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PROJECTIVE TORIC VARIETIES AS FINE MODULI SPACES OF QUIVER REPRESENTATIONS

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Abstract. This paper proves that every projective toric variety is the fine moduli space for stable representations of an appropriate bound quiver. To accomplish this, we study the quiver Q with relations R corresponding to the finite-dimensional algebra $\mathrm{End}\left(\bigoplus_{i=0}^{r} L_i\right)$ where $\mathcal{L} := (\mathscr{O}_X, L_1, \ldots, L_r)$ is a list of line bundles on a projective toric variety X. The quiver Q defines a smooth projective toric variety, called the multilinear series $|\mathcal{L}|$, and a map $X \longrightarrow |\mathcal{L}|$. We provide necessary and sufficient conditions for the induced map to be a closed embedding. As a consequence, we obtain a new geometric quotient construction of projective toric varieties. Under slightly stronger hypotheses on $\mathcal{L}$, the closed embedding identifies X with the fine moduli space of stable representations for the bound quiver (Q, R).

1. Introduction. The dictionary between the geometry of a moduli space X and the family of objects classified by X lies at the heart of modern algebraic geometry. Fine moduli spaces, although substantially rarer than coarse ones, provide the fundamental example of this correspondence. A scheme X is a fine moduli space for the equivalence classes of some objects if and only if there is a universal family of the selected objects over X such that every other family of objects is induced from the universal one by a unique morphism to X. The hyperplane bundle on $\mathbb{P}^d$ and the tautological vector bundle on a Grassmannian are the classic examples of universal families. A universal family is a powerful tool for studying the geometry of X as illustrated by the forgetful morphism between moduli spaces of pointed stable curves, or the Fourier-Mukai transform on abelian varieties. The primary goal of this paper is to realize every projective toric variety as a fine moduli space of stable representations for an appropriate bound quiver.

To be more precise, let X be a projective toric variety over a field $\mathbb{k}$ and consider a list $\mathcal{L} := (\mathscr{O}_X, L_1, \ldots, L_r)$ of line bundles on X; for convenience set $L_0 := \mathscr{O}_X$. The finite-dimensional $\mathbb{k}$-algebra $\mathrm{End}\left(\bigoplus_{i=0}^{r} L_i\right)$ is encoded by a quiver Q, called the complete quiver of sections for $\mathcal{L}$, together with an ideal of relations R in the path algebra $\mathbb{k}Q$. We associate to the quiver Q a unimodular, projective toric variety $|\mathcal{L}|$ called the multilinear series of Q. The variety $|\mathcal{L}|$ can be defined combinatorially, by geometric invariant theory, or via representation theory; see

Proposition 3.8. Since the isomorphism $\Bbbk Q/R \cong \operatorname{End}(\bigoplus_{i=0}^{r} L_i)$ identifies arrows in Q with global sections of line bundles, the quiver Q induces a map from X to $|\mathcal{L}|$. We prove the following:

THEOREM 1.1. *If $L_1, \ldots, L_r$ are basepoint-free line bundles on X, then the induced map $\varphi_{|\mathcal{L}|} \colon X \longrightarrow |\mathcal{L}|$ is a morphism and the image is presented as a geometric quotient.*

When $r = 1$, $\varphi_{|\mathcal{L}|}$ coincides with the morphism from X to the linear series $|L_1|$.

Lists $\mathcal{L}$ for which the induced morphism $\varphi_{|\mathcal{L}|}$ is a closed embedding are ubiquitous; see Proposition 4.14. Hence, Theorem 1.1 produces a wealth of new geometric quotient constructions for a projective toric variety X. In particular, these *geometric* quotients provide new "homogeneous coordinates" for the points on X. In contrast with [9], [19], these quotient constructions are not intrinsic to the toric variety; they depend on the choice of line bundles in $\mathcal{L}$. Although we recover the quotient constructions in [9], [19] for some toric varieties and particular lists $\mathcal{L}$, the homogeneous coordinate systems arising from a multilinear series $|\mathcal{L}|$ are typically larger. Examples suggest that some of these larger coordinate systems appear naturally in the quantum cohomology of X and the derived category of coherent sheaves on X.

To achieve the primary goal, we relate the image $\varphi_{|\mathcal{L}|}(X)$ to an important subscheme of $|\mathcal{L}|$. From the viewpoint of representation theory, the multilinear series $|\mathcal{L}|$ is the fine moduli space of ϑ-stable representations with dimension vector $(1, \ldots, 1)$ for the quiver Q where ϑ is a distinguished weight on Q; see Proposition 3.8. The ideal of relations R in the path algebra $\Bbbk Q$ determines a subscheme of $|\mathcal{L}|$ that coincides with the fine moduli space $\mathcal{M}_\vartheta(Q, R)$ of ϑ-stable representations with dimension vector $(1, \ldots, 1)$ for the bound quiver (Q, R). In other words, the variety $\mathcal{M}_\vartheta(Q, R)$ classifies certain finite-dimensional modules over the $\Bbbk$-algebra $\operatorname{End}(\bigoplus_{i=0}^{r} L_i)$. The universal ϑ-stable representation of (Q, R) over $\mathcal{M}_\vartheta(Q, R)$ decomposes into a direct sum of line bundles called the tautological line bundles. Our main results can be summarized as follows:

THEOREM 1.2. *Let X be a projective toric variety. There exist (many) lists $\mathcal{L}$ of line bundles on X such that the induced morphism $\varphi_{|\mathcal{L}|} \colon X \longrightarrow |\mathcal{L}|$ identifies X with the fine moduli space $\mathcal{M}_\vartheta(Q, R)$. Moreover, the tautological line bundles on $\mathcal{M}_\vartheta(Q, R)$ coincide with the line bundles in the list $\mathcal{L}$.*

This fine moduli interpretation yields a functorial approach to projective toric varieties. Specifically, it allows one to describe the data needed to specify a map from a scheme to a projective toric variety as in [10], [19].

Theorem 1.2 also helps clarify the relationship between descriptions of the derived category $D^b(\mathcal{O}_X - \mathrm{mod})$ and realizations of X as a fine moduli space of quiver representations. [8] shows that $D^b(\mathcal{O}_X - \mathrm{mod})$ is equivalent to the derived

category of finite-dimensional modules over $\operatorname{End}(\bigoplus_{i=0}^{r} \mathscr{F}_i)$ if and only if the coherent sheaves $\mathscr{F}_i$ form a complete strong exceptional collection on X. On certain toric quiver varieties, [1] describe such collections in which the $\mathscr{F}_i$ are line bundles; toric quiver varieties are fine moduli spaces of quiver representations. The influential [22] constructs complete strong exceptional collections of line bundles on several smooth toric surfaces by realizing the surfaces as fine moduli spaces of stable representations of a bound quiver. Given a smooth projective variety with a complete strong exceptional collection of line bundles, [4] establishes that the variety is isomorphic to a connected component of a corresponding moduli space of stable quiver representations. [20] proves that every toric variety has a complete exceptional collection of coherent sheaves and, in contrast, [18] exhibits a smooth toric surface that does not have a complete strong exceptional collection of line bundles. In this context, Theorem 1.2 clearly differentiates between a fine moduli interpretation of a variety and the existence of a complete strong exceptional collection of line bundles.

This paper is organized as follows. Our notation and some standard results from toric geometry and quiver theory are described in §2. In §3, we define a quiver of sections and its associated multilinear series. This generalizes the classical notion of a linear series from a single line bundle to a list of line bundles. The induced map to the multilinear series is studied in §4. In particular, we give necessary and sufficient conditions for the induced map to be a morphism or a closed embedding. Finally, §5 examines representations of a bound quiver of sections and establishes our main results.

2. Background and notation. We collect here standard definitions, results and notation. In this paper, $\mathbb{N}$ denotes the nonnegative integers and $\Bbbk$ is an algebraically closed field of characteristic zero.

2.1. Toric varieties. Let X be a projective toric variety over $\Bbbk$ determined by a strongly convex rational polyhedral fan $\Sigma_X \subseteq N_X \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ where N_X is a lattice of rank d. The dual lattice is $M_X := \operatorname{Hom}_{\mathbb{Z}}(N_X, \mathbb{Z})$ and $T_X := N_X \otimes_{\mathbb{Z}} \Bbbk^*$ is the algebraic torus acting on X. The i-dimensional cones of Σ_X form the set $\Sigma_X(i)$. Since X is projective, $\bigcup_{\sigma \in \Sigma_X} \sigma = N_X \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Sigma_X(d)$ is the set of maximal cones.

Each $\rho \in \Sigma_X(1)$ corresponds to an irreducible T_X-invariant Weil divisor D_ρ on X. These divisors generate the free abelian group $\mathbb{Z}^{\Sigma_X(1)}$ of T_X-invariant Weil divisors and the semigroup $\mathbb{N}^{\Sigma_X(1)}$ of effective T_X-invariant Weil divisors. The quotient of $\mathbb{Z}^{\Sigma_X(1)}$ by the subgroup of principal divisors is the class group (or

Chow group) $\mathrm{Cl}(X)$. The T_X-invariant Cartier divisors $\mathrm{CDiv}(X)$ form a subgroup of $\mathbb{Z}^{\Sigma_X(1)}$. Moreover, there is a commutative diagram

(2.1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_X & \longrightarrow & \mathrm{CDiv}(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_X & \longrightarrow & \mathbb{Z}^{\Sigma_X(1)} & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow & 0
\end{array}
$$

where the rows are exact and the vertical arrows are inclusions; see §3.4 in [15]. The projection from $\mathbb{Z}^{\Sigma_X(1)}$ to $\mathrm{Cl}(X)$ is denoted by $u \mapsto [u]$ and the inclusion of $\mathrm{Pic}(X)$ into $\mathrm{Cl}(X)$ is also denoted by $L \mapsto [L]$. For a line bundle L on X and a global section $s \in H^0(X, L)$, $\mathrm{div}(s)$ denotes the effective Cartier divisor determined by s.

The total coordinate ring of X is the polynomial ring $S_X := \mathbb{k}[x_\rho \colon \rho \in \Sigma_X(1)]$. Following [9], S_X is the semigroup algebra of $\mathbb{N}^{\Sigma_X(1)}$ with the $\mathrm{Cl}(X)$-grading induced by $\deg(x^u) = \deg\left(\prod_\rho x_\rho^{u_\rho}\right) = [u] \in \mathrm{Cl}(X)$. A divisor $D = \sum_{\rho \in \Sigma_X(1)} u_\rho D_\rho$ determines a Laurent monomial

$$
x^u = \prod_{\rho \in \Sigma_X(1)} x_\rho^{u_\rho} \in \mathbb{k}[x_\rho^{\pm 1} \colon \rho \in \Sigma_X(1)]
$$

and we often write the monomial as x^D. The support of D or x^u is the set

$$
\mathrm{supp}(D) = \mathrm{supp}(x^u) = \{\rho \in \Sigma_X(1) \colon u_\rho \neq 0\}.
$$

For a cone $\sigma \in \Sigma_X$, $\widehat{\sigma}$ is the set of one-dimensional cones in Σ_X that are not contained in σ and $x^{\widehat{\sigma}} = \prod_{\rho \in \widehat{\sigma}} x_\rho$ is the associated monomial in S_X. The irrelevant ideal of X is the square-free (i.e. reduced) monomial ideal $B_X := (x^{\widehat{\sigma}} \colon \sigma \in \Sigma_X)$. Theorem 2.1 in [9] shows that the pair (S_X, B_X) encodes a quotient construction of X. Specifically, if $\mathbb{V}(B_X)$ is the subvariety of $\mathbb{A}^{\Sigma_X(1)}$ defined by B_X, then the toric variety X is a categorical quotient of $\mathbb{A}^{\Sigma_X(1)} \setminus \mathbb{V}(B_X)$ by the group $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbb{k}^*)$; the group action is induced by the $\mathrm{Cl}(X)$-grading of S_X.

