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Cones of Hilbert Functions

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We study the closed convex hull of various collections of Hilbert functions. Working over a standard graded polynomial ring with modules that are generated in degree 0, we describe the supporting hyperplanes and extreme rays for the cones generated by the Hilbert functions of all modules, all modules with bounded a-invariant, and all modules with bounded Castelnuovo–Mumford regularity. The first of these cones is infinite-dimensional and simplicial, the second is finite-dimensional but neither simplicial nor polyhedral, and the third is finite-dimensional and simplicial.

1 Introduction

Classifying modules is a universal problem in algebra. Within commutative algebra, the classification of graded modules bifurcates into understanding the space of all modules with a specified Hilbert function and describing the numerical functions that arise as the Hilbert function of some module. As a counterpart to multigraded Quot schemes which parameterize the modules with a fixed Hilbert function (see [13, Section 6.2]), this paper initiates the study of the closed convex cones generated by the Hilbert functions of a given collection of modules.

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There are many collections of modules to consider over the standard graded ring $S := \Bbbk[x_0, x_1, \dots, x_n]$ where $\Bbbk$ is a field. The most naive consists of all finitely generated $\mathbb{N}$-graded S-modules. In this case, every point in the corresponding closed convex cone is a unique countable linear combination of the Hilbert functions of the S-modules $S(-i)/\langle x_0, x_1, \dots, x_n\rangle \cong \Bbbk(-i)$ for $i \in \mathbb{N}$. Hence, within any relevant topological vector space, the closed convex hull is the simplicial cone generated by the Hilbert functions of these artinian modules. In other words, it is simply the infinite-dimensional positive orthant. To actually capture the subtleties of homogeneous coordinate rings, we concentrate on collections of $\mathbb{N}$-graded S-modules that are generated in degree 0. If $E \subseteq \mathbb{Q}^{\mathbb{N}}$ is a topological $\mathbb{Q}$-vector space that contains the Hilbert functions of all artinian S-modules generated in degree 0 and the function $h\colon \mathbb{N} \to \mathbb{Q}$ lies in E, then our first substantive result is the following.

Theorem 1.1. The closed convex hull of the Hilbert functions of S-modules generated in degree 0 and contained in E is the intersection of the closed half-spaces defined by the inequalities

$$(n + j + 1)h(j) \geqslant (j + 1)h(j + 1) \quad \text{for } j \in \mathbb{N}.$$

The extreme rays of this simplicial cone are generated by the Hilbert functions of the S-modules $S/\langle x_0, x_1, \dots, x_n\rangle^i$ where $i \in \mathbb{N}$. □

By design, our approach overcomes limitations in Macaulay's celebrated theorem on Hilbert functions. Although [16, Main Theorem] determines those numerical functions which occur as Hilbert functions of a homogeneous quotient of S, the complexity of this result, as underscored in [2, p. 27; 4, p. 132], makes it unwieldy. The optimal linear conditions are frequently more useful despite not providing a complete characterization. Moreover, because Macaulay's Theorem depends inherently on lex-segment ideals, it cannot be extended to graded rings that do not have analogous ideals. Closed convex hulls enjoy no such restrictions. These two features, in addition to the advantages of endowing the set of Hilbert functions with a geometric structure, motivate our interest in cones of Hilbert functions. In particular, we regard the supporting hyperplanes in Theorem 1.1 (also see Theorem 2.1) as the linearization of Macaulay's Theorem.

To reveal the properties related to Hilbert polynomials, we need a smaller collection of modules—one that does not contain artinian modules of arbitrary length. Requiring that the Hilbert polynomial and Hilbert function agree for all integers greater than a fixed number a is a straightforward method of making such a collection.

Equivalently, we restrict to the finite-dimensional $\mathbb{Q}$-vector space $V_{n,a} \subset \mathbb{Q}^{\mathbb{N}}$ consisting of all functions $h\colon \mathbb{N} \to \mathbb{Q}$ satisfying $\sum_{j \in \mathbb{N}} h(j)t^j = (b_0 + b_1 t + \cdots + b_{a+n}t^{a+n})/(1-t)^n$ for some $b_0, b_1, \ldots, b_{a+n} \in \mathbb{Q}$. In this context, the primary object of interest is the closed convex hull $Q_{n,a} \subset V_{n,a}$ of the Hilbert functions of finitely generated $\mathbb{N}$-graded S-modules that are generated in degree 0 and have no free summands. Our second major result characterizes this cone.

Theorem 1.2. If $T\colon \mathbb{Q}^{\mathbb{N}} \to \mathbb{Q}^{\mathbb{N}}$ is the linear operator defined by $(T[h])(j) := (n+j+1)h(j) - (j+1)h(j+1)$ where $h\colon \mathbb{N} \to \mathbb{Q}$, then the image $T[Q_{n,a}]$ equals the closed convex hull of $\mathbb{N}^{\mathbb{N}} \cap V_{n,a}$. □

The linear operator T and the supporting hyperplanes for $Q_{n,a}$ arise from the linearization of Macaulay's Theorem. Since Proposition 3.2 describes the extreme rays for the image $T[Q_{n,a}]$, we also obtain, in Corollary 3.8, a description for the extreme rays of $Q_{n,a}$. As Example 3.12 demonstrates, the cone $Q_{n,a}$ is generally neither simplicial nor polyhedral.

Alternatively, Castelnuovo–Mumford regularity, which is defined for a module not just its Hilbert function, provides a more sophisticated mechanism for creating a smaller collection. To be explicit, let $R_{n,m}$ be the closed convex hull in $V_{n+1,m}$ of the Hilbert functions of finitely generated $\mathbb{N}$-graded S-modules that are generated in degree 0, have no free summands, and have regularity at most m. If $q_h \in \mathbb{Q}[s]$ denotes the Hilbert polynomial associated to $h \in R_{n,m}$ and $\nabla\colon \mathbb{Q}[s] \to \mathbb{Q}[s]$ is the backward difference operator defined by $\nabla q(s) := q(s) - q(s-1)$ for $q \in \mathbb{Q}[s]$, then our third significant result describes the cone $R_{n,m}$.

Theorem 1.3. The closed convex cone $R_{n,m}$ lies in the subspace $V_{n,m} \subset V_{n+1,m}$ and is the intersection of the closed half-spaces given by the inequalities:

$$\begin{gathered}(n+j+1)h(j) \geqslant (j+1)h(j+1) \quad \text{for } 0 \leqslant j < m,\\ h(m) \geqslant q_h(m), \quad \text{and}\\ (n+1-i)\nabla^i q_h(m) \geqslant (n+m+1-i)\nabla^{i+1} q_h(m) \quad \text{for } 0 \leqslant i < n.\end{gathered}$$

The extreme rays of this simplicial polyhedral cone are generated by the Hilbert functions of these

$$\begin{gathered}\frac{S}{\langle x_0, x_1, \ldots, x_n\rangle}, \frac{S}{\langle x_0, x_1, \ldots, x_n\rangle^2}, \ldots, \frac{S}{\langle x_0, x_1, \ldots, x_n\rangle^m},\\ \frac{S}{\langle x_0, x_1, \ldots, x_n\rangle^{m+1}}, \frac{S}{\langle x_0, x_1, \ldots, x_{n-1}\rangle^{m+1}}, \ldots, \frac{S}{\langle x_0\rangle^{m+1}}\end{gathered} \tag{1}$$

cyclic modules. □

To prove this, we use the natural projection from the cone of Betti tables. It is intriguing that the extreme rays of $R_{n,m}$ correspond to modules with linear free resolutions, arguably the simplest pure Betti tables.

Our progress in describing cones of Hilbert functions points in several promising directions. For instance, how does one describe the closed convex hull for other important collections of S-modules. Since convex cones are closed under linear combinations with positive coefficients and the Hilbert function of a direct sum is the sum of the Hilbert functions, collections of modules that are closed under finite direct sums are likely the most pertinent. In contrast, we would also like to generalize Macaulay's Theorem to other rings by describing the closed convex hull of the Hilbert functions of all module generated in degree zero. Following [10], toric rings are the most prominent candidates among $\mathbb{N}$-graded commutative rings. More generally, what is the analog of Theorem 1.1 when S is replaced by the homogeneous coordinate ring of a projective variety and how do the supporting hyperplanes and extreme rays reflect the geometry of the underlying variety. Considering nonstandard and multigraded polynomial rings branches onto a somewhat different track as [1, 5, 15] establish. Preliminary work for a standard bigraded polynomial ring, or equivalently the Cox ring for a product of projective spaces, indicates that an elementary variant of Theorem 1.1 holds. However, versions over the Cox ring for any smooth projective toric variety appear to be intrinsically more complicated. For geometric applications, one should probably exclude all modules that contain an element annihilated by a power of the irrelevant ideal. Finally, we have not begun to analyze the semigroup within the closed convex cone formed by the Hilbert functions of modules.

