Problems 03
Due: Tuesday, 29 September 2020

1. Let G be a group. The ***commutator*** of two elements f and g in G is the element $[f, g] := f^{-1}g^{-1}fg$. The ***commutator subgroup*** G' of G is the subgroup generated by all the commutators of elements of G: $G' := \langle f^{-1}g^{-1}fg \mid f, g \in G\rangle$.

(i) Show that G' is a normal subgroup of G and the quotient group G/G' is abelian.

(ii) Let $\pi : G \to G/G'$ be the canonical group homomorphism and let A be an abelian group. Show that every group homomorphism $\varphi : G \to A$ factors as $\varphi = \varphi' \circ \pi$ where $\varphi' : G/G' \to A/A'$ is the induced group homomorphism.

(iii) Show that a subgroup H of G contains G' if and only if H is normal and G/H is abelian.

2. Let $m\mathbb{Z}$ be the subgroup of integers $\mathbb{Z}$ generated by m and let $\overline{r}$ denote the left coset in the quotient group $\mathbb{Z}/m\mathbb{Z}$ containing the integer r. Consider the set

$$(\mathbb{Z}/m\mathbb{Z})^{\times} := \{\overline{r} \in \mathbb{Z}/m\mathbb{Z} \mid \gcd(r, m) = 1\}.$$

(i) Show that multiplication of integers induces a group structure on $(\mathbb{Z}/m\mathbb{Z})^{\times}$.

(ii) The ***totient*** $\phi(n)$ of a positive integer n is defined to be the number of positive integers less than or equal to n that are coprime to n. When $\gcd(r, m) = 1$, prove that $r^{\phi(m)} \equiv 1 \pmod{m}$.

(iii) For any prime number p and any integer r, prove that $r^p \equiv r \pmod{p}$.

3. The ***icosahedral group*** I consists of the rotational symmetries of a regular dodecahedron. It acts transitively on the vertices, edges and faces, and $|I| = 60$.

(i) Determine the number of elements in I of each order.

(ii) Determine the cardinality of each conjugacy class in I.

(iii) Show that I is a simple group (i.e. it has no nontrivial normal subgroups).