2.2. Quivers. A quiver Q is specified by two finite sets Q_0 and Q_1, whose elements are called vertices and arrows, together with two maps $\mathrm{hd}, \mathrm{tl} \colon Q_1 \longrightarrow Q_0$ indicating the vertices at the head and tail of each arrow. A nontrivial path in Q is a sequence of arrows $p = a_1 \cdots a_\ell$ with $\mathrm{hd}(a_k) = \mathrm{tl}(a_{k+1})$ for $1 \leq k < \ell$. We set $\mathrm{tl}(p) = \mathrm{tl}(a_1)$ and $\mathrm{hd}(p) = \mathrm{hd}(a_\ell)$. Each $i \in Q_0$ gives a trivial path e_i where $\mathrm{tl}(e_i) = \mathrm{hd}(e_i) = i$. The path algebra $\mathbb{k}Q$ is the $\mathbb{k}$-algebra whose underlying $\mathbb{k}$-vector space has a basis consisting of paths in Q; the product of two basis elements equals the basis element defined by concatenation of the paths if possible or zero otherwise. A cycle is a path p in which $\mathrm{tl}(p) = \mathrm{hd}(p)$. A quiver is acyclic

if it contains no cycles. A vertex is a source if it is not the head of any arrow and a quiver is rooted if it has a unique source.

A walk γ in Q is an alternating sequence $i_0 a_1 i_1 \cdots a_\ell i_\ell$ of vertices $i_1, \ldots, i_\ell$ and arrows $a_1, \ldots, a_\ell$ where a_k is an arrow between i_{k-1} and i_k. If $\mathrm{tl}(a_k) = i_{k-1}$ and $\mathrm{hd}(a_k) = i_k$ then a_k is a forward arrow in γ; otherwise $\mathrm{tl}(a_k) = i_k$, $\mathrm{hd}(a_k) = i_{k-1}$ and a_k is a backward arrow. If $a \in Q_1$ then a^{-1} denotes the walk from $\mathrm{hd}(a)$ to $\mathrm{tl}(a)$. A walk γ is closed if $i_0 = i_\ell$ and a circuit is a closed walk in which the arrows $a_1, \ldots, a_\ell$ are distinct. A quiver is connected if there is a walk between any two vertices. A tree is a connected acyclic quiver. We say that $Q' \subseteq Q$ is a spanning subquiver if $Q'_0 = Q_0$.

The vertex space $\mathbb{Z}^{Q_0}$ is the free abelian group of functions from Q_0 to $\mathbb{Z}$ and the arrow space $\mathbb{Z}^{Q_1}$ is the free abelian group of functions from Q_1 to $\mathbb{Z}$. The characteristic functions $\chi_i \colon Q_0 \longrightarrow \mathbb{Z}$ for $i \in Q_0$ and $\chi_a \colon Q_1 \longrightarrow \mathbb{Z}$ for $a \in Q_1$ form the standard bases for the vertex and arrows spaces. We write $\mathbb{N}^{Q_0}$ and $\mathbb{N}^{Q_1}$ for the semigroups generated by all $\mathbb{N}$-linear combinations of the characteristic functions χ_i and χ_a respectively. The incidence map $\mathrm{inc} \colon \mathbb{Z}^{Q_1} \longrightarrow \mathbb{Z}^{Q_0}$ is defined by $\mathrm{inc}(\chi_a) = \chi_{\mathrm{hd}(a)} - \chi_{\mathrm{tl}(a)}$. A function $\theta \colon Q_0 \longrightarrow \mathbb{Z}$ is an integral weight of Q if $\sum_{i \in Q_0} \theta_i = 0$ and a function $f \colon Q_1 \longrightarrow \mathbb{Z}$ is an integral circulation if

$$\sum_{\substack{a \in Q_1 \\ \mathrm{tl}(a)=i}} f_a = \sum_{\substack{a \in Q_1 \\ \mathrm{hd}(a)=i}} f_a \quad \text{for each } i \in Q_0.$$

The weight lattice $\mathrm{Wt}(Q) \subset \mathbb{Z}^{Q_0}$ and the circulation lattice $\mathrm{Cir}(Q) \subset \mathbb{Z}^{Q_1}$ are generated by the integral weights and circulations respectively. There is an exact sequence

$$(2.2) \qquad 0 \longrightarrow \mathrm{Cir}(Q) \longrightarrow \mathbb{Z}^{Q_1} \xrightarrow{\ \mathrm{inc}\ } \mathrm{Wt}(Q)$$

and the incidence map is surjective when Q is connected; see §4 in [6]. For $a \in Q_1$ and a walk γ, let $\mathrm{mult}_\gamma(a) \in \mathbb{Z}$ equal the number of times a appears as a forward arrow in γ minus the number of times it appears as a backward arrow. Given a walk γ, we set $f(\gamma) := \sum_{a \in Q_1} \mathrm{mult}_\gamma(a) \chi_a \in \mathbb{Z}^{Q_1}$; $f(\gamma) \in \mathrm{Cir}(Q)$ if and only if γ is a closed walk.

2.3. Representations of quivers. Let Q be a connected quiver. A representation $W = (W_i, w_a)$ of Q consists of a $\mathbb{k}$-vector space W_i for each $i \in Q_0$ and a $\mathbb{k}$-linear map $w_a \colon W_{\mathrm{tl}(a)} \longrightarrow W_{\mathrm{hd}(a)}$ for each $a \in Q_1$. The dimension vector of W is $\sum_{i \in Q_0} \dim_{\mathbb{k}}(W_i) \chi_i \in \mathbb{N}^{Q_0}$. In this paper, we will assume that $\dim_{\mathbb{k}}(W_i) = 1$ for all $i \in Q_0$. A map between representations $W = (W_i, w_a)$ and $W' = (W'_i, w'_a)$ is a family $\psi_i \colon W_i \longrightarrow W'_i$ for $i \in Q_0$ of $\mathbb{k}$-linear maps that are compatible with the structure maps, that is $w'_a \psi_{\mathrm{tl}(a)} = \psi_{\mathrm{hd}(a)} w_a$ for all $a \in Q_1$. With composition defined componentwise, we obtain the abelian category of representations of Q. Each rational weight $\theta \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ defines a stability notion for representations and

subquivers of Q. A representation W is θ-stable if, for every proper, nonzero subrepresentation $W' \subset W$, we have $\theta(W') := \sum_{i \in \text{supp}(W')} \theta_i > 0$, where $\text{supp}(W') := \{i \in Q_0 : W'_i \neq 0\}$. The notion of θ-semistability is obtained by replacing $>$ with $\geq$. A subquiver $Q' \subseteq Q$ is θ-stable if it admits a θ-stable representation.

The isomorphism classes of representations are orbits in the representation space $\mathbb{A}^{Q_1} = \text{Spec}(\mathbb{k}[y_a : a \in Q_1]) \cong \bigoplus_{a \in Q_1} \text{Hom}_{\mathbb{k}}(W_{\text{tl}(a)}, W_{\text{hd}(a)})$ by the action of the group $(\mathbb{k}^*)^{Q_0} \cong \prod_{i \in Q_0} \text{GL}(W_i)$ induced by the incidence map; in other words, it acts by $(g \cdot w)_a = g_{\text{hd}(a)} w_a g_{\text{tl}(a)}^{-1}$. Hence, the algebraic torus $G := \text{Hom}_{\mathbb{Z}}(\text{Wt}(Q), \mathbb{k}^*)$ acts faithfully on $\mathbb{A}^{Q_1}$ and the polynomial ring $S_Y := \mathbb{k}[y_a : a \in Q_1]$ has the associated $\text{Wt}(Q)$-grading. For $\theta \in \text{Wt}(Q)$, let $(S_Y)_\theta$ be the θ-graded piece. Following [21], the GIT-quotient $\mathcal{M}_\theta(Q) := \mathbb{A}^{Q_1} /\!\!/_\theta G = \text{Proj}(\bigoplus_{k \in \mathbb{N}} (S_Y)_{k\theta})$ is the categorical quotient $(\mathbb{A}^{Q_1})_\theta^{\text{SS}}/G$, where $(\mathbb{A}^{Q_1})_\theta^{\text{SS}} \subseteq \mathbb{A}^{Q_1}$ is the open subscheme parametrizing θ-semistable representations of Q. Since $\dim(W_i) = 1$ for all $i \in Q_0$, $\mathcal{M}_\theta(Q)$ is also a toric quiver variety as defined in [17]. A weight $\theta \in \text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generic if every θ-semistable representation is θ-stable. In this case, $\mathcal{M}_\theta(Q)$ is the geometric quotient $(\mathbb{A}^{Q_1})_\theta^{\text{S}}/G$, where $(\mathbb{A}^{Q_1})_\theta^{\text{S}}$ parametrizes θ-stable representations of Q. The set of generic weights decomposes into finitely many open chambers, where $\mathcal{M}_\theta(Q)$ is unchanged as θ varies in a chamber; see [12], [28].

For generic θ, Proposition 5.3 in [21] implies that $\mathcal{M}_\theta(Q)$ is the fine moduli space of θ-stable representations of Q. To describe the universal family on $\mathcal{M}_\theta(Q)$, we set $Q_0 = \{0, \ldots, r\}$ and identify the group G with $\{(g_0, \ldots, g_r) \in (\mathbb{k}^*)^{Q_0} : g_0 = 1\}$. This choice determines a G-equivariant vector bundle $\bigoplus_{i \in Q_0} \mathcal{O}_{\mathbb{A}^{Q_1}}$ which descends to $\bigoplus_{i \in Q_0} F_i$ on $\mathcal{M}_\theta(Q)$; see Proposition 5.3 in [21]. The line bundles $F_0, \ldots, F_r$ are called the tautological line bundles on $\mathcal{M}_\theta(Q)$. Since G acts trivially on the 0th component of $\bigoplus_{i \in Q_0} \mathcal{O}_{\mathbb{A}^{Q_1}}$, it follows that F_0 is the trivial line bundle.