1.1 Contents of the paper

Section 2 gives both a combinatorial proof and an algebraic proof for the linearization of Macaulay's Theorem, also known as Theorem 2.1. Our description of the closed convex hull of the Hilbert function of artinian S-modules generated in degree 0, given in Corollary 2.3, and the proof for Theorem 1.1 follow. In Section 3, Proposition 3.2 describes the extreme rays of the closed convex hull of $\mathbb{N}^{\mathbb{N}} \cap V_{n,a}$. After a triple of technical lemmas, we prove Theorem 1.2. The section ends with Corollary 3.8, which explicitly describes the supporting hyperplanes and extreme rays of $Q_{n,a}$, and four examples illustrating this corollary. We prove Theorem 1.3 in Section 4 and we close with Proposition 4.5, which explicitly bounds Betti numbers linearly via Hilbert functions.

1.2 Conventions

We write $\mathbb{N}$ for the set of nonnegative integers and $\Bbbk$ for an arbitrary field. A set is countable if it has the same cardinality as $\mathbb{N}$. Throughout the document, the polynomial ring $S := \Bbbk[x_0, x_1, \dots, x_n]$ has the standard $\mathbb{N}$-grading induced by setting $\deg(x_i) = 1$ for all $0 \leqslant i \leqslant n$. All S-modules are finitely generated and $\mathbb{N}$-graded.

2 Modules Generated in Degree 0

This section considers the closed convex hull of Hilbert functions of S-modules generated in degree 0. The key result, namely Theorem 2.1, describes the linear inequalities satisfied by the Hilbert function of such a module. By working in appropriate infinite-dimensional topological vector spaces, we obtain descriptions of the supporting hyperplanes and the extreme rays for the closed convex hull of Hilbert functions for any collection of modules containing all artinian S-modules.

If M is a finitely generated $\mathbb{N}$-graded S-module, then its Hilbert function is the numerical function $h_M \colon \mathbb{N} \to \mathbb{N}$ defined by $h_M(j) := \dim_{\Bbbk} M_j$.

Theorem 2.1. The Hilbert function of a finitely generated $\mathbb{N}$-graded S-module M generated in degree 0 satisfies the inequalities

$$\frac{h_M(j)}{h_S(j)} \geqslant \frac{h_M(j+1)}{h_S(j+1)} \quad \text{or} \quad (n+j+1)h_M(j) \geqslant (j+1)h_M(j+1)$$

for all $j \in \mathbb{N}$. $\square$

Proof. We have $h_S(j) = \binom{n+j}{j}$ for all $j \in \mathbb{N}$ and the Absorption Identity gives $k\binom{\ell}{k} = \ell\binom{\ell-1}{k-1}$ for all $k, \ell \in \mathbb{Z}$, so the two forms of inequalities are equivalent and it is enough to prove that

$$(n+j+1)h_M(j) \geqslant (j+1)h_M(j+1). \tag{2}$$

Since M is generated in degree 0, there is a surjective homomorphism of $\mathbb{N}$-graded S-modules $\eta \colon S^{(m)} \to M$ where $S^{(m)}$ is the m-fold direct sum of S for some $m \in \mathbb{N}$. By choosing a monomial order on $S^{(m)}$, we see that both M and the quotient of $S^{(m)}$ by the monomial submodule generated by the leading terms of $\ker(\eta)$ have the same Hilbert function. In both cases, the monomials not belonging to the initial submodule form a $\Bbbk$-vector spaces basis (see [7, Theorem 15.3]). Hence, it suffices to establish the inequality (2) in the case $M = S/I$ for some monomial ideal I.

We interpret both sides of the inequality (2) as cardinalities of sets and describe an appropriate injective map. Using the stars-and-bars correspondence (see [20, Section 1.2]), we identify the set $\mathcal{M}_{j+1}$ of monomials in S_{j+1} with the $(j+1)$-subsets of $\{1, 2, \ldots, n+j+1\}$. Consider $\mathcal{X} \subseteq \{1, 2, \ldots, j+1\} \times \mathcal{M}_{j+1}$ consisting of all pairs (i, σ) such that $i \in \sigma$, and let $\mathcal{Y} := \{1, 2, \ldots, n+j+1\} \times \mathcal{M}_j$. Define the map $\Phi\colon \mathcal{X} \to \mathcal{Y}$ by $\Phi(i, \sigma) = (i, \sigma \setminus \{i\})$. This map is injective, because we can reconstruct σ from the pair $(i, \sigma \setminus \{i\})$. If $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$ are the subsets for which the second components correspond to monomials not in I, then we have $|\mathcal{X}'| = (j+1)h_{S/I}(j+1)$ and $|\mathcal{Y}'| = (n+j+1)h_{S/I}(j)$. Since $\sigma \setminus \{j\}$ corresponds to a monomial not in I whenever σ corresponds to a monomial not in I, restricting the map Φ yields the required injection from $\mathcal{X}'$ to $\mathcal{Y}'$. ■

This proof is surprisingly elementary and self-contained. One can provide alternative proofs of Theorem 2.1 by relying on more elaborate algebraic results. To highlight the differences between Theorem 2.1 and Macaulay's work, we include one of these arguments. Unlike the first proof, the second proof clearly depends on the existence of lex-segment ideals.

Secondary Proof. Generalizing Macaulay's characterization of Hilbert functions of $\mathbb{N}$-graded $\Bbbk$-algebras (i.e., [16, Main Theorem] or [6, Theorem 4.2.10]), [14, Corollary 6] implies that the Hilbert function of M is bounded above by the Hilbert function of the quotient of a free module $S^{(m)}$ by a lexicographic submodule. In particular, if $h_M(j)$ is a multiple of $h_S(j) = \binom{n+j}{j}$, then $h_M(j+1)$ is bounded above by the same multiple of $h_S(j+1) = \binom{n+j+1}{j+1} = \frac{n+j+1}{j+1}\binom{n+j}{j} = \frac{n+j+1}{j+1}h_S(j)$. Hence, for an appropriate $k \in \mathbb{N}$, $h_{M^{(k)}}(j)$ is a multiple of $h_S(j)$ and we obtain $kh_M(j+1) = h_{M^{(k)}}(j+1) \leqslant \left(\frac{n+j+1}{j+1}\right) h_{M^{(k)}}(j) = k\left(\frac{n+j+1}{j+1}\right) h_M(j)$. ■

Remark 2.2. If one replaces the symmetric algebra S with an exterior algebra (see [1, Corollary 4.18]), then the analog of Theorem 2.1 also holds. However, these inequalities do not hold in all rings. For example, if $R := \Bbbk[x_0, x_1]/\langle x_0^2, x_0x_1\rangle$ and $M := R/\langle x_0\rangle \cong \Bbbk[x_1]$, then we have $h_M(1)/h_R(1) = \frac{1}{2} < 1 = h_M(2)/h_R(2)$. □

Let $\mathsf{c}_0 \subset \mathbb{Q}^{\mathbb{N}}$ be the Banach space consisting of all convergent real sequences $h\colon \mathbb{N} \to \mathbb{Q}$ such that $h(j) \to 0$ as $j \to \infty$ equipped with the sup norm (see [12, p. 31]). For any finitely generated $\mathbb{N}$-graded artinian S-module M, we have $h_M \in \mathsf{c}_0$, because the sequence is eventually zero.

Corollary 2.3. The closed convex hull in c_0 of the Hilbert functions of artinian S-modules generated in degree 0 is the intersection of the closed half-spaces defined by the inequalities $(n+j+1)h(j) \geqslant (j+1)h(j+1)$ for $j \in \mathbb{N}$. Moreover, the extreme rays of this cone are generated by the Hilbert functions of the S-modules $S/\langle x_0, x_1, \ldots, x_n\rangle^i$ where $i \in \mathbb{N}$. □

Proof. Theorem 2.1 shows that the Hilbert function of any S-module M generated in degree 0 is contained in the intersection of the closed half-spaces determined by the inequalities $(n+j+1)h_M(j) \geqslant (j+1)h_M(j+1)$ for all $j \in \mathbb{N}$. For brevity, set $\mathfrak{m} := \langle x_0, x_1, \ldots, x_n\rangle$. For $i \in \mathbb{N}$, we have

$$h_{S/\mathfrak{m}^i}(j) = \begin{cases} \dbinom{n+j}{j} & \text{if } j < i, \\ 0 & \text{if } j \geqslant i. \end{cases} \tag{3}$$

These Hilbert functions are linearly independent in c_0, so each $S/\mathfrak{m}^i$ corresponds to an extreme ray of the closed convex cone K generated by $\{(h_{S/\mathfrak{m}^i}(j)) : i \in \mathbb{N}\}$. Equation (3) also yields $(n+i)h_{S/\mathfrak{m}^i}(i-1) > (i)h_{S/\mathfrak{m}^i}(i)$ and, together with the Absorption Identity, shows that $(n+j+1)h_{S/\mathfrak{m}^i}(j) = (j+1)h_{S/\mathfrak{m}^i}(j+1)$ for all $j \neq i-1$. Since $\frac{1}{\ell!}\binom{n+\ell-1}{n} \to 0$ as $\ell \to \infty$, the sequences

$$\sum_{\substack{k=0 \\ k\neq i}}^{\ell} \frac{1}{k!} h_{S/\mathfrak{m}^k}(j)$$

converge as $\ell \to \infty$ and the limit lies on the closed hyperplane

$$(n+j+1)h(j) = (j+1)h(j+1)$$

if and only if $j = i-1$. Hence, K is intersection of the closed half-spaces defined by the inequalities $(n+j+1)h(j) \geqslant (j+1)h(j+1)$ for $j \in \mathbb{N}$. Finally, the Krein–Milman Theorem (e.g., [12, Theorem 1, p. 187; 12, Corollary, p. 189]) establishes that every extreme ray of the cone of Hilbert functions for artinian S-modules generated in degree 0 corresponds to an S-module $S/\mathfrak{m}^i$ for some $i \in \mathbb{N}$. ■

Remark 2.4. The proof of Corollary 2.3 exploits only the topological vector space structure of the Banach space c_0. □

Remark 2.5. Since every Cohen–Macaulay module has an artinian reduction (cf. [6, Corollary 4.1.10]), Corollary 2.3 leads immediately to a description of the closed convex hull of the Hilbert function of Cohen–Macaulay S-modules generated in degree 0. □

By working in a larger space, we can extend Corollary 2.3. Let $E \subseteq \mathbb{Q}^{\mathbb{N}}$ be a topological $\mathbb{Q}$-vector space that contains the Hilbert functions of all artinian S-modules generated in degree 0. For example, if E is the weighted ℓ^{∞}-space consisting of all bounded sequences $h\colon \mathbb{N} \to \mathbb{Q}$ with respect to the norm $\|h\| := \sup_j |n^{-j} h(j)|$, then the Hilbert function of every finitely generated $\mathbb{N}$-graded S-module is contained in E.

