Problem Set #5

1. Use Laplace transforms to solve:

$$\frac{d^4x}{dt^4} - 4\frac{d^3x}{dt^3} + 6\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + x = 0\,,$$

where $x(0) = 0$, $x'(0) = 1$, $x''(0) = 0$, $x'''(0) = 1$.

2. Use Laplace transforms to solve: $y''' + y'' = e^t + t + 1$, $y(0) = y'(0) = y''(0) = 0$.

3. Use Laplace transforms to solve: $f''(t) + 2f'(t) + f(t) = 4e^{-t}$, $f(0) = 2$, $f'(0) = -1$.

4. For $\lambda > 0$, use Laplace transforms to solve: $y'' + \lambda^2 y = \cos(\lambda t)$, $y(0) = 1$, $y(\pi/2\lambda) = -1$.

5. Solve the following initial value problem which describes the deflection of a uniform static cantilever beam:

$$\frac{d^4w}{dx^4} = \begin{cases} 1 & 0 \le x < 1 \\ 0 & 1 \le x \end{cases} \qquad \text{where } w(0) = w'(0) = 0 \text{ and } w''(2) = w'''(2) = 0.$$

6. **(a)** Compute $e^t * t$.
(b) Show that $f * (g * h) = (f * g) * h$.

7. Use the convolution property to find the following:

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}(t)$ **(b)** $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-a)(s-b)}\right\}(t)$ where $a \neq b$.

8. Solve $y'' + y = g(t)$ where $y(0) = y'(0) = 0$ and $g(t) := \begin{cases} \sin(t) & 0 \le t < \pi \\ 0 & \pi \le t\,. \end{cases}$

9. Solve the following integral equation: $te^{-at} = \displaystyle\int_0^t x(\tau)x(t-\tau)\,d\tau$.

10. Solve $y'' + ty' - 2y = 4$, $y(0) = -1$ and $y'(0) = 0$.

11. The ***sawtooth wave*** is the piecewise linear function defined by $\text{saw}(t) := t - \lfloor t \rfloor$, where $\lfloor t \rfloor$ is the largest integer not greater than t. Sketch the graph of $\text{saw}(t)$ and compute its Laplace transform.

12. Solve $y'' + 4\pi^2 y = 2\pi\,\text{saw}(t)$ where $y(0) = y'(0) = 0$.