3. Quivers of sections. The goal of this section is to extend the classical notion of a linear series from a single line bundle to a list of line bundles. Let $\mathcal{L} := (L_0, \ldots, L_r)$ be a list of distinct line bundles on the projective toric variety X. A T_X-invariant section $s \in H^0(X, L_j \otimes L_i^{-1})$ is indecomposable if the divisor $\text{div}(s)$ cannot be expressed as a sum $\text{div}(s') + \text{div}(s'')$ where $s' \in H^0(X, L_k \otimes L_i^{-1})$ and $s'' \in H^0(X, L_j \otimes L_k^{-1})$ are nonzero T_X-invariant sections and $0 \leq k \leq r$. A quiver of sections associated to $\mathcal{L}$ is a quiver Q in which the vertices $Q_0 = \{0, \ldots, r\}$ correspond to the line bundles in $\mathcal{L}$ and the arrows from i to j correspond to a subset of the indecomposable T_X-invariant sections in $H^0(X, L_j \otimes L_i^{-1})$. If every indecomposable T_X-invariant section in $H^0(X, L_j \otimes L_i^{-1})$ for $0 \leq i, j \leq r$ corresponds to an arrow then Q is the complete quiver of sections for $\mathcal{L}$. Since X is projective, the unique element in $H^0(X, L_i \otimes L_i^{-1}) = H^0(X, \mathcal{O}_X)$ defines the trivial path e_i in Q. Moreover, if $L_j \neq L_i$ then projectivity implies that both $H^0(X, L_j \otimes L_i^{-1})$ and $H^0(X, L_j^{-1} \otimes L_i)$ cannot be nonzero. It follows that Q is acyclic.

Conventions 3.1. Let Q be a quiver of sections associated to $\mathcal{L} = (L_0, \ldots, L_r)$.

(a) By definition, Q only depends on the line bundles $L_j \otimes L_i^{-1}$ where $0 \leq i, j \leq r$. Consequently, for any line bundle L' on X, we have $Q = Q'$ where Q' is a quiver of sections associated to $\mathcal{L}' = (L_0 \otimes L', \ldots, L_r \otimes L')$. To eliminate this redundancy, we will assume that $L_0 = \mathcal{O}_X$. By reordering the elements in $\mathcal{L}$ if necessary, we may also assume that $j < i$ implies $H^0(X, L_j \otimes L_i^{-1}) = 0$.

(b) We will assume that $H^0(X, L_i) \neq 0$ for $0 \leq i \leq r$. If Q is the complete quiver of sections for $\mathcal{L}$, then this implies that Q is connected and rooted at $0 \in Q_0$.

Since each $a \in Q_1$ corresponds to a T_X-invariant section $s \in H^0(X, L_j \otimes L_i^{-1})$, we simply write $\mathrm{div}\,(a) := \mathrm{div}\,(s) \in \mathrm{CDiv}\,(X)$. More generally, for a path $p = a_1 \cdots a_\ell$ in Q, we set $\mathrm{div}\,(p) := \mathrm{div}\,(a_1) + \cdots + \mathrm{div}\,(a_\ell)$. This labelling of paths induces relations on Q. Specifically, the *ideal of relations* is the two-sided ideal R in the path algebra $\Bbbk Q$ generated by differences $p - p' \in \Bbbk Q$ such that $\mathrm{tl}\,(p) = \mathrm{tl}\,(p')$, $\mathrm{hd}\,(p) = \mathrm{hd}\,(p')$ and $\mathrm{div}\,(p) = \mathrm{div}\,(p')$. Since the arrows in Q correspond to indecomposable sections, R is an admissible ideal. The pair (Q, R) is called a *bound quiver of sections*; the phrase "bound quiver" is a synonym for "quiver with relations".

Example 3.2. If L_1 is a nontrivial line bundle on X, then the complete quiver of sections for $\mathcal{L} = (\mathcal{O}_X, L_1)$ has two vertices and $\dim_\Bbbk H^0(X, L_1)$ arrows. The ideal of relations R is the zero ideal.

The general correspondence between bound quivers and finite-dimensional $\Bbbk$-algebras has the following useful incarnation for a quiver of sections.

PROPOSITION 3.3. *If (Q, R) is the complete bound quiver of sections for the list $\mathcal{L} = (L_0, \ldots, L_r)$ then the quotient algebra $\Bbbk Q/R$ is isomorphic to $\mathrm{End}\,(\bigoplus_{i=0}^r L_i)$.*

Proof. The map sending a path $p = a_1 \cdots a_\ell$ in Q to the product of the corresponding sections $s_1 \cdots s_\ell \in H^0(X, L_{\mathrm{hd}\,(p)} \otimes L_{\mathrm{tl}\,(p)}^{-1}) = \mathrm{Hom}\,(L_{\mathrm{tl}\,(p)}, L_{\mathrm{hd}\,(p)})$ determines a homomorphism of $\Bbbk$-algebras

$$\eta\colon \Bbbk Q \to \mathrm{End}\,\Big(\bigoplus\nolimits_{i=0}^r L_i\Big) = \bigoplus\nolimits_{i,j=0}^r \mathrm{Hom}\,(L_i, L_j).$$

The map is surjective because Q is a complete quiver. Moreover, η sends paths p, p' in Q satisfying $\mathrm{tl}\,(p) = \mathrm{tl}\,(p')$ and $\mathrm{hd}\,(p) = \mathrm{hd}\,(p')$ to the same element in $\mathrm{Hom}\,(L_{\mathrm{tl}\,(p)}, L_{\mathrm{hd}\,(p)})$ if and only if $\mathrm{div}\,(p) = \mathrm{div}\,(p')$. Thus, we have $\mathrm{Ker}\,(\eta) = R$. □

Remark 3.4. Let (Q, R) be a (not necessarily complete) bound quiver of sections associated to $\mathcal{L} = (L_0, \ldots, L_r)$. If $\{e_0, \ldots, e_r\}$ is a complete set of primitive orthogonal idempotents in $\mathrm{End}\,(\bigoplus_{i=0}^r L_i)$ and $s_0, \ldots, s_m$ are the indecomposable sections corresponding to the arrows in Q, then $\Bbbk Q/R$ is the subalgebra of $\mathrm{End}\,(\bigoplus_{i=0}^r L_i)$ generated by $\{e_0, \ldots, e_r, s_0, \ldots, s_m\}$.

Remark 3.5. [8], [21], [4] work with the opposite quiver. In particular, a "Bondal quiver" is a complete quiver of sections in which arrows have the opposite orientation.

A quiver of sections Q comes equipped with a distinguished lattice. The map sending $a \in Q_1$ to $\mathrm{div}\,(a) \in \mathrm{CDiv}\,(X)$ extends to give a $\mathbb{Z}$-linear map

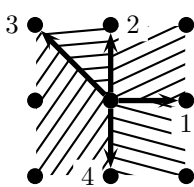$$\mathrm{div}\colon \mathbb{Z}^{Q_1} \longrightarrow \mathrm{CDiv}\,(X),$$

where $\mathrm{div}\,(v) := \sum_{a \in Q_1} v_a \,\mathrm{div}\,(a)$ for $v = \sum_{a \in Q_1} v_a \chi_a$. The *section lattice* $\mathbb{Z}(Q)$ is the image of the map $\pi := (\,\mathrm{inc}, \mathrm{div})\colon \mathbb{Z}^{Q_1} \longrightarrow \mathrm{Wt}\,(Q) \oplus \mathrm{CDiv}\,(X)$; by definition, we have $\pi(\chi_a) = (\chi_{\mathrm{hd}\,(a)} - \chi_{\mathrm{tl}\,(a)}, \mathrm{div}\,(a))$. The projections onto the components are denoted by $\pi_1\colon \mathbb{Z}(Q) \longrightarrow \mathrm{Wt}\,(Q)$ and $\pi_2\colon \mathbb{Z}(Q) \longrightarrow \mathrm{CDiv}\,(X)$ respectively. These maps fit in to the commutative diagram

(3.1)

$$
\begin{array}{ccc}
& \mathbb{Z}^{Q_1} & \\
& \swarrow \quad \searrow^{\mathrm{inc}} & \\
\mathrm{div} \swarrow \quad \mathbb{Z}(Q) & \xrightarrow{\ \pi_1\ } & \mathrm{Wt}\,(Q) \\
\downarrow^{\pi_2} & & \downarrow^{\mathrm{pic}} \\
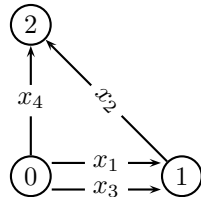\mathrm{CDiv}\,(X) & \longrightarrow & \mathrm{Pic}\,(X),
\end{array}
$$

where $\mathrm{pic}\,(\theta) := \bigotimes_{i \in Q_0} L_i^{\theta_i}$ for $\theta = \sum_{i \in Q_0} \theta_i \chi_i$ and the map $\mathrm{CDiv}\,(X) \longrightarrow \mathrm{Pic}\,(X)$ is taken from (2.1).

Example 3.6. Let $X = \mathbb{F}_1 = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1))$ be the Hirzebruch surface determined by the fan in Figure 1 (a). For $(k, \ell) \in \mathbb{Z}^2$, set $\mathscr{O}_X(k, \ell) := \mathscr{O}_X(kD_1 + \ell D_4) \in \mathrm{Pic}\,(X)$. The complete quiver of sections for $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1))$ appears in Figure 1 (b). The ideal of relations is $R = (0)$. The section lattice $\mathbb{Z}(Q)$ is generated by the columns of the matrix
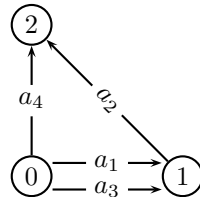
(3.2)

$$
\begin{bmatrix}
-1 & 0 & -1 & -1 \\
1 & -1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

(a) Fan   (b) Quiver of sections   (c) Listing the arrows

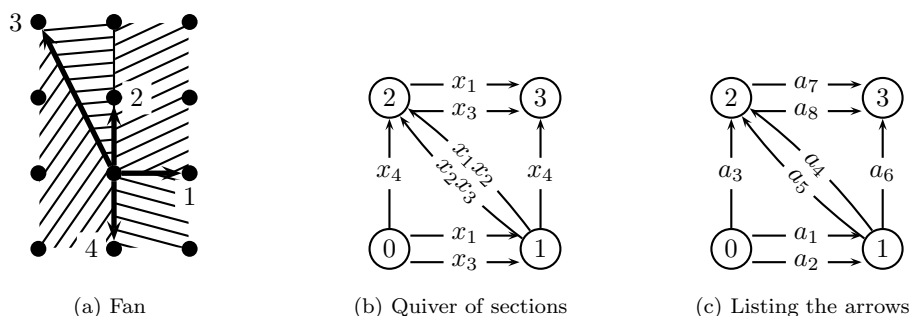Figure 1.   Hirzebruch surface $\mathbb{F}_1$.

(a) Fan   (b) Quiver of sections   (c) Listing the arrows

Figure 2.   Hirzebruch surface $\mathbb{F}_2$.