Proof of Theorem 1.1. Set $\mathfrak{m} := \langle x_0, x_1, \ldots, x_n \rangle$. Since every Hilbert function in E can expressed uniquely as a nonnegative countable linear combination of the Hilbert functions of the S-modules $S/\mathfrak{m}^i$ for $i \in \mathbb{N}$, the cone of all Hilbert functions in E is generated by $\{h_{S/\mathfrak{m}^i}(j) : i \in \mathbb{N}\}$. Hence, the assertions follow from Corollary 2.3. ∎

Remark 2.6. Every point in the cone is a unique countable linear combination of the Hilbert functions of the S-modules $S/\langle x_0, x_1, \ldots, x_n \rangle^i$ where $i \in \mathbb{N}$, so the closed convex cones described in Theorem 1.1 and Corollary 2.3 are both simplicial (in the sense of Choquet theory (see [18, Section 10])). □

3 Modules with Bounded a-Invariant

In this section, we replace the ambient infinite-dimensional vector space E appearing in Section 2 with a finite-dimensional vector space. We accomplish this by concentrating on S-modules with bounded a-invariant (cf. [6, Definition 4.4.4]). In other words, we insist that the Hilbert function and Hilbert polynomial agree for all integers greater than a. To determine the supporting hyperplanes and extreme rays, we related the cone of Hilbert functions with bounded a-invariant to the cone of nonnegative sequences.

Fix $n \in \mathbb{N}$ and let $a \in \mathbb{Z}$ satisfy $a \geqslant -n$. Consider the finite-dimensional subspace $V_{n,a} \subset \mathbb{Q}^{\mathbb{N}}$ consisting of all sequences $h\colon \mathbb{N} \to \mathbb{Q}$ such that the associated generating functions are rational functions of the form

$$\sum_{j \in \mathbb{N}} h(j) t^j = \frac{b_0 + b_1 t + \cdots + b_{a+n} t^{a+n}}{(1-t)^n} \in \mathbb{Q}(t)$$

for some $b_0, b_1, \ldots, b_{a+n} \in \mathbb{Q}$. Following [20, Corollary 4.3.1], this condition on the generating function is equivalent to the existence of $q_h \in \mathbb{Q}[s]$ such that $q_h(j) = h(j)$ for all $j > a$. As an abuse of terminology, we refer to $q_h \in \mathbb{Q}[s]$ as the Hilbert polynomial of $h \in V_{n,a}$. We identify a sequence $h \in V_{n,a}$ with its generating function $\sum_j h(j) t^j \in \mathbb{Q}(t)$ and regard $V_{n,a}$ as a subspace of $\mathbb{Q}(t)$.

Definition 3.1. Let $P_{n,a}$ denote the closed convex hull in $V_{n,a}$ of the intersection $\mathbb{N}^{\mathbb{N}} \cap V_{n,a}$. Informally, we say that $P_{n,a} \subset V_{n,a}$ is the *cone of nonnegative sequences*. □

For $j \gg 0$, the sign of the Hilbert function is determined by the sign of the leading coefficient of the Hilbert polynomial. Hence, the inequalities $h(j) \geqslant 0$ for $j \in \mathbb{N}$, which define the cone $P_{n,a}$, assert that the leading coefficient of the Hilbert polynomial is positive.

To give the dual description of $P_{n,a}$, it is convenient to introduce a family of polynomials. For a sequence $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of integers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0$, we define

$$p_\lambda(s) := \prod_{i=1}^{r} (s - \lambda_{r-i+1} - 2i + 2)(s - \lambda_{r-i+1} - 2i + 1) \in \mathbb{Z}[s].$$

Following [20, Section 1.7], we view λ as an integer partition of $\sum_{i=1}^{r} \lambda_i$ with at most r parts. Hence, the set $\{p_\lambda : \text{the integer partition } \lambda \text{ has at most } r \text{ parts}\}$ consists of all monic polynomials of degree $2r$ with nonnegative integer roots that appear in consecutive pairs.

To give a uniform description of the extreme rays, we introduce an auxiliary parameter.

Proposition 3.2. Set $\hat{a} := a + \max(1, -a)$. The extreme rays of the nonnegative cone $P_{n,a}$ correspond to the polynomials $1, t, \ldots, t^a$ and the power series

$$\sum_{j \geqslant \hat{a}} \left(p_\lambda(j - \hat{a}) \prod_{\ell=1}^{\hat{a}-a-1} (j + \ell) \right) t^j, \quad \sum_{j \geqslant \hat{a}} \left(p_\mu(j - \hat{a} - 1) \prod_{\ell=0}^{\hat{a}-a-1} (j + \ell) \right) t^j,$$

where λ ranges over all integer partitions with at most $\lfloor (n - \hat{a} + a)/2 \rfloor$ parts and μ ranges over all integer partitions with at most $\lfloor (n - \hat{a} + a - 1)/2 \rfloor$ parts. $\square$

Proof. The Binomial Theorem yields both $t^k = (1-t)^{-n} \sum_i \binom{n}{i} (-1)^i t^{k+i}$ for $0 \leqslant k \leqslant a$ and

$$(1-t)^{-\ell} = (1-t)^{-n} \sum_i \binom{n-\ell}{i} (-1)^i t^i \quad \text{for } 1 \leqslant \ell \leqslant n.$$

When $a \geqslant 0$, the rational functions $t^a, t^{a-1}, \ldots, 1, (1-t)^{-1}, (1-t)^{-2}, \ldots, (1-t)^{-n}$ form a triangular basis for $V_{n,a}$. Let $(c_{-a}, c_{-a+1}, \ldots, c_0, c_1, c_2, \ldots, c_n)$ denote the coordinates of $h \in P_{n,a}$ with respect to this ordered basis. When $a < 0$, just the rational functions $(1-t)^a, (1-t)^{a-1}, \ldots, (1-t)^{-n}$ form a triangular basis for $V_{n,a}$. For consistency, let

$(c_{-a}, c_{-a+1}, \dots, c_n)$ denote the coordinates of $h \in P_{n,a}$ in this situation. Since the Generalized Binomial Theorem implies that $(1-t)^{-\ell} = \sum_j \binom{\ell+j-1}{\ell-1} t^j$, we obtain the inequalities

$$c_{-j} + \sum_{\ell=1}^{n} \binom{\ell+j-1}{\ell-1} c_\ell = h(j) \geqslant 0 \text{ for } 0 \leqslant j \leqslant a \quad \text{and}$$

$$\times \sum_{\ell=\hat{a}-a}^{n} \binom{\ell+j-1}{\ell-1} \quad c_\ell = h(j) \geqslant 0 \text{ for } j \geqslant \hat{a}.$$

The binomial coefficient $\binom{\ell+s-1}{\ell-1}$ is a polynomial in $\mathbb{Q}[s]$ of degree $\ell-1$. Hence, for $s \gg 0$, the leading coefficient of $\sum_\ell \binom{\ell+s-1}{\ell-1} c_\ell$ determines its sign and we obtain $c_n \geqslant 0$.

Since $\dim(V_{n,a}) = n+a+1$, any extreme ray $h \in P_{n,a}$ must satisfy $h(j) = 0$ for at least $n+a$ distinct $j \in \mathbb{N}$. Suppose that we have at least n equalities $h(j) = 0$ with $j \geqslant \hat{a}$. It follows that $c_\ell = 0$ for $1 \leqslant \ell \leqslant n$, because the polynomials $\{\binom{\ell+s-1}{\ell-1} : 1 \leqslant \ell \leqslant n\}$ form a triangular basis for the vector space of all polynomials in $\mathbb{Q}[s]$ with degree at most $n-1$. To obtain a ray, we must also have $c_{-j} = h(j) = 0$ for all but one j satisfying $0 \leqslant j \leqslant a$. Hence, we have $a \geqslant 0$ and the extreme rays in this case correspond to the polynomials $1, t, \dots, t^a$.

