Problems 04

Due: Tuesday, 06 October 2020

1. Let p be a prime number. Prove that a group of order $2p$ is either cyclic or dihedral.

2. Prove that there are no simple groups of order 80, 96, or 1000.

3. Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denote the extended complex plane. Consider the two functions $f, g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by $f(z) := z + 2$ and $g(z) := z/(2z + 1)$ respectively.

(i) Prove that f and g are bijections and hence elements of the symmetric group on $\widehat{\mathbb{C}}$.

(ii) Show that any nonzero power of f maps the interior of the unit circle $|z| = 1$ to the exterior. Similarly, show that any nonzero power of g maps the exterior of the unit circle to the punctured interior (a point is removed from the interior).

(iii) Prove that the subgroup of the symmetric group on $\widehat{\mathbb{C}}$ generated by functions f and g is free.