Example 3.7. Let $X = \mathbb{F}_2 = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(2))$ be the Hirzebruch surface determined by the fan in Figure 2 (a). For $(k, \ell) \in \mathbb{Z}^2$, we write $\mathscr{O}_X(k, \ell) := \mathscr{O}_X(kD_1 + \ell D_4) \in \operatorname{Pic}(X)$. The complete quiver of sections for $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1, 0), \mathscr{O}_X(0, 1), \mathscr{O}_X(1, 1))$ appears in Figure 2 (b). If we order the arrows as in Figure 2 (c), then the ideal of relations is $R = (a_2 a_4 - a_1 a_5, a_4 a_8 - a_5 a_7, a_2 a_6 - a_3 a_8, a_1 a_6 - a_3 a_7)$. The section lattice $\mathbb{Z}(Q)$ is generated by the columns of the following matrix

$$(3.3) \qquad \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} ;$$

the ith column corresponds to a_i for $1 \leq i \leq 8$.

Classically, one associates a projective space, called the linear series, to a nonzero subspace of global sections of a line bundle. Generalizing this construction, we associate a toric variety Y_Q to an appropriate quiver of sections Q. Following §1.4 in [17], any connected, rooted, acyclic quiver Q defines a complete fan Σ_Q in the $\mathbb{R}$-vector space $\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Cir}(Q), \mathbb{R})$ with underlying lattice $N_Y := \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Cir}(Q), \mathbb{Z})$. Each arrow $a \in Q_1$ determines a ray $\rho_a \in \Sigma_Q(1)$ where the unique generator of $\rho_a \cap N_Y$ is the evaluation map $\operatorname{ev}_a \colon \operatorname{Cir}(Q) \longrightarrow \mathbb{R}$ defined by $\operatorname{ev}_a(f) = f_a$ for $f \in \operatorname{Cir}(Q)$. The rays $\rho_{a_1}, \ldots, \rho_{a_\ell} \in \Sigma_Q(1)$ span a cone in Σ_Q if and only if there exists a spanning tree rooted at the source of Q that does not contain $a_1, \ldots, a_\ell$. Hence, maximal cones in Σ_Q correspond to spanning trees rooted at the source, and have dimension $|Q_1| - |Q_0| + 1$. Since Σ_Q is also the triangulation associated to the region in the chamber complex corresponding to our acyclic orientation of the underlying graph of Q (Theorem 3.1 in [7] relates triangulations with chambers; Lemma 7.1 in [14] defines the chamber complex

and region associated to an acyclic orientation), it follows that Σ_Q is a fan. Let Y_Q be the toric variety determined by Σ_Q.

The toric variety Y_Q, which is a toric quiver variety as defined in [17], has several other characterizations. Following [3], a toric variety Y is unimodular if M_Y is a unimodular sublattice of $\mathbb{Z}^{\Sigma_Y(1)}$; see (2.1). This is equivalent to saying that Y is smooth and any other variety obtained from Y by toric flips and flops is also smooth.

PROPOSITION 3.8. *Let Q be a connected, rooted, acyclic quiver. If $Q_0 = \{0, \ldots, r\}$ where 0 is the unique source then the following varieties coincide:*

(a) *the toric variety Y_Q defined by the fan Σ_Q;*

(b) *the geometric quotient of $\mathbb{A}^{Q_1} \setminus \mathbb{V}(B_Y)$ by the group $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Wt}(Q), \Bbbk^*)$, where $\mathbb{A}^{Q_1} = \mathrm{Spec}(S_Y)$, $S_Y := \Bbbk[y_a \colon a \in Q_1]$ and*

$$B_Y := \left(\prod_{a \in Q_1'} y_a \colon Q' \text{ is a spanning tree of } Q \text{ rooted at } 0 \right) = \bigcap_{i=1}^{r} (y_a \colon \mathrm{hd}(a) = i);$$

(c) *the fine moduli space $\mathcal{M}_\theta(Q)$ of θ-stable representations for any rational weight $\theta \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ lying in the open GIT-chamber containing $\vartheta := \sum_{i \in Q_0} (\chi_i - \chi_0)$.*

Moreover, this variety is unimodular and projective.

Proof. Let W be a ϑ-semistable representation of Q. If $W' \subset W$ is a proper nonzero subrepresentation, then we have $\vartheta(W') = \sum_{i \in \mathrm{supp}(W')} \vartheta_i \geq 0$. Since $\vartheta_i = 1$ for $i \neq 0$ and $\vartheta_0 = -r$, it follows that $W_0' = 0$ and $\vartheta(W') > 0$. Therefore, ϑ is generic, the open chamber containing ϑ is well-defined, and results in §4 of [21] show that $\mathcal{M}_\vartheta(Q)$ is a smooth projective variety. Since the map $\mathrm{inc} \colon \mathbb{Z}^{Q_1} \longrightarrow \mathrm{Wt}(Q)$ is totally unimodular (e.g. Proposition 5.3 in [6] or Example 2 in §19.3 of [26]), it follows that $\mathrm{Cir}(Q)$ is a unimodular sublattice of $\mathbb{Z}^{Q_1}$ and the toric variety defined by Σ_Q is unimodular. Theorem 2.1 in [9] establishes the equivalence between (a) and (b), and the discussion preceding Theorem 1.7 in [17] establishes the equivalence between (a) and (c). □

When Q is a connected, rooted, acyclic quiver of sections on X, the toric variety Y_Q is called the *multilinear series* of Q. When the quiver of sections is unambiguous, we simply write Y for the multilinear series. If Q is the complete quiver of sections for a list a of line bundles $\mathcal{L}$, then we write $|\mathcal{L}| := Y$ for the *complete multilinear series*.

Remark 3.9. Proposition 3.8 implies that the diagram (2.1) becomes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_Y & \longrightarrow & \mathrm{CDiv}(Y) & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Cir}(Q) & \longrightarrow & \mathbb{Z}^{Q_1} & \xrightarrow{\ \mathrm{inc}\ } & \mathrm{Wt}(Q) & \longrightarrow & 0,
\end{array}
$$

where the vertical maps are isomorphisms. The open chamber containing ϑ is the ample cone $\mathrm{Amp}_{\mathbb{Q}}(Y)$ of $\mathbb{Q}$-divisor classes on Y and the closure of this chamber is

$$\mathrm{Nef}_{\mathbb{Q}}(Y) = \bigcap_{Q' \subseteq Q} \left\{ \sum_{a \in Q'_1} \lambda_a[D_a]: \ \lambda_a \in \mathbb{Q}_{\geq 0} \right\},$$

where the intersection runs over all spanning trees Q' of Q rooted at 0 and D_a is the irreducible T_Y-invariant Weil divisor associated to $a \in Q_1$. Since Y is smooth, the ample line bundle $\mathscr{O}_Y(\vartheta)$ determined by $\vartheta \in \mathrm{Wt}(Q)$ is very ample.

Example 3.10. Let Q be a quiver with $Q_0 = \{0, 1\}$ and $Q_1 = \{a_0, \ldots, a_m\}$ such that $\mathrm{tl}(a_k) = 0$ and $\mathrm{hd}(a_k) = 1$ for all $0 \leq k \leq m$. Since every arrow forms a spanning tree rooted at 0, the irrelevant ideal of Y is $B_Y = (y_{a_0}, \ldots, y_{a_m})$. Hence, Y is the geometric quotient of $\mathbb{A}^{Q_1} \setminus \{0\}$ by $G := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Wt}(Q), \Bbbk^*)$. Choosing $\chi_1 - \chi_0$ as a basis for $\mathrm{Wt}(Q)$, we see that $G \cong \Bbbk^*$ and the G-action is induced by the matrix is $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$. Therefore, we have $Y \cong \mathbb{P}^m$. In particular, if Q is the complete quiver of sections for $\mathcal{L} = (\mathscr{O}_X, L_1)$ described in Example 3.2, then the complete multilinear series $|\mathcal{L}|$ is canonically isomorphic to the linear series $|L_1|$.

Example 3.11. Let $X = \mathbb{F}_1$ and $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1))$ as in Example 3.6. If we identify $\mathrm{Cir}(Q)$ with $\mathbb{Z}^2$ by choosing the circuits $(a_1 a_3^{-1}, a_3 a_2 a_4^{-1})$ as an ordered basis, then the unique generator of $\rho_k \cap N_Y$, where $\rho_k \in \Sigma_Q(1)$ corresponds to $a_k \in Q_1$, is the kth column of the matrix $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$. Figure 1 (c) gives $S_Y = \Bbbk[y_1, y_2, y_3, y_4]$ and $B_Y = (y_1, y_3) \cap (y_2, y_4)$. Hence, the quotient construction of Y from Proposition 3.8 (b) coincides with the quotient construction of X encoded by the pair (S_X, B_X); see §2.1. Therefore, the multilinear series Y equals X.

Example 3.12. Let $X = \mathbb{F}_2$ and $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1), \mathscr{O}_X(1,1))$ as in Example 3.7. If we identify $\mathrm{Cir}(Q)$ with $\mathbb{Z}^5$ by choosing the circuits

$$(a_1 a_2^{-1}, a_1 a_4 a_3^{-1}, a_1 a_5 a_3^{-1}, a_1 a_6 a_7^{-1} a_3^{-1}, a_1 a_6 a_8^{-1} a_3^{-1})$$

as an ordered basis, then the unique generator of $\rho_k \cap N_Y$, where $\rho_k \in \Sigma_Q(1)$ corresponds to $a_k \in Q_1$, is the kth column of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Figure 2 (c) implies that $S_Y = \Bbbk[y_1, \ldots, y_8]$ and $B_Y = (y_1, y_2) \cap (y_3, y_4, y_5) \cap (y_6, y_7, y_8)$. Hence, the multilinear series Y is a smooth 5-dimensional toric variety

with 8 irreducible T_Y-invariant Weil divisors and 18 T_Y-fixed points. The ample cone of Y is

$$\mathrm{Amp}_{\mathbb{Q}}(Y) = \{\theta = (\theta_0, \theta_1, \theta_2, \theta_3) \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} \colon \theta_1 > 0, \, \theta_2 > 0, \, \theta_3 > 0\}\,.$$

Since $(-3, -1, 1, 3) \notin \mathrm{Amp}_{\mathbb{Q}}(Y)$, the dualizing line bundle on Y is not ample.

4. Multilinear series. In this section, we study morphisms from the toric variety X to the multilinear series Y induced by a quiver of sections. To begin, we give necessary and sufficient conditions for a quiver of sections Q on X to define a morphism from X to the multilinear series Y. The divisors labelling the arrows in Q define a ring homomorphism $\Phi_Q \colon S_Y \longrightarrow S_X$ between the total coordinate rings of X and Y given by $\Phi_Q(y_a) = x^{\mathrm{div}\,(a)}$. The *base ideal* of Q is the ideal B_Q in S_X generated by the image $\Phi_Q(B_Y)$.