Now, suppose that $c_n > 0$ and that we have at most $n - \hat{a} + a$ equalities $h(j) = 0$ with $j \geqslant \hat{a}$. To obtain a ray, we must have $h(j) = 0$ for all $0 \leqslant j \leqslant a$ and $h(j) = 0$ for exactly $n - \hat{a} + a$ distinct j satisfying $j \geqslant \hat{a}$. Hence, the polynomial

$$q(s) := \sum_{\ell=\hat{a}-a}^{n} \binom{\ell+s+\hat{a}-1}{\ell-1} c_\ell \in \mathbb{Q}[s]$$

has $n - \hat{a} + a$ distinct nonnegative integer roots. This polynomial also has $\hat{a} - a - 1$ distinct negative integer roots, namely $-1, -2, \dots, 1 - \hat{a} + a$. Since $\deg(q) = n-1$, it is uniquely determined by its leading coefficient and these integer roots. Furthermore, the real function q changes sign at each root and the evaluation of q at every nonnegative integer is nonnegative, so the nonnegative roots of q must come in consecutive pairs. When $n - \hat{a} + a$ is odd, we need an even number of sign changes arising from the nonnegative roots, so 0 itself must be a root of q. Thus, the extreme rays in this case correspond to the power series

$$\sum_{j \geqslant \hat{a}} \left(p_\lambda(j-\hat{a}) \prod_{\ell=1}^{\hat{a}-a-1} (j+\ell) \right) t^j \quad \text{or} \quad \sum_{j \geqslant \hat{a}} \left(p_\mu(j-\hat{a}-1) \prod_{\ell=0}^{\hat{a}-a-1} (j+\ell) \right) t^j,$$

where $n - \hat{a} + a$ is even and the integer partition λ has $(n - \hat{a} + a)/2$ parts or $n - \hat{a} + a$ is odd and the integer partition μ has $(n - \hat{a} + a - 1)/2$ parts.

The remaining extreme rays of $P_{n,a}$ lie in the hyperplane $c_n = 0$ or equivalently $V_{n-1,a}$. Therefore, induction on n completes the proof. ■

The more important cone in $V_{n,a}$ is generated by Hilbert functions. Specifically, if M is any finitely generated $\mathbb{N}$-graded S-module without free summands (i.e., $\dim(M) < n+1$), then the Hilbert function $h_M \colon \mathbb{N} \to \mathbb{Q}$ is contained in $V_{n,a}$ for all $a \gg 0$ (see [6, Corollary 4.1.8]). Moreover, we have $h_M \in V_{n,a}$ if and only if the Hilbert function $h_M(j)$ equals the Hilbert polynomial $q_M(j)$ for all $j > a$ (see [6, Corollary 4.1.12]). When M is a $\Bbbk$-algebra, the parameter a is the a-invariant (see [6, Definition 4.4.4]).

Definition 3.3. Let $\Omega_{n,a}$ denote the closed convex hull in $V_{n,a}$ of the Hilbert functions of finitely generated $\mathbb{N}$-graded S-modules that are generated in degree 0 and have no free summands. Informally, we say that $\Omega_{n,a} \subset V_{n,a}$ is the *cone of Hilbert functions with bounded a-invariant*. □

To encode the inequalities appearing in Theorem 2.1, we introduce the linear operator $T \colon \mathbb{Q}^{\mathbb{N}} \to \mathbb{Q}^{\mathbb{N}}$ defined by $(T[h])(j) := (n + j + 1)h(j) - (j+1)h(j+1)$ for any $h \colon \mathbb{N} \to \mathbb{Q}$. For an associated generating function, we have $T[\sum_j h(j)t^j] = \sum_j (T[h](j))t^j$. Despite our notation, the operator T depends on the parameter n. Our first lemma shows that the restriction of T to $V_{n,a}$ has an elegant reinterpretation.

Lemma 3.4. The subspace $V_{n,a}$ is T-invariant and $T = (n+1) - (1-t)\frac{d}{dt}$. Moreover, the rational functions $(1-t)^i$ where $-n \leqslant i \leqslant a$ form an eigenbasis for $V_{n,a}$ and the eigenvalue of T corresponding to $(1-t)^i$ is $n+1+i$. □

Proof. The Binomial Theorem yields $(1-t)^i = (1-t)^{-n} \sum_j \binom{i+n}{j} (-1)^j t^j$, so the rational functions $(1-t)^i$ for $-n \leqslant i \leqslant a$ also form triangular basis for $V_{n,a}$. Hence, we have

$$\begin{aligned} \left((n+1) - (1-t)\frac{d}{dt}\right)\left[\sum_j h(j)t^j\right] &= \sum_j (n+1)h(j)t^j - \sum_j jh(j)t^{j-1} + \sum_j jh(j)t^j \\ &= \sum_j ((n+1+j)h(j) - (j+1)h(j+1))t^j \\ &= T\left[\sum_j h(j)t^j\right] \end{aligned}$$

and $T[(1-t)^i] = (n+1)(1-t)^i - (1-t)(i)(1-t)^{i-1}(-1) = (n+1+i)(1-t)^i$. ■

The second lemma calculates the image under T of the Hilbert function for certain cyclic modules.

Lemma 3.5. Let $\ell \in \mathbb{N}$ and $i \in \mathbb{N}$ satisfy $0 \leqslant \ell \leqslant n+1$ and $1 \leqslant i \leqslant a+n-\ell+2$. For the cyclic module $M := S/\langle x_0, x_1, \dots, x_{\ell-1}\rangle^i$, we have $h_M \in V_{n,a}$, and the equation

$$T\left[\sum_j h_M(j)t^j\right] = T\left[(1-t)^{\ell-1-n}\sum_{k=0}^{i-1}\binom{\ell-1+k}{k}t^k\right] = i\binom{\ell-1+i}{i}t^{i-1}(1-t)^{\ell-1-n}$$

holds. □

Proof. The monomials not in the ideal $\langle x_0, x_1, \dots, x_{\ell-1}\rangle^i$ form a $\Bbbk$-vector space basis for M (see [7, Theorem 15.3]). Since these basis elements in degree j are the disjoint union of monomials in $\Bbbk[x_0, x_1, \dots, x_{\ell-1}]_k \cdot \Bbbk[x_\ell, x_{\ell+1}, \dots, x_n]_{j-k}$ where $0 \leqslant k \leqslant i-1$, we have $\sum_j h_M(j)t^j = (1-t)^{\ell-1-n}\sum_{k=0}^{i-1}\binom{\ell-1+k}{k}t^k$, so $h_M \in V_{n,a}$ when $i-1+\ell-1 \leqslant a+n$. Combining Lemma 3.4 with the Absorption Identity, we obtain

$$\begin{aligned}
T\left[\sum_j h_M(j)t^j\right] &= \left((n+1)-(1-t)\frac{d}{dt}\right)\left[(1-t)^{\ell-1-n}\sum_{k=0}^{i-1}\binom{\ell-1+k}{k}t^k\right] \\
&= \ell(1-t)^{\ell-1-n}\sum_{k=0}^{i-1}\binom{\ell-1+k}{k}t^k - (1-t)^{\ell-n}\sum_{k=1}^{i-1}k\binom{\ell-1+k}{k}t^k \\
&= (1-t)^{\ell-1-n}\left(\ell+\sum_{k=1}^{i-1}(\ell+k)\binom{\ell-1+k}{k}t^k - \sum_{k=0}^{i-2}(k+1)\binom{\ell+k}{k+1}t^k\right) \\
&= (1-t)^{\ell-1-n}(\ell-1+i)\binom{\ell+i-2}{i-1}t^{i-1} \\
&= i\binom{\ell-1+i}{i}t^{i-1}(1-t)^{\ell-1-n}
\end{aligned}$$

as required. ■

We concluded our trilogy of lemmas with an elementary positivity result.

Lemma 3.6. Any polynomial $f \in \mathbb{Q}[s]$ of degree r with r distinct negative integer roots and a positive leading coefficient is a nonnegative $\mathbb{Q}$-linear combination of the polynomials $\binom{s+k}{k}$ for $0 \leqslant k \leqslant r$. □

Proof. We proceed by induction on r. If $r=0$, then f is the product of the leading coefficient of f and the polynomial $\binom{s+0}{0}$ which establishes the base case. Assume that $r>0$. Since f has r distinct negative integer roots, the smallest root of f equals $-r-\ell$

for some $\ell \in \mathbb{Z}$ satisfying $\ell \geqslant 0$. It follows that $f(s) = (s + r + \ell)g(s)$ where $g \in \mathbb{Q}[s]$ has degree $r-1$, $r-1$ distinct negative integer roots, and a positive leading coefficient. The induction hypothesis implies that there exists nonnegative $c_0, c_1, \ldots, c_{r-1} \in \mathbb{Q}$ such that

$$g(s) = c_0 \binom{s+0}{0} + c_1 \binom{s+1}{1} + \cdots + c_{r-1} \binom{s+r-1}{r-1}.$$

Hence, the Absorption Identity yields

$$\begin{aligned} f(s) = (s+r+\ell)g(s) &= \sum_{k=0}^{r-1} (s+r+\ell) c_k \binom{s+k}{k} \\ &= \sum_{k=0}^{r-1} (s+k+1) c_k \binom{s+k}{k} + \sum_{k=0}^{r-1} (r-1-k+\ell) c_k \binom{s+k}{k} \\ &= \sum_{k=0}^{r-1} (k+1) c_k \binom{s+k+1}{k+1} + \sum_{k=0}^{r-1} (r-1-k+\ell) c_k \binom{s+k}{k} \\ &= \sum_{k=1}^{r} k c_{k-1} \binom{s+k}{k} + \sum_{k=0}^{r-1} (r-1-k+\ell) c_k \binom{s+k}{k}, \end{aligned}$$

which completes the induction. ■

We can now prove Theorem 1.2 by showing that $T[Q_{n,a}] = P_{n,a}$.

