j \in V_j$ for all $j \in J$.

3.2.10 Definition. The submodule of W generated by the union of a family $\{V_j\}_{j\in J}$ of submodules is called the ***sum*** of the family and denoted by $\sum_{j\in J} V_J$. If $\gamma_j : V_\alpha \to W$ is the canonical injection and $\gamma : \bigoplus_{j\in J} V_j \to W$ is defined by $\gamma(v_j) = \sum_{j\in J} \gamma_j(v_j)$, then $\sum_{j\in J} V_j$ is the image of γ.

3.2.11 Proposition (Characterization of direct sums). *For any family $\{V_\alpha\}_{\alpha\in A}$ of submodules in an R-module W, the following properties are equivalent:*

(a) *The submodule $\sum_{j\in J} V_j$ is the direct sum of the family $\{V_j\}_{j\in J}$.*
(b) *The relation $\sum_{j\in J} v_j = 0$ where $v_j \in V_j$ implies that $v_j = 0$ for all $j \in J$.*
(c) *For all $k \in J$, the intersection of V_k and $\sum_{j\neq k} V_j$ is* 0.

Proof.
(a) ⇔ (b): Since $\sum_{j\in J} v_j = \sum_{j\in J} u_j$ if and only if $\sum_{j\in J}(v_j - u_j) = 0$, we see that (a) and (b) are equivalent.
(a) ⇒ (c): The uniqueness of the expression for an element in the sum $\sum_{j\in J} V_j$ implies that the intersections are 0.
(c) ⇒ (b): The relation $\sum_{j\in J} v_j = 0$ can be written as $v_k = \sum_{j\neq k} -v_j$ and (c) implies that $v_k = 0$. □

3.3 Complementary Submodules

3.3.1 Definition. Let $\varphi : U \to V$ and $\psi : V \to W$ be two R-module homomorphisms. The pair (φ, ψ) or diagram

$$U \xrightarrow{\varphi} V \xrightarrow{\psi} W$$

is an ***exact*** sequence if $\operatorname{Ker}(\psi) = \operatorname{Im}(\varphi)$. Consider the diagram

$$U \xrightarrow{\varphi} V \xrightarrow{\psi} W \xrightarrow{\theta} X\,.$$

This diagram is ***exact at*** V if the pair (φ, ψ) is exact. It is exact at W if the pair (ψ, θ) is exact. If the diagram is exact at each module, it is an exact sequence. Exact sequences with an arbitrary number of terms are defined similarly.

For the pair (φ, ψ) to form an exact sequence, one must have $\psi \circ \varphi = 0$ because this property yields the inclusion $\operatorname{Im}(\varphi) \subseteq \operatorname{Ker}(\psi)$.

3.3.2 Remarks.

- The sequence $0 \longrightarrow U \xrightarrow{\varphi} V$ is exact if and only if φ be injective.
- The sequence $U \xrightarrow{\varphi} V \longrightarrow 0$ is exact if and only if φ be surjective.
- Let U be a submodule of an R-module V. The canonical inclusion $\iota : U \to V$ and the canonical surjection $\pi : V \to V/U$ form the exact sequence $0 \longrightarrow U \xrightarrow{\iota} V \xrightarrow{\pi} V/U \longrightarrow 0\,.$
- An R-module homomorphism $\varphi : U \to V$ has an exact sequence

$$0 \longrightarrow \operatorname{Ker}(\varphi) \longrightarrow V \xrightarrow{\varphi} W \longrightarrow \operatorname{Coker}(\varphi) \longrightarrow 0\,.$$

Multiplication by 2 yields the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/\langle 2 \rangle \longrightarrow 0$$

of $\mathbb{Z}$-modules.

3.3.3 Proposition. *For any two submodules U and W of an R-module V, there exists two exact sequences*

$$0 \longrightarrow U \cap W \xrightarrow{\left[\begin{smallmatrix} \eta_U \\ \eta_W \end{smallmatrix}\right]} U \oplus W \xrightarrow{[\,\zeta_U \;\; -\zeta_W\,]} U + W \longrightarrow 0$$

$$0 \longrightarrow \frac{V}{U \cap W} \xrightarrow{\left[\begin{smallmatrix} \beta_U \\ \beta_W \end{smallmatrix}\right]} \frac{V}{U} \oplus \frac{V}{W} \xrightarrow{[\,\alpha_U \;\; -\alpha_W\,]} \frac{V}{U + W} \longrightarrow 0$$

where the component maps $\eta_U : U \cap W \to U$, $\eta_W : U \cap W \to W$, $\zeta_U : U \to U + W$, $\zeta_W : W \to U + W$ are the canonical injections, the maps $\beta_U : V/(U \cap W) \to V/U$ and $\beta_W : V/(U \cap W) \to V/W$ are induced by the identity map on V, and the components $\alpha_U : V/U \to V/(U + W)$, $\alpha_W : V/W \to V/(U + W)$ are the canonical surjections.

Proof. From the definition of $U + W$, we see that the map $[\,\zeta_U \;\; -\zeta_W\,]$ is surjective. Since the maps η_U and η_W are injective, the map $\left[\begin{smallmatrix} \eta_U \\ \eta_W \end{smallmatrix}\right]$ is also injective. To say that $[\,\zeta_U \;\; -\zeta_W\,](u, w) = 0$ for some $u \in U$ and some $w \in W$ means that $\zeta_U(u) - \zeta_W(w) = 0$. Thus, there exists $v \in U \cap W$ such that $v = \zeta_U(u) = \zeta_W(w)$, so $u = \eta_U(v)$ and $w = \eta_W(v)$. We conclude that $\operatorname{Ker}([\,\zeta_U \;\; -\zeta_W\,]) = \operatorname{Im}(\left[\begin{smallmatrix} \eta_U \\ \eta_W \end{smallmatrix}\right])$, so the first sequence is exact.