PROPOSITION 4.1. *Let Q be a connected, rooted, acyclic quiver of sections on X. If Y is the multilinear series of Q, then the following are equivalent:*
  (a) *the map Φ_Q determines a morphism $\varphi_Q \colon X \longrightarrow Y$;*
  (b) *the irrelevant ideal B_X is contained in the radical $\mathrm{rad}\,(B_Q)$;*
  (c) *for all $\sigma \in \Sigma_X$, there exists a spanning tree $Q' \subseteq Q$ rooted at the unique source in Q such that $\mathrm{supp}\,(\mathrm{div}\,(a)) \subseteq \hat{\sigma}$ for all $a \in Q'_1$.*

Proof. The toric variety X is a categorical quotient of $\mathbb{A}^{\Sigma_X(1)} \setminus \mathbb{V}(B_X)$ under the action of the group $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \Bbbk^*)$; see §2.1. Similarly, Proposition 3.8 shows that Y is the geometric quotient of $\mathbb{A}^{Q_1} \setminus \mathbb{V}(B_Y)$ under the action of the group G. The ring map $\Phi_Q \colon S_Y \longrightarrow S_X$ is induced by $\mathrm{div} \colon \mathbb{Z}^{Q_1} \longrightarrow \mathbb{Z}^{\Sigma_X(1)}$, so it defines a morphism $\mathbb{A}^{\Sigma_X(1)} \longrightarrow \mathbb{A}^{Q_1}$ which is equivariant with respect to the action of the dense tori on $\mathbb{A}^{\Sigma_X(1)}$ and $\mathbb{A}^{Q_1}$. Since combining (2.1) and (3.1) yields the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}^{Q_1} & \xrightarrow{\ \mathrm{inc}\ } & \mathrm{Wt}(Q) \\
\downarrow & & \downarrow {\scriptstyle [\,\mathrm{pic}\,]} \\
\mathbb{Z}^{\Sigma_X(1)} & \longrightarrow & \mathrm{Cl}(X),
\end{array}
$$

it follows that this morphism is also equivariant with respect to the actions of the groups $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \Bbbk^*)$ and $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Wt}(Q), \Bbbk^*)$ on $\mathbb{A}^{\Sigma_X(1)}$ and $\mathbb{A}^{Q_1}$ respectively. Thus, Φ_Q induces the morphism $\varphi_Q \colon X \longrightarrow Y$ if and only if the preimage of the irrelevant set $\mathbb{V}(B_Y)$ is contained in the irrelevant set $\mathbb{V}(B_X)$; see Theorem 3.2 in [10]. Since the preimage of $\mathbb{V}(B_Y)$ is cut out by the base ideal B_Q, this is equivalent to B_X being contained in $\mathrm{rad}\,(B_Q)$. In other words, conditions (a) and (b) are equivalent. The definition of B_X and the explicit description of B_Y from Proposition 3.8 (b) gives the equivalence between (b) and (c). □

A quiver of sections Q is *basepoint-free* if it is connected, rooted, acyclic, and satisfies any of the equivalent conditions in Proposition 4.1. If the complete quiver of sections for $\mathcal{L}$ is basepoint-free, then $\varphi_{|\mathcal{L}|}\colon X \longrightarrow |\mathcal{L}|$ denotes the associated morphism to the complete multilinear series.

COROLLARY 4.2. *If Q is a basepoint-free quiver of sections then each line bundle L_i on X is basepoint-free. Conversely, if each L_i is basepoint-free and Q is a complete quiver of sections for $\mathcal{L} = (\mathscr{O}_X, L_1, \ldots, L_r)$ then Q is basepoint-free.*

Proof. Since Q is basepoint-free, it satisfies condition (c) from Proposition 4.1. Hence, for $i \in Q_0$ and $\sigma \in \Sigma_X(d)$, there exists a path $p = a_1 \ldots a_\ell$ in Q such that $\mathrm{tl}\,(p) = 0$, $\mathrm{hd}\,(p) = i$, and $\mathrm{supp}\,(\mathrm{div}\,(a_k)) \subseteq \widehat{\sigma}$ for all $1 \leq k \leq \ell$. In other words, L_i admits a T_X-invariant section that does not vanish at the T_X-fixed point indexed by σ for all $\sigma \in \Sigma_X(d)$. Therefore, for each $i \in Q_0$, L_i is basepoint-free. When Q is complete, we can reverse the argument for the first part. $\square$

Given a basepoint-free quiver of sections Q, the image of φ_Q can be described explicitly. Let $\mathbb{N}(Q)$ be the image of $\mathbb{N}^{Q_1}$ under the map $\pi\colon \mathbb{Z}^{Q_1} \longrightarrow \mathrm{Wt}\,(Q) \oplus \mathrm{CDiv}\,(X)$. Observe that $\mathbb{N}(Q)$ is a subsemigroup of the section lattice $\mathbb{Z}(Q)$. Since S_Y is the semigroup algebra of $\mathbb{N}^{Q_1}$, the map π induces a surjective $\mathbb{k}$-algebra homomorphism from S_Y to $\mathbb{k}[\mathbb{N}(Q)]$ where $\mathbb{k}[\mathbb{N}(Q)]$ is the semigroup algebra of $\mathbb{N}(Q)$. The kernel of this induced map is the toric ideal

$$I_Q := (y^u - y^v \in S_Y \colon u, v \in \mathbb{N}^{Q_1}, u - v \in \mathrm{Ker}\,(\pi)).$$

Since $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Wt}\,(Q), \mathbb{k}^*)$, the affine toric variety $\mathbb{V}(I_Q) \subseteq \mathbb{A}^{Q_1}$ cut out by the ideal I_Q is G-invariant, however it need not be normal. The ideal I_Q is analogous to the toric ideal defined by the augmented vertex-edge incidence matrix of the McKay quiver in [11]. With this notation, we obtain the following:

PROPOSITION 4.3. *Let Q be a basepoint-free quiver of sections. If $\varphi_Q\colon X \longrightarrow Y$ is the induced morphism, then the image of φ_Q is:*

(a) *the subscheme of Y corresponding to the $\mathrm{Wt}\,(Q)$-graded B_Y-saturated ideal I_Q;*

(b) *the geometric quotient of $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Y)$ by $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Wt}\,(Q), \mathbb{k}^*)$;*

(c) *the GIT-quotient $\mathbb{V}(I_Q)/\!\!/_\theta\, G$ for any $\theta \in \mathrm{Amp}_{\mathbb{Q}}(Y)$.*

Proof. Proposition 3.8 (b) implies that the closed subsets of Y are in bijection with the G-invariant closed subsets of $\mathbb{A}^{Q_1} \setminus \mathbb{V}(B_Y)$, and hence with the B_Y-saturated ideals of S_Y. The image of the map from $\mathbb{A}^{\Sigma_X(1)}$ to $\mathbb{A}^{Q_1}$ induced by the homomorphism of semigroups $\mathrm{div}\colon \mathbb{N}^{Q_1} \longrightarrow \mathrm{CDiv}\,(X)$ is cut out by the toric ideal $\mathrm{Ker}\,(\Phi_Q)$. Since the action of $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Wt}\,(Q), \mathbb{k}^*)$ on $\mathbb{A}^{Q_1}$ is induced by the map $\mathrm{inc}\colon \mathbb{Z}^{Q_1} \longrightarrow \mathrm{Wt}\,(Q)$, the toric ideal I_Q associated to the map $\pi = (\mathrm{inc}, \mathrm{div})\colon \mathbb{N}^{Q_1} \longrightarrow \mathrm{Wt}\,(Q) \oplus \mathrm{CDiv}\,(X)$ is clearly $\mathrm{Wt}\,(Q)$-homogeneous and contained in $\mathrm{Ker}\,(\Phi_Q)$. Moreover, because I_Q is also maximal among $\mathrm{Wt}\,(Q)$-homogeneous ideals contained in $\mathrm{Ker}\,(\Phi_Q)$, it cuts out the $\mathrm{Wt}\,(Q)$-homogeneous

part of $\mathrm{Ker}\,(\Phi_Q)$. As I_Q is prime and hence B_Y-saturated, (a) and (b) follow. The equivalence of (b) and (c) follows directly from the equivalence between (b) and (c) in Proposition 3.8. $\square$

Remark 4.4. Proposition 4.3 (c) holds for rational weights in a cone that may strictly contain $\mathrm{Amp}_{\mathbb{Q}}\,(Y)$ because the GIT-chamber decomposition for the G-action on $\mathbb{V}(I_Q)$ is a coarsening of that for the G-action on $\mathbb{A}^{Q_1}$.

Proof of Theorem 1.1. The complete quiver of sections for $\mathcal{L} = (\mathscr{O}_X, L_1, \ldots, L_r)$ is basepoint-free by Corollary 4.2, so $\varphi_{|\mathcal{L}|}$ is a morphism. The explicit description of the image as a geometric quotient is presented in Proposition 4.3 (b). $\square$

Example 4.5. Corollary 4.2 shows that the complete quiver of sections for the list $\mathcal{L} = (\mathscr{O}_X, L_1)$ is basepoint-free if and only if L_1 is basepoint-free. Since the semigroup $\mathbb{N}(Q)$ is generated by $((-1,1), \mathrm{div}\,(s)) \in \mathbb{Z}^2 \oplus \mathrm{CDiv}\,(X)$ for T_X-invariant sections $s \in H^0(X, L_1)$, it is isomorphic to the semigroup generated by the effective divisors $\mathrm{div}\,(s)$ in $\mathrm{CDiv}\,(X)$. Hence, the ideal I_Q defines the image of X in the projective space $|L_1| := \mathbb{P}(H^0(X, L_1))$. It follows that $\mathbb{V}(I_Q)$ is the affine cone over $\varphi_{|L_1|}(X)$, and $\varphi_{|\mathcal{L}|} = \varphi_{|L_1|}$.

Example 4.6. Let $X = \mathbb{F}_1$ and $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1))$ as in Example 3.6. Example 3.11 shows that the multilinear series is $Y = |\mathcal{L}| = X$. By examining Figure 1 (b), we see that the complete quiver of sections for $\mathcal{L}$ satisfies condition (c) in Proposition 4.1. By definition, $\mathbb{N}(Q)$ is generated by the columns of the matrix in (3.2) so the toric ideal is $I_Q = (0) \subset S_Y$. Thus, $\varphi_{|\mathcal{L}|}$ is an isomorphism.

Example 4.7. Let $X = \mathbb{F}_2$ and $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1), \mathscr{O}_X(1,1))$ as in Example 3.7. The multilinear series $Y = |\mathcal{L}|$ is described in Example 3.12. By examining Figure 2 (b), we see that the complete quiver of sections for $\mathcal{L}$ satisfies condition (c) in Proposition 4.1. By definition, $\mathbb{N}(Q)$ is generated by the columns of the matrix in (3.3), so

$$I_Q = (y_2 y_4 - y_1 y_5, y_1 y_6 - y_3 y_7, y_2 y_6 - y_3 y_8, y_2 y_7 - y_1 y_8, y_5 y_7 - y_4 y_8) \subset S_Y.$$

In this case, we have $\varphi_{|\mathcal{L}|}(X) = \mathbb{V}(I_Q) /\!/_\theta\, G$ for all rational weights θ in the GIT-chamber

$$\Theta := \left\{ (\theta_0, \theta_1, \theta_2, \theta_3) \in \mathrm{Wt}\,(Q) \otimes_{\mathbb{Z}} \mathbb{Q} : \begin{array}{cc} \theta_3 > 0, & \theta_1 + \theta_3 > 0 \\ \theta_2 + \theta_3 > 0, & \theta_1 + \theta_2 + \theta_3 > 0 \end{array} \right\};$$

see Remark 4.4.