Proof of Theorem 1.2. Theorem 2.1 together with Lemma 3.4 prove that $T[Q_{n,a}] \subseteq P_{n,a}$, so it suffices to show that all of the extreme rays of $P_{n,a}$ are images under T of elements in $Q_{n,a}$. Lemma 3.5 establishes that images under T of Hilbert functions for the artinian modules $S/\langle x_0, \ldots, x_n \rangle^i$ where $1 \leqslant i \leqslant a+1$ are scalar multiples of the polynomials $1, t, \ldots, t^a$. As in Proposition 3.2, let $\hat{a} := a + \max(1, -a)$, fix an appropriate integer partition λ or μ, and let $F(t)$ equal either

$$\sum_{j \geqslant \hat{a}} \left(p_\lambda(j - \hat{a}) \prod_{\ell=1}^{\hat{a}-a-1} (j+\ell) \right) t^j \quad \text{or} \quad \sum_{j \geqslant \hat{a}} \left(p_\mu(j - \hat{a} - 1) \prod_{\ell=0}^{\hat{a}-a-1} (j+\ell) \right) t^j.$$

We need only exhibit a module M such that the image of its Hilbert series under T is a scalar multiple of $F(t)$. Since $b := \hat{a} + \lambda_1 + 2r$ is the largest root of $p_\lambda(j - \hat{a} - 1)$, there is a unique decomposition $F(t) = F_1(t) + F_2(t)$ where $F_1(t)$ is a polynomial of degree less than b and $F_2(t)$ is a power series in which only the terms of degree larger than b have nonzero coefficients. It follows from Lemma 3.5 that the image of the Hilbert series an appropriate direct sum M_1 of the artinian modules $S/\langle x_0, \ldots, x_n \rangle^i$ for $\hat{a} \leqslant i \leqslant b$ maps to

$c_1 F_1(t)$ for some positive $c_1 \in \mathbb{Z}$. Thus, if there exists a module M_2 such that its Hilbert series maps to $c_2 F_2(t)$ for some positive $c_2 \in \mathbb{Z}$, then the Hilbert series of the module $M = M_1^{(c_2)} \oplus M_2^{(c_1)}$ maps to $c_1 c_2 F(t)$ under T.

Establishing the existence of M_2 reduces by Lemma 3.5 to proving that $F_2(t)$ equals a finite nonnegative $\mathbb{Q}$-linear combination of the power series $t^{b+1}(1-t)^{-(k+1)} = t^{b+1}\sum_{j\geqslant 0}\binom{j+k}{k}t^j$ for $0 \leqslant k \leqslant n$. By construction, we have $F_2(t) = t^{b+1}\sum_{j\geqslant 0} f_2(j)t^j$ where f_2 is a polynomial of degree $r \leqslant n$ with r distinct negative integer roots and a positive leading coefficient. Therefore, Lemma 3.6 completes the argument by showing that f_2 is a nonnegative $\mathbb{Q}$-linear combination of the polynomials $\binom{s+k}{k}$ for $0 \leqslant k \leqslant r \leqslant n$. ■

Remark 3.7. The proof of Theorem 1.2 is constructive. However, the procedure for creating a module M that generates an extreme ray is rarely effective, because the number of cyclic summands used is so large. Although each cyclic summand used has the simple form $S/\langle x_0, x_1, \ldots, x_{\ell-1}\rangle^i$ for some $i \in \mathbb{N}$ and $1 \leqslant \ell \leqslant n+1$, the Hilbert function of each individual summand does not belong to $V_{n,a}$. □

Corollary 3.8. The closed convex cone $Q_{n,a}$ is the intersection of the closed half-spaced defined by the inequalities $(n+j+1)h(j) \geqslant (j+1)h(j+1)$ for $j \in \mathbb{N}$ and the limiting inequality which asserts that leading coefficient of the associated Hilbert polynomial is positive. If $\hat{a} := a + \max(1, -a)$, then the extreme rays of $Q_{n,a}$ are generated by the Hilbert functions of the cyclic modules $S/\langle x_0, x_1, \ldots, x_n\rangle^i$ for $1 \leqslant i \leqslant a+1$ and the inverse images under T of the power series

$$\sum_{j\geqslant \hat{a}}\left(p_\lambda(j-\hat{a})\prod_{\ell=1}^{\hat{a}-a-1}(j+\ell)\right)t^j, \quad \sum_{j\geqslant \hat{a}}\left(p_\mu(j-\hat{a}-1)\prod_{\ell=0}^{\hat{a}-a-1}(j+\ell)\right)t^j,$$

where λ ranges over all integer partitions with at most $\lfloor (n-\hat{a}+a)/2 \rfloor$ parts and μ ranges over all integer partitions with at most $\lfloor (n-\hat{a}+a-1)/2 \rfloor$ parts. □

Proof. This follows immediately from Proposition 3.2 and Theorem 1.2. ■

We end this section with some examples illustrating Corollary 3.8. When the dimension of the ambient vector space $V_{n,a}$ is small enough, we can visualize the cone $Q_{n,a}$.

Example 3.9. If $\dim(V_{n,a}) = 1$, then we have $n = -a$. The cone $Q_{n,-n}$ is the positive c_n-axis generated by $(1-t)^{-n}$ which corresponds to the S-module $S/\langle x_0\rangle$. □

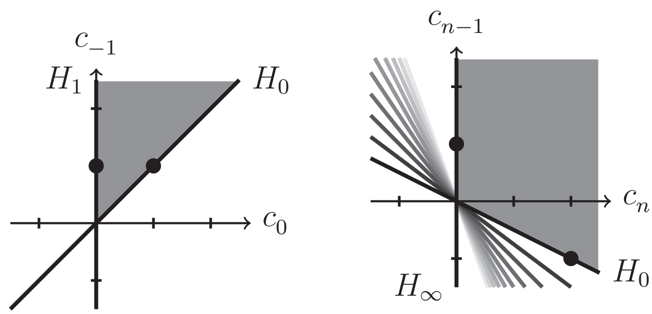

Fig. 1. Cones of Hilbert functions when $\dim(V_{n,a}) = 2$.

Example 3.10. If $n = 0$, then we have $S = \Bbbk[x_0]$ and $a \geqslant 0$. Since the associated generating functions for elements of $V_{0,a}$ have the form $c_{-a}t^a + c_{-a+1}t^{a-1} + \cdots + c_{-1}t + c_0$, the linear half-spaces defining $Q_{0,a}$ are $c_{-j} \geqslant c_{-j-1}$ for $0 \leqslant j < a$ and $c_{-a} \geqslant 0$. The extreme rays are generated by $1 + t + \cdots + t^{i-1}$ for $0 \leqslant i \leqslant a+1$ which corresponds to the S-module $S/\langle x_0\rangle^i$. In particular, $Q_{0,a}$ is a simplicial polyhedral cone. □

Example 3.11. If $\dim(V_{n,a}) = 2$, then we have $a = -n+1$. The case $n = 0$ is described in Example 3.10, so we may assume that $n \geqslant 1$. Since we have

$$\frac{c_{n-1}}{(1-t)^{n-1}} + \frac{c_n}{(1-t)^n} = \sum_{j \in \mathbb{N}} \left(c_{n-1} \binom{n+j-2}{n-2} + c_n \binom{n+j-1}{n-1} \right) t^j$$

the linear half-spaces defining $Q_{n,-n+1}$ are $2(n-1)c_{n-1} + (n+j-1)c_n \geqslant 0$ for $j \in \mathbb{N}$. In this degenerate case, the two linear half-spaces $2c_{n-1} + c_n \geqslant 0$ and $c_n \geqslant 0$ coming from $j = 0$ and $j = \infty$ suffice. The extreme rays are generated by $(1-t)^{-n+1}$

$$-(1-t)^{-n+1} + 2(1-t)^{-n}$$

which correspond to the S-modules $S/\langle x_0, x_1\rangle$ and $S/\langle x_0\rangle^2$. Once again, $Q_{n,-n+1}$ is a simplicial polyhedral cone.