Since the maps α_U and α_W are surjective, the map $[\,\alpha_U \;\; -\alpha_W\,]$ is also surjective. If $\left[\begin{smallmatrix} \beta_U \\ \beta_W \end{smallmatrix}\right](v) = 0$ for some $v \in V/(U \cap W)$, then we have $\beta_U(v) = \beta_W(v) = 0$ which implies that v is the class of an element in $U \cap W$. Finally, to say that $[\,\alpha_U \; -\alpha_W\,](u,w) = 0$ for some $u \in V/U$ and some $w \in V/W$ means that $\alpha_U(u) = \alpha_W(w)$. Thus, there exists $u', w' \in V$ whose classes modulo U and W are u, w respectively and $u' - w' \in U + W$. It follows that there are $u'' \in U$ and $w'' \in W$ such that $u' - w' = u'' - w''$, so there also exists $v' \in V$ such that $u' - u'' = w' - w'' = v'$. Let v be the class of v' modulo $U \cap W$. Since $\beta_U(v) = u$ and $\beta_W(v) = w$, we conclude that $\mathrm{Ker}([\,\alpha_U \; -\alpha_W\,]) = \mathrm{Im}(\left[\begin{smallmatrix}\beta_U \\ \beta_W\end{smallmatrix}\right])$, so the second sequence is exact. □

The characterization of direct sums of submodules implies that U and W are complementary submodules of V if and only if $U + W = V$ and $U \cap W = 0$.

3.3.4 Definition. In an R-module V, two submodules U and W are ***complementary*** if $V = U \oplus W$.

3.3.5 Example. All submodules of the $\mathbb{Z}$-module $\mathbb{Z}^1$ have the form $\langle m\rangle$ for some nonnegative integer M, but $\langle m\rangle \cap \langle n\rangle = \big\langle \mathrm{lcm}(m,n)\big\rangle$. Thus, $\mathbb{Z}, 0$ are the only pair of complementary submodules in $\mathbb{Z}$. ◇

Any two complements of a submodule are isomorphic.

3.3.6 Proposition. *When $V = U \oplus W$, the restriction $\pi|_U : U \to V/W$ of the canonical map $\pi : V \to V/W$ is an isomorphism.*

Proof. The map $\pi|_U$ is surjective because $U + W = V$. It is injective because its kernel is $U \cap W = 0$. □

3.3.7 Proposition. *Given an exact sequence of R-modules*

$$0 \longrightarrow U \xrightarrow{\;\varphi\;} V \xrightarrow{\;\psi\;} W \longrightarrow 0\,,$$

the following are equivalent:

(a) *There exists an R-linear map $\theta : V \to U$ such that $\theta \circ \varphi = \mathrm{id}_U$.*
(b) *There exists an R-linear map $\sigma : W \to V$ such that $\psi \circ \sigma = \mathrm{id}_W$.*
(c) *There exists an isomorphism $V \cong U \oplus W$.*

The homomorphisms θ and σ are said to **split** *the exact sequence.*

Sketch of Proof.

(b) ⇒ (c) is similar to (a) ⇒ (c).

(a) ⇒ (c): For any $v \in V$, we have

$$\theta\big(v - (\varphi \circ \theta)(v)\big) = \theta(v) - (\theta \circ \varphi)\big(\theta(v)\big) = 0\,,$$

so $v-(\varphi\circ\theta)(v) \in \mathrm{Ker}(\theta)$ and $V = \mathrm{Ker}(\theta)+\mathrm{Im}(\varphi)$. Consider $u \in U$ such that $\varphi(u) \in \mathrm{Ker}(\theta) \cap \mathrm{Im}(\varphi)$. It follows that $0 = \theta\big(\varphi(u)\big) = u$, so $\varphi(u) = \varphi(0) = 0$. We conclude that $V \cong \mathrm{Im}(\varphi) \oplus \mathrm{Ker}(\theta)$.

(b) ⇒ (a) is similar to (a) ⇒ (b).

(a) ⇒ (b): For any $w \in W$, there exists $v \in V$ such that $\psi(v) = w$ because ψ is surjective. Define $\sigma : W \to V$ by $\sigma(w) := v-(\varphi\circ\theta)(v)$. If $\psi(v) = u = \psi(v')$, then we have $v - v' \in \mathrm{Ker}(\psi) = \mathrm{Im}(\varphi)$, so

$$v - (\varphi \circ \theta)(v) - \big(v' - (\varphi \circ \theta)(v')\big) = (v - v') - (\varphi \circ \theta)(v - v')$$

belongs to $\mathrm{Im}(\varphi) \cap \mathrm{Ker}(\theta) = 0$ and the map σ is well-defined. By construction, we have $\psi \circ \sigma = \mathrm{id}_W$.

(c) ⇒ (b) is similar to (c) ⇒ (a).

(c) ⇒ (a): When $V \cong U \oplus W$, the mapping property of product shows that the canonical surjection $\varpi : V \to U$ satisfies $\varpi \circ \varphi = \mathrm{id}_U$. □