With additional hypotheses, we can enlarge the commutative diagram (3.1).

COROLLARY 4.8. *If Q is basepoint-free and $\dim \varphi_Q(X) = \dim X$, then*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_X & \longrightarrow & \mathbb{Z}(Q) & \xrightarrow{\pi_1} & \mathrm{Wt}\,(Q) & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \mathrm{pic}} & & \\
0 & \longrightarrow & M_X & \longrightarrow & \mathrm{CDiv}\,(X) & \longrightarrow & \mathrm{Pic}\,(X) & \longrightarrow & 0
\end{array}
$$

is a commutative diagram with exact rows. In particular, the projection π_2 induces an isomorphism between $\mathrm{Ker}\,(\pi_1)$ and M_X.

Proof. Combining (3.1) with the top row of (2.1), it is enough to prove that π_2 yields an isomorphism between $\mathrm{Ker}\,(\pi_1)$ and M_X. The morphism $\varphi_Q \colon X \longrightarrow \varphi_Q(X)$ corresponds to the map of semigroup algebras $S_Y/I_Q = \Bbbk[\mathbb{N}(Q)] \longrightarrow \Bbbk[\mathbb{N}^{\Sigma_X(1)}] = S_X$ induced by π_2. Since $\dim \varphi_Q(X) = \dim X$, it identifies the dense tori in X and $\varphi_Q(X)$. Therefore, π_2 identifies the character lattices M_X and $\mathrm{Ker}\,(\pi_1)$. □

Next, we give a criterion for $\varphi_Q \colon X \longrightarrow Y$ to be a closed embedding. For $\sigma \in \Sigma_X(d)$, let $y^{\widehat{\sigma}} := \prod_{\mathrm{supp}\,(\mathrm{div}\,(a)) \subseteq \widehat{\sigma}} y_a$ be the associated monomial in S_Y. The localization of an S_Y-module F at the element $y^{-\widehat{\sigma}}$ is denoted by $F[y^{-\widehat{\sigma}}]$. The weight $\vartheta := \sum_{i \in Q_0} (\chi_i - \chi_0)$ appearing below is defined in Proposition 3.8 (c).

PROPOSITION 4.9. *Let Q be a basepoint-free quiver of sections on X. The map $\varphi_Q \colon X \longrightarrow Y$ is a closed embedding if and only if the line bundle*

$$
L := L_0^{\vartheta_0} \otimes \cdots \otimes L_r^{\vartheta_r} = \bigotimes_{i \in Q_0} L_i
$$

is ample and $((S_Y/I_Q)[y^{-\widehat{\sigma}}])_{[0]} \cong (S_X[x^{-\widehat{\sigma}}])_{[0]}$ for all $\sigma \in \Sigma_X(d)$.

Proof. The very ample line bundle $\mathscr{O}_Y(\vartheta)$ from Remark 3.9 provides a closed embedding $Y \to \mathbb{P}^m := \mathbb{P}(H^0(Y, \mathscr{O}_Y(\vartheta)))$. Hence, $\varphi_Q \colon X \to Y$ is a closed embedding if and only if the composition $X \to \mathbb{P}^m$, determined by $\varphi_Q^*(\mathscr{O}_Y(\vartheta)) = \mathrm{pic}\,(\vartheta) = L$ and the subspace $(S_Y/I_Q)_\vartheta \cong \Phi_Q((S_Y)_\vartheta) \subseteq (S_X)_{[L]} \cong H^0(X, L)$ of global sections, is a closed embedding. The morphism $\varphi_Q \colon X \to Y$ corresponds to the map of semigroup algebras $S_Y/I_Q = \Bbbk[\mathbb{N}(Q)] \to \Bbbk[\mathbb{N}^{\Sigma_X(1)}] = S_X$ induced by $\pi_2 \colon \mathbb{Z}(Q) \to \mathrm{CDiv}\,(X)$ which implies that $(S_Y/I_Q)_\vartheta \cong \Phi_Q((S_Y)_\vartheta)$. Moreover, the map π_2 identifies the monomial basis of $(S_Y/I_Q)_\vartheta$ indexed by the set $\mathcal{V} := \mathbb{N}(Q) \cap \pi_1^{-1}(\vartheta)$ with a subset of lattice points in the polytope associated to L. Since Q is basepoint-free, Proposition 4.1 (c) implies that for each cone $\sigma \in \Sigma_X(d)$, there exists a monomial $y^{\widehat{\sigma}} \in (S_Y)_\vartheta$ such that $x^{v_\sigma} := \Phi_Q(y^{\widehat{\sigma}})$ satisfies $\mathrm{supp}\,(x^{v_\sigma}) \subseteq \widehat{\sigma}$. By Theorem 2.13 in [25], we deduce that $X \to \mathbb{P}^m$ is a closed embedding if and only if $v_\sigma \neq v_\tau$ holds for each pair $\sigma \neq \tau \in \Sigma_X(d)$, and the semigroup $M_X \cap \sigma^\vee$ is generated by $\mathcal{V} - v_\sigma$ for all $\sigma \in \Sigma_X(d)$. Corollary 2.14 in

[25] proves that the first condition is equivalent to L being ample. Lemma 2.2 in [9] shows that the semigroup algebra of $M_X \cap \sigma^\vee$ is isomorphic to the coordinate ring $(S_X[x^{-\widehat{\sigma}}])_{[0]}$ of the affine chart on X determined by $v_\sigma \in \mathcal{V}$. Similarly, the semigroup algebra of $\mathcal{V} - v_\sigma$ is isomorphic to the coordinate ring $((S_Y/I_Q)[y^{-\widehat{\sigma}}])_{[0]}$ of the affine chart on $\mathbb{V}(I_Q) /\!\!/_\vartheta G$ determined by $v_\sigma \in \mathcal{V}$. Therefore, the second condition is equivalent to $((S_Y/I_Q)[y^{-\widehat{\sigma}}])_{[0]} \cong (S_X[x^{-\widehat{\sigma}}])_{[0]}$ for all $\sigma \in \Sigma_X(d)$. $\square$

A quiver of sections Q is *very ample* if it is complete, basepoint-free, and $\varphi_Q \colon X \longrightarrow Y$ is a closed embedding. For convenience, we record an instance of Proposition 4.9.

COROLLARY 4.10. *Let $\mathcal{L} = (\mathscr{O}_X, L_1, \ldots, L_r)$ be a list of basepoint-free line bundles and set $L := \bigotimes_{i \in Q_0} L_i$. Assume that the map*

$$H^0(X, L_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} H^0(X, L_r) \longrightarrow H^0(X, L)$$

is surjective. The morphism $\varphi_{|\mathcal{L}|} \colon X \longrightarrow Y$ is a closed embedding if and only if L is very ample.

Proof. Since each L_i is basepoint-free, Corollary 4.2 implies that the complete quiver of sections Q for $\mathcal{L}$ is basepoint-free. From the proof of Proposition 4.9, we know that $\varphi_Q \colon X \longrightarrow Y$ is a closed embedding if and only if the map to projective space, determined by the line bundle L and the subspace $(S_Y/I_Q)_\vartheta \subseteq H^0(X, L)$, is a closed embedding. For $1 \leq i \leq r$, identify the set $\mathcal{V}_i := \mathbb{N}(Q) \cap \pi_1^{-1}(\chi_i - \chi_0)$ with a monomial basis of $(S_Y/I_Q)_{\chi_i - \chi_0}$. The map $\pi_2 \colon \mathbb{N}(Q) \longrightarrow \mathbb{N}^{\Sigma_X(1)}$ identifies $\mathcal{V}_i$ with the lattice points in the polytope associated to L_i, so

$$(S_Y/I_Q)_{\chi_i - \chi_0} \cong (S_X)_{[L_i]} \cong H^0(X, L_i).$$

Since $L_0 = \mathscr{O}_X$ and $\vartheta = \sum_{i=1}^r (\chi_i - \chi_0)$, we may identify the image of the map

$$H^0(X, L_0^{\vartheta_0}) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} H^0(X, L_r^{\vartheta_r}) = H^0(X, L_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} H^0(X, L_r) \longrightarrow H^0(X, L)$$

with a subspace of $(S_Y/I_Q)_\vartheta \cong H^0(X, L)$. Surjectivity implies that this subspace equals $H^0(X, L)$. Lastly, we observe that the complete linear series $|L|$ determines a closed embedding if and only L is very ample. $\square$

Example 4.11. Corollary 4.10 shows that the complete quiver of sections for the list $\mathcal{L} = (\mathscr{O}_X, L_1)$ is very ample if and only the line bundle L_1 is very ample.

Example 4.12. Let $X = \mathbb{F}_2$ and $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1))$ as in Example 3.6. Since $L_1 \otimes L_2 = \mathscr{O}_X(1,1)$ is very ample and the map

$$H^0(X, L_1) \otimes_{\Bbbk} H^0(X, L_2) \longrightarrow H^0(X, \mathscr{O}_X(2,2))$$

is surjective, the complete quiver of sections for $\mathcal{L}$ is very ample by Corollary 4.10.

Example 4.13. Let $X = \mathbb{F}_2$ and $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1), \mathscr{O}_X(1,1))$ as in Example 3.7. Since the line bundle $L_1 \otimes L_2 \otimes L_3 = \mathscr{O}_X(2,2)$ is very ample on X and the map $H^0(X, L_1) \otimes_\Bbbk H^0(X, L_2) \otimes_\Bbbk H^0(X, L_3) \longrightarrow H^0(X, \mathscr{O}_X(2,2))$ is surjective, Corollary 4.10 implies that the complete quiver of sections for $\mathcal{L}$ is very ample.