In Figure 1, the case $Q_{0,1}$ appears on the left and the cases $Q_{n,-n+1}$ for $n \geq 1$ appear on the right. The supporting hyperplanes H_j are represented by thick black lines (that fade to white as j increase), the cone is represented by the gray region, and the generators of the extreme rays are represented by black circles. □

Example 3.12. If $n = 3$ and $a = -1$, then $\dim(V_{3,-1}) = 3$. Since we have

$$\frac{c_1}{(1-t)} + \frac{c_2}{(1-t)^2} + \frac{c_3}{(1-t)^3} = \sum_{j \in \mathbb{N}} \left(c_1 \binom{j}{0} + c_2 \binom{j+1}{1} + c_3 \binom{j+2}{2} \right) t^j,$$

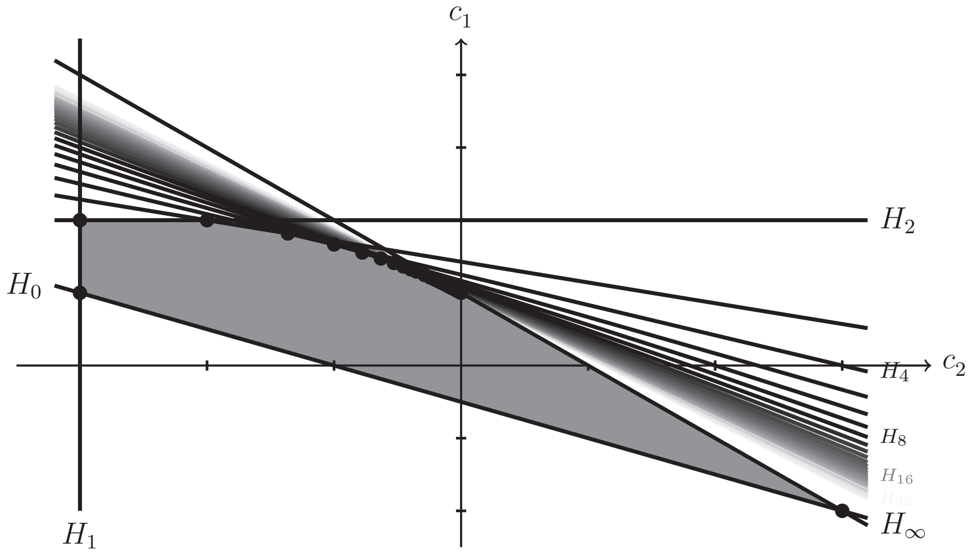

Fig. 2. Cyclic cross-section of $Q_{3,-1}$.

the linear half-spaces defining $Q_{3,-1}$ are

$$(3+j+1)h(j)-(j+1)h(j+1)=3c_1+2(j+1)c_2+\tfrac{1}{2}(j+1)(j+2)c_3\geqslant 0\quad \text{for } j\geqslant 0.$$

To visualize this closed convex cone, we intersect with the hyperplane $c_1+c_2+c_3=1$; for a cyclic module, we have $h(0)=1$. Points in this cross-section are determined by the coordinates (c_2,c_1), and the linear half-spaces in these coordinates are

$$H_j:(j-1)(j+4)c_1+(j-2)(j+1)c_2-(j+1)(j+2)\leqslant 0$$

for $j\geqslant 0$. As $j\to\infty$, we also obtain $H_\infty: c_1+c_2-1\leqslant 0$. In Figure 2, the supporting hyperplanes corresponding to H_j are represented by thick black lines (that fade to white as j increases) and the cross-section of the cone is represented by the gray region.

The extreme points of the cross-section are represented by black circles in Figure 2. More precisely, the supporting hyperplanes corresponding to H_i and H_{i+1} meet at the point $(c_2,c_1)=\frac{3}{i^2+2}(-(i+2),\frac{1}{3}(i+1)(i+2))$ for $i\geqslant 0$, and the supporting hyperplanes corresponding to H_0 and H_∞ meet at the point $(c_2,c_1)=(3,-2)$. As $i\to\infty$, we also obtain the point $(0,1)$. Hence, the extreme rays of $Q_{3,-1}$ are generated by

$$\frac{1}{(1-t)},\quad -\frac{2}{(1-t)}+\frac{3}{(1-t)^2},\quad \text{and}\quad \frac{3}{i^2+2}\left(\frac{(i+1)(i+2)}{3(1-t)}-\frac{(i+2)}{(1-t)^2}+\frac{2}{(1-t)^3}\right).$$

Moreover, these extreme rays correspond to integer partitions with at most 1-part:

$$T\left[\frac{1}{(1-t)}\right] = \sum_{j\in\mathbb{N}} t^j \qquad \longleftrightarrow \qquad \lambda = \varnothing$$

$$T\left[-\frac{2}{(1-t)} + \frac{3}{(1-t)^2}\right] = \sum_{j\in\mathbb{N}} jt^j \qquad \longleftrightarrow \qquad \lambda = \varnothing$$

$$T\left[\frac{(i+1)(i+2)}{3(1-t)} - \frac{(i+2)}{(1-t)^2} + \frac{2}{(1-t)^3}\right] = \sum_{j\in\mathbb{N}} (j-i)(j-i-1)t^j \qquad \longleftrightarrow \qquad \lambda = (i)\,.$$

The cone $Q_{3,-1}$ is neither simplicial nor polyhedral.

The minimal number of generators for the modules lying on the extreme rays is unbounded. Specifically, by considering the linear term, we see that the smallest multiple of the rational function

$$\frac{3}{i^2+2}\left(\frac{(i+1)(i+2)}{3(1-t)} - \frac{(i+2)}{(1-t)^2} + \frac{2}{(1-t)^3}\right)$$

that could be the Hilbert function of a module has constant term i^2+2. Hence, any module that corresponds to a point on this ray has at least i^2+2 generators in degree 0. □

4 Modules with Bounded Regularity

This final section examines our third cone of Hilbert functions. By bounding the Castelnuovo–Mumford regularity of S-modules, we provide an alternative condition which guarantees that the Hilbert functions lie in a finite-dimensional vector space. To enumerate the supporting hyperplanes and extreme rays for the cone of Hilbert functions with bounded regularity, we use the natural projection from the cone of Betti tables.

For a finitely generated $\mathbb{N}$-graded S-module M, the graded Betti numbers are $\beta_{i,j}(M) := \dim_{\Bbbk}(\mathrm{Tor}_i(M, \Bbbk)_j)$, and we have $\beta_{i,j}(M) = 0$ for all $i > n+1$ (see [8, Theorem 1.1]). The graded Betti numbers of M determine its Hilbert series via the formula

$$\sum_{j\in\mathbb{N}} h_M(j)t^j = \frac{\sum_{j\in\mathbb{N}}\sum_{i=0}^{n+1} \beta_{i,j}(M)t^j}{(1-t)^{n+1}}. \tag{4}$$

The Betti table $\beta(M)$ is the matrix in $\bigoplus_{j=-\infty}^{\infty}\bigoplus_{i=0}^{n+1}\mathbb{Q}$ whose entry in the jth row and ith column is $\beta_{i,i+j}(M)$; see [8, Proposition 1.9] for an explanation of this convention. The Castelnuovo–Mumford regularity is the largest index of a nonzero row in the Betti table $\beta(M)$ or equivalently $\mathrm{reg}(M) := \max\{j \in \mathbb{Z} : \beta_{i,i+j}(M) \neq 0\}$. The Hilbert function $h_M(j)$

equals the Hilbert polynomial $q_M(j)$ for all $j > \operatorname{reg}(M)$ (see [8, Theorem 4.2]). Hence, if $m \geqslant \operatorname{reg}(M)$, then Equation (4) shows that $h_M \in V_{n+1,m}$.

Definition 4.1. Let $R_{n,m}$ denote the closed convex hull in $V_{n+1,m}$ of the Hilbert functions of finitely generated $\mathbb{N}$-graded S-modules that are generated in degree 0, have no free summands, and have Castelnuovo–Mumford regularity at most m. Informally, we say that $R_{n,m} \subset V_{n+1,m}$ is the *cone of Hilbert functions with bounded regularity*. □

As in Section 3, let $q_h \in \mathbb{Q}[s]$ be the Hilbert polynomial of the sequence $h \in R_{n,m}$. The backward difference operator $\nabla\colon \mathbb{Q}[s] \to \mathbb{Q}[s]$ is defined by $\nabla q(s) := q(s) - q(s-1)$ where $q \in \mathbb{Q}[s]$. We write ∇^i for the i-fold composition of ∇ with itself.

Proof of Theorem 1.3. We first show that the cone $R_{n,m}$ is generated by the Hilbert functions of the cyclic modules appearing in the list (1). Our indirect proof exploits the Betti tables for certain modules over the smaller polynomial ring $S' := S/\langle x_n \rangle = \Bbbk[x_0, x_1, \ldots, x_{n-1}]$.

Let Ψ be the linear map from the rational vector space of Betti tables for S-modules to the rational vector space of Betti tables for S'-modules defined by $(\Psi(\beta))_{i,j} := j\beta_{i+1,j}$ (cf. [19, Definition 4.5]). Following [3, Definition 2.1], the pure Betti table with degree sequence $d_0 < d_1 < \cdots < d_e$ satisfies $\beta_{i,d_i} = \prod_{j \neq i} \frac{1}{|d_j - d_i|}$ for $0 \leqslant i \leqslant e$. Hence, the map Ψ sends the pure Betti table with degree sequence $0 < d_1 < d_2 < \cdots < d_e$ to the pure Betti table with degree sequence $d_1 < d_2 < \cdots < d_e$. Since [3, Theorems 3.7 and 4.1] establish that the closed convex cones of Betti tables are generated by the pure Betti tables, the map Ψ induces a surjection from the closed convex cone of Betti tables for S-modules to the closed convex cone of Betti tables for S'-modules. Moreover, the kernel of Ψ is generated by the Betti table for the free module S. Therefore, the Betti tables for the modules associated to the generators of the cone $R_{n,m}$ correspond to the Betti tables for finitely generated $\mathbb{N}$-graded S'-modules that are generated in degree at least 0 and have regularity at most m.

