Problems 02
Due: Tuesday, 22 September 2020

1. Let H and K be two subgroups of a group G. For any element $g \in G$, the set
$$H\,g\,K := \{f \in G \mid f = h\,g\,k \text{ for some } h \in H, k \in K\}$$
is called a ***double coset***.
(i) Prove that the double cosets partition G.
(ii) Do all double cosets have the same cardinality?
(iii) When G has finite order, must the cardinality of a double coset divide $|G|$?

2. Let G be a group and let $\operatorname{Aut}(G)$ be its automorphism group. Given an element $g \in G$, consider the map $\gamma_g : G \to G$ defined, for all $f \in G$, by $\gamma_g(f) := g\,f\,g^{-1}$.
(i) For all $g \in G$, show that γ_g is an automorphism.
(ii) Prove that the map $\Gamma : G \to \operatorname{Aut}(G)$ defined, for all $g \in G$, by $\Gamma(g) := \gamma_g$ is a group homomorphism.
(iii) Show that $\operatorname{Ker}(\Gamma) = Z(G)$.
(iv) Prove that $\operatorname{Inn}(G) := \operatorname{Im}(\Gamma)$ is a normal subgroup of $\operatorname{Aut}(G)$.

3. Fix $n \in \mathbb{N}$. Two permutations $\sigma, \tau \in \mathfrak{S}_n$ have the ***same cycle structure*** if, for all $k \in \mathbb{N}$, their factorizations into disjoint cycles have the same number of cycles of length k. The ***cycle type*** of a permutation is the list λ of cycles lengths from its factorization into disjoint cycles arranged in non-increasing order.
(i) For all permutations $\sigma, \tau \in \mathfrak{S}_n$, prove that the conjugate permutation $\sigma\,\tau\,\sigma^{-1}$ has the same cycle structure as τ and is obtained by applying σ to the entries in the cycles of τ.
(ii) Prove that two permutations are conjugate if and only if they have the same cycle type.

Remarks. If $\sigma = (4\ 3\ 1)(6\ 2\ 5)$ and $\tau = (3\ 1)(7\ 2\ 4)$, then we have
$$\sigma\,\tau\,\sigma^{-1} = \big(\sigma(3)\ \sigma(1)\big)\big(\sigma(7)\ \sigma(2)\ \sigma(4)\big) = (1\ 4)(7\ 5\ 3) = (4\ 1)(7\ 5\ 3)\,.$$
The cycle type of $(4\ 2\ 1)(5)(6)(8\ 7\ 3)(9) \in \mathfrak{S}_9$ is $(3,3,1,1,1)$ and the cycle type of $(5\ 1)(8\ 6\ 3\ 4)(9\ 7\ 2) \in \mathfrak{S}_9$ is $(4,3,2)$. Since every element in $[n]$ appears in a unique cycle, the positive integers in the cycle type sum to n.