To see that every list of basepoint-free line bundles belongs to some very ample quiver of sections, we prove:

PROPOSITION 4.14. *Let $L_1, \ldots, L_{r-1}$ be basepoint-free line bundles on X. If the subsemigroup of $\mathrm{Pic}\,(X)$ generate by $L_1, \ldots, L_{r-1}$ contains an ample line bundle, then there exists a line bundle L_r such that the complete quiver of sections for $\mathcal{L} = (\mathscr{O}_X, L_1, \ldots, L_r)$ is very ample.*

Proof. By choosing $b_1, \ldots, b_{r-1} \in \mathbb{N}$ sufficiently large, we may assume that the line bundle $L_r := L_1^{b_1} \otimes \cdots \otimes L_{r-1}^{b_{r-1}}$ is very ample and $\mathscr{O}_X$-regular with respect to $L_1, \ldots, L_{r-1}$; for the multigraded definition of regularity see [23], [16]. Let Q be the complete quiver of sections for $\mathcal{L} = (\mathscr{O}_X, L_1, \ldots, L_r)$. Since L_r is very ample and $L_0, \ldots, L_{r-1}$ are basepoint-free, it follows that $L := \bigotimes_{i \in Q_0} L_i$ is very ample. Since Theorem 2.1 in [16] proves that $H^0(X, L_0) \otimes_\Bbbk \cdots \otimes_\Bbbk H^0(X, L_r) \longrightarrow H^0(X, L)$ is surjective, Corollary 4.10 completes the proof. □

THEOREM 4.15. *If Q is a very ample quiver of sections, then we can recover the line bundles $L_0, \ldots, L_r$ as the restriction of the tautological line bundles on $Y = |\mathcal{L}|$.*

Proof. If we identify $\mathrm{Wt}\,(Q)$ with $\mathbb{Z}^r$ by choosing the weights $(\chi_1 - \chi_0, \ldots, \chi_r - \chi_0)$ as an ordered basis, then the projection map $\mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}^r$ induces an isomorphism between G and the subgroup $\{(g_0, \ldots, g_r) \in (\Bbbk^*)^{r+1} \colon g_0 = 1\}$ of $(\Bbbk^*)^{Q_0}$. This isomorphism determines a G-equivariant vector bundle $\bigoplus_{i \in Q_0} \mathscr{O}_{\mathbb{A}^{Q_1}}$; specifically, the ith component corresponds to the S_Y-module $S_Y(\chi_i - \chi_0)$, where we have $(S_Y(\theta'))_\theta := (S_Y)_{\theta' + \theta}$. If follows that the tautological line bundles on Y are $\mathscr{O}_Y, \mathscr{O}_Y(\chi_1 - \chi_0), \ldots, \mathscr{O}_Y(\chi_r - \chi_0)$. Restricting to $\mathbb{V}(I_Q)$, we obtain a G-equivariant vector bundle $\bigoplus_{i \in Q_0} \mathscr{O}_{\mathbb{V}(I_Q)}$ where the ith component corresponds to $(S_Y/I_Q)(\chi_i - \chi_0)$. Since $\varphi_Q \colon X \longrightarrow Y$ is a closed embedding, Proposition 4.9 implies that $((S_Y/I_Q)[y^{-\widehat{\sigma}}])_{[0]} \cong (S_X[x^{-\widehat{\sigma}}])_{[0]}$ for all $\sigma \in \Sigma_X(d)$. Hence, the S_Y-module $(S_Y/I_Q)(\chi_i - \chi_0)$ corresponds to $\mathrm{pic}\,(\chi_i - \chi_0) = L_i$ on $X \cong \varphi_Q(X)$. □

Remark 4.16. If we identify $\mathrm{Wt}\,(Q)$ with $\mathbb{Z}^r$ by choosing $(\chi_0 - \chi_1, \ldots, \chi_0 - \chi_r)$ as an ordered basis then the restriction of the tautological line bundles would yield the inverse line bundles $\mathscr{O}_X, L_1^{-1}, \ldots, L_r^{-1}$.

5. Representations of bound quivers. This section connects the ideal of relations on a quiver of sections to the geometry of its multilinear series. Throughout this section, let (Q, R) be a complete, bound quiver of sections for $\mathcal{L} =$

$(\mathscr{O}_X, L_1, \ldots, L_r)$ of line bundles on X and let $Y = |\mathcal{L}|$ be the multilinear of Q.

The ideal of relations defines an algebraic subset of $\mathbb{A}^{Q_1}$. More precisely, the map sending the path $p = a_1 \cdots a_\ell$ in Q to the monomial $y_{a_1} \cdots y_{a_\ell} \in S_Y$ extends to give a $\Bbbk$-linear map from $\Bbbk Q$ into S_Y. Let I_R be the ideal in S_Y generated by the image of R under this map; it is a binomial ideal because R is spanned by differences $p - p' \in \Bbbk Q$. Since these differences satisfy $\mathrm{tl}(p) = \mathrm{tl}(p')$, $\mathrm{hd}(p) = \mathrm{hd}(p')$, and $\mathrm{div}(p) = \mathrm{div}(p')$, I_R is homogeneous with respect to the $\mathrm{Wt}(Q)$-grading on S_Y and is contained in I_Q.

The following examples illustrate various possible relations between I_R and I_Q.

Example 5.1. Let $X = \mathbb{P}^1$ and let $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(2))$; Example 3.10 describes $|\mathcal{L}|$. There are no paths of length greater than 1 in Q, so R and I_R are both the zero ideal. Since I_Q is the toric ideal associated to the matrix

$$\begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix},$$

we have $I_Q = (y_0 y_2 - y_1^2)$. Thus, $\mathbb{V}(I_Q)$ is a closed subvariety of $\mathbb{V}(I_R) = \mathbb{A}^3 = \mathbb{A}^{Q_1}$.

Example 5.2. Let $X = \mathbb{F}_1$ and $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1))$. Since Example 3.6 shows $R = (0)$ and Example 4.7 shows $I_Q = (0)$, it follows that $I_R = I_Q = (0)$; in other words, $\mathbb{V}(I_Q) = \mathbb{V}(I_R) = \mathbb{A}^4 = \mathbb{A}^{Q_1}$.

Example 5.3. Let $X = \mathbb{F}_2$ and $\mathcal{L} = (\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1), \mathscr{O}_X(1,1))$. Using the description of R in Example 3.7, we see that

$$\begin{aligned} I_R &= (y_2 y_4 - y_1 y_5, y_4 y_8 - y_5 y_7, y_2 y_6 - y_3 y_8, y_1 y_6 - y_3 y_7) \\ &= (y_2 y_4 - y_1 y_5, y_1 y_6 - y_3 y_7, y_2 y_6 - y_3 y_8, y_2 y_7 - y_1 y_8, y_5 y_7 - y_4 y_8) \\ &\quad \cap (y_3, y_4, y_5, y_6). \end{aligned}$$

The description of I_Q given in Example 4.7 implies that I_Q is a primary component of I_R. Geometrically, $\mathbb{V}(I_Q)$ is the unique component of $\mathbb{V}(I_R)$ not lying in a linear subspace.

Let $W = (W_i, w_a)$ be a representation of Q. For any nontrivial path $p = a_1 \cdots a_\ell$, the evaluation of W on p is the $\Bbbk$-linear map $w_p \colon W_{\mathrm{tl}(p)} \longrightarrow W_{\mathrm{hd}(p)}$ defined by the composition $w_p = w_{a_1} \cdots w_{a_\ell}$. This definition extends to $\Bbbk$-linear combinations of paths with a common head and a common tail. A representation of the bound quiver (Q, R) is a representation W of Q such that $w_p = w_{p'}$ for all $p - p' \in R$. Consequently, a point in the representation space $\mathbb{A}^{Q_1}$ for Q corresponds to a representation for (Q, R) if and only it lies in the subscheme $\mathbb{V}(I_R)$. By

Theorem III.1.6 in [2], the category of representations of (Q, R) with dimension vector $\sum_{i \in Q_0} \chi_i \in \mathbb{N}^{Q_0}$ is equivalent to the category of $(\Bbbk Q/R)$-modules that are isomorphic as $(\bigoplus_{i \in Q_0} \Bbbk e_i)$-modules to $\bigoplus_{i \in Q_0} \Bbbk e_i$.

The ideal I_R is homogeneous with respect to the $\mathrm{Wt}\,(Q)$-grading on S_Y, so the subscheme $\mathbb{V}(I_R)$ is G-invariant where $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Wt}\,(Q), \Bbbk^*)$. The GIT-chamber decomposition of $\mathrm{Wt}\,(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ arising from the G-action on $\mathbb{V}(I_R)$ coarsens that for the G-action on $\mathbb{A}^{Q_1}$; see Remark 4.4. Let Θ denote the GIT-chamber arising from the G-action on $\mathbb{V}(I_R)$ containing $\vartheta = \sum_{i \in Q_0}(\chi_i - \chi_0)$. Proposition 5.3 in [21] shows that, for $\theta \in \Theta$, the GIT-quotient

$$\mathcal{M}_\theta(Q, R) := \mathbb{V}(I_R) /\!\!/_\theta G = \mathrm{Proj}\left(\bigoplus_{k \in \mathbb{N}} (\tfrac{S_Y}{I_R})_{k\theta}\right)$$

is the fine moduli space for θ-stable representations of (Q, R). Equivalently, if $\bigoplus_{i \in Q_0} \Bbbk e_i$ denotes the subalgebra of $\mathrm{End}\,(\bigoplus_{i \in Q_0} L_i)$ generated by the primitive orthogonal idempotents e_i for $i \in Q_0$, then Proposition 3.3 implies that $\mathcal{M}_\theta(Q, R)$ is the fine moduli space of θ-stable $\mathrm{End}\,(\bigoplus_{i \in Q_0} L_i)$-modules that are isomorphic as $(\bigoplus_{i \in Q_0} \Bbbk e_i)$-modules to $\bigoplus_{i \in Q_0} \Bbbk e_i$.

THEOREM 5.4. *If Q is a very ample quiver of sections, then the following are equivalent:*

(a) *the ideal I_Q equals ideal quotient $(I_R : B_Y^\infty)$;*
(b) *for all $\theta \in \Theta$, the map φ_Q induces an isomorphism from X to $\mathcal{M}_\theta(Q, R)$.*

Proof. The equivalence of (b) and (c) in Proposition 3.8 implies that the moduli space $\mathcal{M}_\theta(Q, R) = \mathbb{V}(I_R) /\!\!/_\theta G$ is the geometric quotient of $\mathbb{V}(I_R) \setminus \mathbb{V}(B_Y)$ by the group G. Since Q is a very ample quiver of sections, Proposition 4.9 proves that the map φ_Q induces an isomorphism from X to $\varphi_Q(X)$ and Proposition 4.3 establishes that $\varphi_Q(X)$ is the geometric quotient of $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Y)$ by G. Since I_Q is prime, the locally closed subscheme $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Y)$ equals $\mathbb{V}(I_R) \setminus \mathbb{V}(B_Y)$ if and only if we have $I_Q = I_R : B_Y^\infty$. □

A list $\mathcal{L} = (\mathscr{O}_X, L_1, \ldots, L_r)$ of line bundles on X is *fine* if the complete bound quiver of sections for $\mathcal{L}$ is very ample and satisfies either of the equivalent conditions in Theorem 5.4. The next result shows that every projective toric variety has many fine lists.

THEOREM 5.5. *Let $L_1, \ldots, L_{r-2}$ be basepoint-free line bundles on X. If the subsemigroup of $\mathrm{Pic}\,(X)$ generated by $L_1, \ldots, L_{r-2}$ contains an ample line bundle, then there exist line bundles L_{r-1} and L_r such that the list $\mathcal{L} = (\mathscr{O}_X, L_1, \ldots, L_r)$ is fine.*

Proof. We divide the proof into three parts.