Consider a finitely generated $\mathbb{N}$-graded S'-module M' that is generated in degrees at least 0 and has regularity at most m. Any such module M' has the same Hilbert function as the S'-module

$$M'' := \bigoplus_{j=0}^{m-1} \Bbbk(-j)^{\oplus h_{M'}(j)} \oplus M'_{\geqslant m}, \tag{5}$$

where the truncation $M'_{\geqslant m}$ equals $\bigoplus_{j \geqslant m} M'_j$. Eisenbud and Goto [9, Proposition 1.1 and Theorem 1.2] establish that $M'_{\geqslant m}$ has a linear resolution such that $\beta_{i,i+m}(M'_{\geqslant m}) =$

$\beta_{i,i+m}(M'')$ for $0 \leqslant i \leqslant n$. The Koszul complex is also linear, in addition to being the minimal free resolution of the S'-module $\Bbbk$, so it follows that $\beta_{i,i+j}(M'') = h_{M'}(j)\binom{n}{i}$ for $0 \leqslant j < m$. Since the Betti table of a direct sum is the sum of Betti tables and the relation (4) holds, we deduce that the Hilbert function of M' can be expressed as a nonnegative integer combination of Hilbert functions of modules with linear resolutions.

As each cyclic modules appearing in the list (1) is the quotient of S by a Borel-fixed ideal, the minimal free resolution is given by an appropriate Eliahou–Kervaire resolution (see [17, Section 2.3]). Miller and Sturmfels [17, Theorem 2.18] implies that the map Ψ sends $\beta(S/\langle x_0, x_1, \dots, x_\ell\rangle^d)$ to a pure resolution with degree sequence $d < d+1 < \cdots < d+\ell$. In other words, the image of $\beta(S/\langle x_0, x_1, \dots, x_\ell\rangle^d)$ is a Betti table of a linear resolution. Taking the inverse image under Ψ for our expression for the Hilbert function of M', we conclude that each generator of the cone $R_{n,m}$ is a nonnegative rational combination of the Hilbert functions of the cyclic modules appearing in the list (1).

We next describe the supporting hyperplanes to the cone $R_{n,m}$. As in Proposition 3.2, the Binomial Theorem establishes both $t^k = (1-t)^{-n-1}\sum_i \binom{n+1}{i}(-1)^i t^{k+i}$ for $0 \leqslant k \leqslant m$ and

$$(1-t)^{-\ell} t^{m+1} = (1-t)^{-n-1} \sum_i \binom{n-\ell+1}{i} (-1)^i t^{i+m+1} \quad \text{for } 1 \leqslant \ell \leqslant n+1.$$

Hence, the rational functions $1, t, \dots, t^m, (1-t)^{-1}t^{m+1}, (1-t)^{-2}t^{m+1}, \dots, (1-t)^{-n-1}t^{m+1}$ form a triangular basis for $V_{n+1,m}$. Let $(c_0, c_1, \dots, c_m, c_{-1}, c_{-2}, \dots, c_{-n-1})$ denote the coordinates of $h \in V_{n+1,m}$ with respect to this ordered basis. Lemma 3.5 implies that the Hilbert series of the S-module $M_{n,i} := S/\langle x_0, x_1, \dots, x_n\rangle^i$ for $1 \leqslant i \leqslant m+1$ is

$$\sum_j h_{M_{n,i}}(j) t^j = \sum_{k=0}^{i-1} \binom{n+k}{k} t^k = \sum_{k=0}^{i-1} \binom{n+k}{n} t^k,$$

so the coordinates are $c_k = \binom{n+k}{k}$ for $0 \leqslant k \leqslant i-1$ and $c_k = 0$ for $i \leqslant k \leqslant m$ or $k < 0$. Similarly, the Hilbert series of $M_{\ell,m+1} := S/\langle x_0, x_1, \dots, x_{n+1-\ell}\rangle^{m+1}$ for $1 \leqslant \ell \leqslant n+1$ is

$$\begin{aligned}
\sum_j h_{M_{\ell,m+1}}(j) t^j &= (1-t)^{1-\ell} \sum_{k=0}^{m} \binom{n+1-\ell+k}{k} t^k \\
&= \left((1-t)^{-\ell} \sum_{k=0}^{m} \binom{n-\ell+k}{k} t^k \right) - \binom{n+1-\ell+m}{m} (1-t)^{-\ell} t^{m+1} \\
&= \sum_j h_{M_{\ell+1,m+1}}(j) t^j - \binom{n+1-\ell+m}{m} (1-t)^{-\ell} t^{m+1},
\end{aligned}$$

so the coordinates are $c_k = \binom{n+k}{n}$ for $0 \leqslant k \leqslant m$, $c_{-k} = \binom{n+1-k+m}{m}$ for $1 \leqslant k \leqslant \ell$, and $c_{-k} = 0$ for $\ell \leqslant k \leqslant n+1$. Since the coordinate vectors are all truncations of the coordinate vector of $h_{M_{n+1,m+1}} \in R_{n,m}$, the inequalities defining this cone are simply

$$\frac{c_k}{\binom{n+k}{n}} \geqslant \frac{c_{k+1}}{\binom{n+k+1}{n}} \text{ for } 0 \leqslant k \leqslant m-1, \quad \frac{c_m}{\binom{n+m}{n}} \geqslant \frac{c_{-1}}{\binom{n+m}{m}},$$
$$\frac{c_{-k}}{\binom{n+1-k+m}{m}} \geqslant \frac{c_{-k-1}}{\binom{n-k+m}{m}} \text{ for } 1 \leqslant k \leqslant n, \quad \text{and} \quad c_{-n-1} = 0.$$

The equation $c_{-n-1} = 0$ implies that $R_{n,m} \subset V_{n,m}$.

To complete the proof, we explicitly relate the coordinates to the Hilbert function. For $h \in V_{n+1,m}$, we have

$$\sum_j h(j)t^j = c_0 + c_1 t + \cdots + c_m t^m + c_{-1}\frac{t^{m+1}}{(1-t)} + c_{-2}\frac{t^{m+1}}{(1-t)^2} + \cdots + c_{-n-1}\frac{t^{m+1}}{(1-t)^{n+1}},$$

so $h(j) = c_j$ for $0 \leqslant j \leqslant m$ and the Generalized Binomial Theorem shows that

$$q_h(s) = \sum_{k=1}^{n+1} c_{-k}\binom{k+s-m-2}{k-1} \in \mathbb{Q}[s].$$

The Addition Formula for binomial coefficients yields $\nabla^i q_h(s) = \sum_{k=i+1}^{n+1} c_{-k}\binom{k+s-m-2-i}{k-1-i}$ from which we obtain $\nabla^i q_h(m) = c_{-i-1}$ for $0 \leqslant i \leqslant n$. Using the Absorption Identity, the inequalities defining the cone $R_{n,m} \subset V_{n,m}$ become

$$(n+j+1)h(j) \geqslant (j+1)h(j+1) \quad \text{for } 0 \leqslant j \leqslant m-1$$
$$h(m) \geqslant q_h(m), \quad \text{and}$$
$$(n+1-i)\nabla^i q_h(m) \geqslant (n+m+1-i)\nabla^{i+1} q_h(m) \quad \text{for } 0 \leqslant i \leqslant n-1$$

as required. ■

Remark 4.2. The coefficients appearing the supporting hyperplanes of $R_{n,m}$ have an intrinsic interpretation in terms of the Hilbert function of the underlying ring. Specifically, the Hilbert polynomial of S is $q_S(s) = \binom{n+s}{n}$ and the Addition Formula yields $\nabla^i q_S(m) = \binom{n+m-i}{m}$, so $R_{n,m}$ is the intersection of the closed half-spaces given by the inequalities:

$$\frac{h(j)}{h_S(j)} \geqslant \frac{h(j+1)}{h_S(j)} \text{ for } 0 \leqslant j < m, \quad \frac{h(m)}{h_S(m)} \geqslant \frac{q_h(m)}{q_S(m)},$$
$$\frac{\nabla^i q_h(m)}{\nabla^i q_S(m)} \geqslant \frac{\nabla^{i+1} q_h(m)}{\nabla^{i+1} q_S(m)} \text{ for } 0 \leqslant i < n, \quad \text{and} \quad \nabla^n q_h(m) = 0.$$

This form for the inequalities may be more amenable to generalization. □

Remark 4.3. Corollary 3.8 and Theorem 1.3, together with Lemma 3.4, establish that $R_{n,m} \subseteq Q_{n,m}$. Moreover, the artinian cyclic modules $S/\langle x_0, x_1, \dots, x_n\rangle^i$ for $1 \leqslant i \leqslant m+1$ generate extreme rays in both cones. However, the simplicial cone $R_{n,m}$ is generally a proper subcone of $Q_{n,m}$. □

The techniques used in the proof of Theorem 1.3 lead to descriptions of other cones closed related to $R_{n,m}$.

