Problem Set #4

1. Consider a mass of 1 kg in a undamped mass-spring system with spring constant 4 kg $\cdot$ s^{-2} and external driving force $F(t) = \cos(t)$. Assume that the mass is displaced $\frac{1}{2}$ m from equilibrium and released. Describe the behaviour of the system.

2. Consider a mass of 1.000 kg in a mass-spring system with spring constant 4.000 kg $\cdot$ s^{-2}, damping constant 0.1000 kg $\cdot$ s^{-1}, and external driving force $F(t) = \cos(2t)$. Assume that the mass is displaced 0.5000 m from equilibrium and released. Describe the behaviour of the system.

3. Consider a mass of 1 kg in a mass-spring system with spring constant 4 kg $\cdot$ s^{-2}, damping constant 4 kg $\cdot$ s^{-1}, and external driving force $F(t) = \cos(2t)$. Assume that the mass is displaced $\frac{1}{2}$ m from equilibrium and released. Describe the behaviour of the system.

4. **(a)** Let $f(t)$ be a function such that $\mathscr{L}\{f(t)\}(s)$ exists and let $b > 0$. Establish the ***scaling*** property of the Laplace transform: $\mathscr{L}\{f(bt)\}(s) = \frac{1}{b}\mathscr{L}\{f(t)\}\left(\frac{s}{b}\right)$.

(b) Suppose that $f(t)$ is a function such that $\mathscr{L}\{f(t)\}(s) = \frac{s^2-s+1}{(2s+1)(s-1)}$. Determine the Laplace transform of $g(t) = f(2t)$.

5. Let $u(t)$ be the unit step function and let $f(t)$ be a function such that $\mathscr{L}\{f(t)\}(s)$ exists. Establish the ***time shifting*** property of the Laplace transform: $\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathscr{L}\{f(t)\}(s)$.

6. Use the definition of the Laplace transform, compute $\mathscr{L}\{f(t)\}(s)$ where

$$f(t) = \begin{cases} 1-t & \text{for } 0 \le t \le 1, \\ 0 & \text{for } 1 < t. \end{cases}$$

Check your answer by using the time shifting property.

7. Find the inverse Laplace transform of $\dfrac{2s+2}{s^2+2s+5}$ and $\dfrac{6}{(s-2)^4}$.

8. Find the inverse Laplace transform of $\dfrac{8s^2-4s+12}{s(s^2+4)}$ and $\dfrac{e^{-2s}}{s^2+s-2}$.

9. The ***frequency differentiation*** property states that $\mathscr{L}\{tf(t)\}(s) = (-1)\frac{d}{ds}\Big(\mathscr{L}\{f(t)\}(s)\Big)$. Use this property, to find the inverse Laplace transform of the following functions:

(a) $\ln\left(\dfrac{s+2}{s-5}\right)$ **(b)** $\arctan(s^{-1})$

10. Use the Laplace transform to solve: $y'' - y' - 6y = 0$, $y(0) = 1$, $y'(0) = -1$.

11. Use the Laplace transform to solve: $y'' - 2y' + 2y = \cos(t)$, $y(0) = 1$, $y'(0) = 0$.

12. Use the Laplace transform to solve the initial value problem:

$$y'' + 4y = \begin{cases} 1 & \text{for } 1 \le t < \pi, \\ 0 & \text{for } \pi \le t, \end{cases} \qquad y(0) = 1, y'(0) = 0.$$