Problem Set #6

1. Verify that the vector functions $\vec{\boldsymbol{x}}_1(t) := \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}$, $\vec{\boldsymbol{x}}_2(t) := \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{bmatrix}$, and $\vec{\boldsymbol{x}}_3(t) := \begin{bmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{bmatrix}$ are solutions to $\vec{\boldsymbol{x}}'(t) = \mathbf{A}\,\vec{\boldsymbol{x}}(t)$ where $\mathbf{A} := \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and that $\vec{\boldsymbol{x}}_0(t) := \begin{bmatrix} 5t+1 \\ 2t \\ 4t+2 \end{bmatrix}$ is a particular solution to $\vec{\boldsymbol{x}}'(t) = \mathbf{A}\,\vec{\boldsymbol{x}}(t) + \vec{\boldsymbol{y}}$ where $\vec{\boldsymbol{y}} = \begin{bmatrix} -9t \\ 0 \\ -18t \end{bmatrix}$. Find a general solution to $\vec{\boldsymbol{x}}'(t) = \mathbf{A}\,\vec{\boldsymbol{x}}(t) + \vec{\boldsymbol{y}}$.

2. Solve $\vec{\boldsymbol{y}}'(x) = \mathbf{A}\,\vec{\boldsymbol{y}}(x)$ where $A = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$ and $\vec{\boldsymbol{y}}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

3. Solve $\vec{\boldsymbol{w}}'(z) = \mathbf{B}\,\vec{\boldsymbol{w}}(z)$ where $\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$.

4. Solve $\vec{\boldsymbol{f}}'(x) = \mathbf{K}\,\vec{\boldsymbol{f}}(x)$ where $\mathbf{K} = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$ and $\vec{\boldsymbol{f}}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

5. For $0 < \delta < 1$, solve $\vec{\boldsymbol{y}}'(y) = \mathbf{A}\,\vec{\boldsymbol{y}}(x)$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix}$.

6. Solve $\vec{\boldsymbol{x}}'(t) = \mathbf{M}\,\vec{\boldsymbol{x}}(t)$ where $\mathbf{M} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

7. Consider two interconnected tanks. Tank A initially contains 100 L of water and 20 g of salt, and tank B initially contains 200 L of water and 75 g of salt. The liquid inside each tank is kept well stirred. Liquid flows from tank A to tank B at a rate of $3\,\mathrm{L}\cdot\mathrm{min}^{-1}$ and liquid flows from tank B to tank A at rate of $2\,\mathrm{L}\cdot\mathrm{min}^{-1}$. A salt brine with concentration $7\,\mathrm{g}\cdot\mathrm{L}^{-1}$ of salt flows into tank A at a rate of $5\,\mathrm{L}\cdot\mathrm{min}^{-1}$ and the solution drains out at $4\,\mathrm{L}\cdot\mathrm{min}^{-1}$. Moreover, a salt brine with concentration $3\,\mathrm{g}\cdot\mathrm{L}^{-1}$ of salt flows into tank A at a rate of $7\,\mathrm{L}\cdot\mathrm{min}^{-1}$ and the solution drains out at $8\,\mathrm{L}\cdot\mathrm{min}^{-1}$. Determine the amount of salt in each tank at any time.

8. **(a)** Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$ has the repeated eigenvalue $r = -1$, and all the eigenvectors are a scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Use part (a) to obtain a nontrivial solution $\vec{x}_1(t)$ to the system $\vec{x}'(t) = \mathbf{A}\vec{x}(t)$.

(c) Obtain a second linearly independent solution to $\vec{x}'(t) = \mathbf{A}\vec{x}(t)$ by considering a function of the form $\vec{x}_2(t) = te^{-t}\vec{u}_1 + e^{-t}\vec{u}_2$.

Hint. Substitute $\vec{x}_2(t)$ into $\vec{x}'(t) = \mathbf{A}\vec{x}(t)$ and derive $(\mathbf{A}+\mathbf{I})\vec{u}_1 = 0$ and $(\mathbf{A}+\mathbf{I})\vec{u}_2 = \vec{u}_1$. Since $\vec{u}_1$ is an eigenvector, solve for $\vec{u}_2$.

9. Solve $\vec{x}''(t) = \mathbf{A}\,\vec{x}(t)$ where $\mathbf{A} = \begin{bmatrix} 17 & 21 \\ -42 & -46 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, and $\vec{x}'(0) = \begin{bmatrix} 0 \\ -20 \end{bmatrix}$.

10. Solve the given linear system of differential equations:

$$\begin{cases} x'' &= -13x - 3y - 12z \\ y'' &= -12x - 4y - 12z \\ z'' &= -3x + 3y - 4z \end{cases}$$

11. Describe the behaviour of a coupled mass-spring system consisting of two identical masses connected between three identical springs, when one masses is held at its equilibrium position while the other mass is displaced two units from equilibrium and then both masses are released from rest.