Remark 4.4. Restricting to modules of dimension at most d and regularity at most m yields a subcone of $R_{n,m}$ generated by the Hilbert functions of the cyclic modules:

$$\frac{S}{\langle x_0, x_1, \dots, x_n\rangle}, \frac{S}{\langle x_0, x_1, \dots, x_n\rangle^2}, \dots, \frac{S}{\langle x_0, x_1, \dots, x_n\rangle^m},$$
$$\frac{S}{\langle x_0, x_1, \dots, x_n\rangle^{m+1}}, \frac{S}{\langle x_0, x_1, \dots, x_{n-1}\rangle^{m+1}}, \dots, \frac{S}{\langle x_0, x_1, \dots, x_{n-d}\rangle^{m+1}}.$$

For the dual description, we need to add the equalities

$$\nabla^d q_h(m) = \nabla^{d+1} q_h(m) = \cdots = \nabla^n q_h(m) = 0.$$

Similarly, one can describe the restriction to modules with projective dimension at most ℓ and regularity at most m by relating it to $R_{\ell-1,m}$ via the backward difference operator $\nabla^{n+1-\ell}$. □

The techniques also yield explicit bounds for the Betti numbers of modules with a fixed Hilbert function and bounded regularity.

Proposition 4.5. If the S-module M is generated in degree 0, has no free summands, and has Castelnuovo–Mumford regularity at most m, then the Betti numbers are bounded by the inequalities

$$\beta_{i,i+j}(M) \leqslant \frac{1}{i+j}\binom{n}{i-1}\Big((n+1+j)h_M(j) - (j+1)h_M(j+1)\Big) = \frac{1}{i+j}\binom{n}{i-1}\big(T[h_M]\big)(j)$$

for $0 \leqslant j < m$, $1 \leqslant i \leqslant n+1$, and

$$\beta_{i,i+m}(M) \leqslant \frac{n+m+1}{i+m}\binom{n}{i-1}h_M(m) + \sum_{k=1}^{i}(-1)^k\binom{n+1}{i-k}h_M(m+k)$$

for $1 \leqslant i \leqslant n+1$. Moreover, these bounds are sharp for some positive multiple of the Hilbert function h_M. □

Proof. Hulett [14, Theorem 1] establishes that the module M'', defined in Equation (5), has the largest possible Betti numbers among all S'-modules with a given Hilbert function (up to scaling) and regularity at most m. The matrix with respect to the standard basis of the linear map Ψ has nonnegative entries, so the Betti table $\Psi^{-1}(\beta(M''))$ is maximal among all S-modules that are generated in degree 0, have no free summands, have regularity at most m, and have a given Hilbert function. We compute $\Psi^{-1}(\beta(M''))$ from the expansion of h_M as a nonnegative linear combination of the extreme rays. Ordering the extreme rays as in the list (1), the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{-1}, \alpha_{-2}, \ldots, \alpha_{-n}$ in the unique such expansion are

$$\alpha_j = \frac{h_M(j)}{\binom{n+j}{n}} - \frac{h_M(j+1)}{\binom{n+j+1}{n}} \text{ for } 0 \leqslant j < m, \quad \alpha_m = \frac{h_M(m) - q_M(m)}{\binom{n+m}{n}}, \quad \text{and}$$

$$\alpha_{-i} = \frac{\nabla^{i-1} q_M(m)}{\binom{n+m+1-i}{m}} - \frac{\nabla^{i} q_M(m)}{\binom{n+m-i}{m}} \text{ for } 1 \leqslant i \leqslant n.$$

Since each of the extreme rays corresponds to a cyclic S-module with a linear resolution and well-known Betti numbers (see [6, Theorem 4.1.15]), the Absorption Identity gives

$$\begin{aligned}
\beta_{i,i+j}(M) &\leqslant \alpha_j \cdot \beta_{i,i+j}\left(\frac{S}{\langle x_0, x_1, \ldots, x_n\rangle^{j+1}}\right) \\
&= \left(\frac{h_M(j)}{\binom{n+j}{n}} - \frac{h_M(j+1)}{\binom{n+j+1}{n}}\right) \cdot \left(\frac{i}{i+j}\right)\binom{n+j+1}{n+1}\binom{n+1}{i} \\
&= ((n+j+1)h_M(j) - (j+1)h_M(j+1))\left(\frac{1}{i+j}\right)\binom{n}{n-i} \\
&= \frac{1}{i+j}\binom{n}{n-i}(T[h_M])(j)
\end{aligned}$$

for $0 \leqslant j < m$, $1 \leqslant i \leqslant n+1$. Since we have $\nabla^n q_M(m) = 0$, the Absorption Identity and the Addition Formula give

$$\begin{aligned}
\beta_{i,i+m}(M) &\leqslant \alpha_m \cdot \beta_{i,i+m}\left(\frac{S}{\langle x_0, x_1, \ldots, x_n\rangle^{m+1}}\right) + \sum_{k=1}^{n} \alpha_{-k} \cdot \beta_{i,i+m}\left(\frac{S}{\langle x_0, x_1, \ldots, x_{n-k}\rangle^{m+1}}\right) \\
&= \left(\frac{h_M(m) - q_M(m)}{\binom{n+m}{n}}\right) \cdot \left(\frac{i}{i+m}\right)\binom{n+m+1}{n+1}\binom{n+1}{i} \\
&\quad + \sum_{k=1}^{n}\left(\frac{\nabla^{k-1} q_M(m)}{\binom{n+m+1-k}{m}} - \frac{\nabla^k q_M(m)}{\binom{n+m-k}{m}}\right) \cdot \left(\frac{i}{i+m}\right)\binom{n+1-k+m}{n+1-k}\binom{n+1-k}{i} \\
&= \frac{(n+m+1)h_M(m)}{i+m}\binom{n}{i-1} - \frac{(n+m+1)q_M(m)}{i+m}\binom{n}{i-1} \\
&\quad + \sum_{k=1}^{n} \frac{(n+1-k)\nabla^{k-1} q_M(m)}{i+m}\binom{n-k}{i-1} \\
&\quad - \sum_{k=1}^{n} \frac{(n+m+1-k)\nabla^k q_M(m)}{i+m}\binom{n-k}{i-1} \\
&= \frac{n+m+1}{i+m}\binom{n}{i-1} h_M(m) + \sum_{k=0}^{n} \frac{\nabla^k q_M(m)}{i+m}\left((n-k)\binom{n-k-1}{i-1}\right. \\
&\quad \left. -(n+m+1-k)\binom{n-k}{i-1}\right) \\
&= \frac{n+m+1}{i+m}\binom{n}{i-1} h_M(m) - \sum_{k=0}^{n} \nabla^k q_M(m)\binom{n-k}{i-1}
\end{aligned}$$

for $1 \leqslant i \leqslant n+1$. Combining the binomial identity $\sum_{k=0}^{r}\binom{r-k}{\ell}\binom{k}{i} = \binom{r+1}{\ell+i+1}$ with the higher-order difference formula yields

$$\begin{aligned}
\sum_{k=0}^{n} \nabla^k q_M(m)\binom{n-k}{i-1} &= \sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{n-k}{i-1} q_M(m-\ell) \\
&= \sum_{\ell=0}^{n}(-1)^{\ell}\binom{n+1}{\ell+i} q_M(m-\ell).
\end{aligned}$$

Since $\nabla^{n+1} q_M(s) = 0$ and $q_M(j) = h_M(j)$ for all $j > m$, we obtain

$$\begin{aligned}\sum_{k=0}^{n} \nabla^k q_M(m) \binom{n-k}{i-1} &= \sum_{\ell=-i}^{n+1-i} (-1)^\ell \binom{n+1}{\ell+i} q_M(m-\ell) - \sum_{\ell=-i}^{-1} (-1)^\ell \binom{n+1}{\ell+i} q_M(m-\ell) \\ &= \nabla^{n+1} q_M(m+i) - \sum_{\ell=1}^{i} (-1)^\ell \binom{n+1}{i-\ell} q_M(m+\ell) \\ &= -\sum_{\ell=1}^{i} (-1)^\ell \binom{n+1}{i-\ell} h_M(m+\ell),\end{aligned}$$

which establishes the second family of inequalities. Because the inequalities are equalities for an appropriate direct sum of the modules appearing in the list (1), we conclude that the bound is sharp for some positive multiple of the Hilbert function h_M. ■

We end by illustrating the final proposition in an example.

Example 4.6. Let $n = 3$ and let M be an S-module generated in degree 0 and satisfying $h_M(j) = 3j + 1$ for all $j \in \mathbb{N}$. If the Castelnuovo–Mumford regularity of M is bounded by 1 or 2, respectively, then Proposition 4.5 produces

	0	1	2	3	4
0	1	.	.	.	.
1	.	3	2	.	.

	0	1	2	3	4
0	1	.	.	.	.
1	.	3	6	$\frac{9}{2}$	$\frac{6}{5}$
2	.	4	$\frac{9}{2}$	$\frac{6}{5}$	.

as the entrywise bounds on the Betti tables. □

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