(1) *Choosing the line bundles L_{r-1} and L_r.* This part is similar to Proposition 4.9. By choosing sufficiently large positive integers $b_1, \ldots, b_{r-2}$, we may assume that the line bundle $L_{r-1} := L_1^{b_1} \otimes \cdots \otimes L_{r-2}^{b_{r-2}}$ is $\mathscr{O}_X$-regular with respect to $L_1, \ldots, L_{r-2}$. Set $L_r := L_{r-1}^2$. By increasing the b_i if necessary, we may

also assume that L_r is very ample. Let Q be the complete quiver of sections for $\mathcal{L} = (\mathcal{O}_X, L_1, \ldots, L_r)$. Since L_r is very ample and $L_1, \ldots, L_{r-1}$ are basepoint-free, $L := \bigotimes_{i \in Q_0} L_i$ is very ample. Since Theorem 2.1 in [16] implies that $H^0(X, L_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} H^0(X, L_r) \longrightarrow H^0(X, L)$ is surjective, Corollary 4.10 implies that Q is very ample.

(2) Proof that $I_Q = (I_R : (\prod_{a \in Q_1} y_a)^\infty)$. By definition, I_Q is the toric ideal associated to the map $\pi : \mathbb{N}^{Q_1} \longrightarrow \mathrm{Wt}(Q) \oplus \mathrm{CDiv}(X)$. It suffices by Lemma 12.2 in [27] to construct a subset $\mathcal{C}$ that generates the kernel of π as an abelian group and satisfies $(y^{v_+} - y^{v_-} : v_+ - v_- = v \in \mathcal{C}) \subseteq I_R$. Since L_{r-1} is $\mathcal{O}_X$-regular and the b_i are positive, Theorem 2.1 in [16] shows that

$$H^0(X, L_{r-1} \otimes L_i^{-1}) \otimes_{\Bbbk} H^0(X, L_{r-1}) \longrightarrow H^0(X, L_r \otimes L_i^{-1})$$

is surjective for all $1 \leq i \leq r - 1$. Hence, every path in Q from 0 to r passes through $r - 1 \in Q_0$. Moreover, because Q is complete, the set $\mathcal{A}$ of the arrows from $r - 1$ to r in Q corresponds to the set of nonzero T_X-invariant elements in $H^0(X, L_{r-1})$. Since the set $\mathcal{P}$ of paths from 0 to $r-1$ in Q are labelled by nonzero T_X-invariant elements in $H^0(X, L_{r-1})$, there is a surjective function $\Psi : \mathcal{P} \to \mathcal{A}$ such that $\mathrm{div}(p) = \mathrm{div}(\Psi(p))$ for all $p \in \mathcal{P}$. For $(a, a', p, p') \in \mathcal{A}^2 \times \mathcal{P}^2$, we have $\mathrm{div}(p) + \mathrm{div}(a) = \mathrm{div}(p') + \mathrm{div}(a)$ if and only if $pa - p'a' \in R$; set

$$\mathcal{C} := \{f(p) + f(a) - f(p') - f(a') : pa - p'a' \in R\} \subseteq \mathrm{Cir}(Q),$$

where $f(\gamma)$ is the element in $\mathbb{Z}^{Q_1}$ associated to a walk γ in Q defined in §2.2.

To analyze $\mathcal{C}$, we use an "elongation" operation on circuits in Q. Because Q is acyclic, we may assume every circuit $\gamma = \alpha_1 \alpha_2^{-1} \alpha_3 \cdots \alpha_{2\ell-1} \alpha_{2\ell}^{-1}$ is an alternating sequence of forward paths $\alpha_1, \alpha_3, \ldots, \alpha_{2\ell-1}$ and backward paths $\alpha_2^{-1}, \alpha_4^{-1}, \ldots, \alpha_{2\ell}^{-1}$. Since Q is connected, there exists at least one path from the unique source 0 to each $i \in Q_0$. Similarly, the choice of L_r implies that there is at least one path from each $i \in Q_0$ to r. Let $\widehat{\gamma}$ denote a closed walk obtained from the circuit γ via the following procedure: for $i = 1, 3, \ldots, 2\ell - 1$, choose a path β_i from $\mathrm{hd}(\alpha_i)$ to r; for $i = 2, 4, \ldots, 2\ell$, choose a path β_i from 0 to $\mathrm{tl}(\alpha_i)$; let $\beta_0 = \beta_{2\ell}$ and set

$$\widehat{\gamma} := \beta_0 \alpha_1 \beta_1 \beta_1^{-1} \alpha_2^{-1} \beta_2^{-1} \beta_2 \alpha_3 \beta_3 \beta_3^{-1} \cdots \alpha_{2\ell}^{-1} \beta_{2\ell}^{-1}.$$

Observe that $\widehat{\gamma} = \widehat{p}_1 \widehat{p}_2^{-1} \widehat{p}_3 \cdots \widehat{p}_{2\ell-1} \widehat{p}_{2\ell}^{-1}$ is an alternating sequence of forward and backward paths between 0 and r, where $\widehat{p}_i = \beta_{i-1} \alpha_i \beta_i$ for odd i, $\widehat{p}_i^{-1} = \beta_{i-1}^{-1} \alpha_i^{-1} \beta_i^{-1}$ for even i. Thus, we have $\mathrm{div}(\gamma) = \mathrm{div}(\widehat{\gamma}) \in \mathrm{CDiv}(X)$ and

$$f(\gamma) = f(\widehat{\gamma}) = \sum_{i=0}^{\ell} (f(\widehat{p}_{2i-1})) - \sum_{i=0}^{\ell} (f(\widehat{p}_{2i})) \in \mathrm{Cir}(Q),$$

where $f(\widehat{p}_i) \in \mathbb{N}^{Q_1}$.

To see that $\mathcal{C}$ spans the lattice $\mathrm{Ker}\,(\pi)$, fix $u \in \mathrm{Ker}\,(\pi)$. Since $\pi = (\mathrm{inc}, \mathrm{div})$, the exact sequence (2.2) implies that $u \in \mathrm{Cir}\,(Q)$. Theorem 5.2 in [6] shows that the circulation lattice is generated by the circuits, so $u = \sum_i f(\gamma_i)$ where γ_i is a circuit in Q. To establish that $u \in \mathcal{C}$, it is enough to show that $f(\gamma) \in \mathcal{C}$ for any circuit γ in Q. By using the elongation operation and regrouping the sum, we have $f(\gamma) = \sum_i (f(\widehat{p}_{2i-1}) - f(\widehat{p}_{2i}))$ where each $\widehat{p}_i$ is a path from 0 to r. Since each path in Q from 0 to r passes through $r-1$, it follows that $\widehat{p}_i = p_i a_i$ where $p_i \in \mathcal{P}$ and $a_i \in \mathcal{A}$. Hence, we have $f(\gamma) = \sum_i (f(p_{2i-1}) + f(a_{2i-1}) - f(p_{2i}) - f(a_{2i}))$. We decompose this expression into a sum of elements in $\mathcal{C}$ by exploiting properties of the sets $\mathcal{A}$ and $\mathcal{P}$ arising from our choice of L_{r-1}. To be specific, let I' be the toric ideal associated to the map $\mathbb{N}^{\mathcal{A}} \longrightarrow \mathrm{CDiv}\,(X)$ given by $\chi_a \mapsto \mathrm{div}\,(a)$. The identification of $\mathcal{A}$ with the nonzero T_X-invariant elements in $H^0(X, L_{r-1})$ implies that I' is the ideal of $\varphi_{|L_{r-1}|}(X)$. Since L_{r-1} is $\mathscr{O}_X$-regular, Theorem 1.1 in [16] establishes that I' is generated by quadrics. Hence, if $\Bbbk[z_a \colon a \in \mathcal{A}] = \Bbbk[\mathbb{N}^{\mathcal{A}}]$ and

$$\mathcal{R} := \{(a_0, a_1, a_2, a_3) \in \mathcal{A}^4 \colon \mathrm{div}\,(a_0) + \mathrm{div}\,(a_1) = \mathrm{div}\,(a_2) + \mathrm{div}\,(a_3)\},$$

then we have $I' = (z_{a_0} z_{a_1} - z_{a_2} z_{a_3} \colon (a_0, a_1, a_2, a_3) \in \mathcal{R})$. For $0 \le i \le \ell$, we define $c_i := f(p_{2i-1}) + f(\Psi(p_{2i})) - f(p_{2i}) - f(\Psi(p_{2i-1}))$. Since $\mathrm{div}\,(p_i) = \mathrm{div}\,(\Psi(p_i))$ for $1 \le i \le \ell$, each c_i belongs to $\mathcal{C}$ and we have

$$f(\gamma) = \sum_{i=0}^{\ell} (f(\Psi(p_{2i-1})) + f(a_{2i-1}) - f(\Psi(p_{2i})) - f(a_{2i})) + \sum_{i=0}^{\ell} c_i.$$

Given $v \in \mathbb{Z}^{\mathcal{A}}$ satisfying $\mathrm{div}\,(v) = 0$, Theorem 5.3 in [27] applied to the generators of the toric ideal I' yields

$$v = \sum_{(a_0, a_1, a_2, a_3) \in \mathcal{R}'} (\chi_{a_0} + \chi_{a_1} - \chi_{a_2} - \chi_{a_3})$$

$$= \sum_{(a_0, a_1, a_2, a_3) \in \mathcal{R}'} (f(a_0) + f(a_1) - f(a_2) - f(a_3)),$$

where $\mathcal{R}'$ is a multiset of elements from $\mathcal{R}$. Taking this decomposition of the element $v := \sum_{i=0}^{\ell} (f(\Psi(p_{2i-1})) + f(a_{2i-1}) - f(\Psi(p_{2i})) - f(a_{2i})) \in \mathbb{Z}^{\mathcal{A}}$ that satisfies $\mathrm{div}\,(v) = 0$, then substituting into the right-hand side of the expression for $f(\gamma)$ above gives

$$f(\gamma) = \sum_{(a_0, a_1, a_2, a_3) \in \mathcal{R}'} (f(a_0) + f(a_1) - f(a_2) - f(a_3)) + \sum_i c_i,$$

where $\mathcal{R}'$ is some multiset of elements from $\mathcal{R}$. For any $\alpha := (a_0, a_1, a_2, a_3) \in \mathcal{R}'$, set $c'_\alpha := f(\widetilde{a}_0) + f(a_2) - f(\widetilde{a}_2) - f(a_0)$ where $\widetilde{a}_j \in \mathcal{P}$ satisfies $\Psi(\widetilde{a}_j) = a_j$ for $j = 0, 2$.

Hence, we obtain

$$f(\gamma) = \sum_{(a_0, a_1, a_2, a_3) \in \mathcal{R}'} (f(\tilde{a}_0) + f(a_1) - f(\tilde{a}_2) - f(a_3)) - \sum_\alpha c'_\alpha + \sum_i c_i,$$

where $\mathcal{R}'$ is a multiset of elements from $\mathcal{R}$. Since each summand belongs to $\mathcal{C}$, we see that $f(\gamma)$ and hence u lies in the lattice spanned by $\mathcal{C}$ and $I_Q = (I_R\colon (\prod_{a \in Q_1} y_a)^\infty)$.

(3) Proof that $(I_R\colon B_Y^\infty) = (I_R\colon (\prod_{a \in Q_1} y_a)^\infty)$. For $u \in \mathbb{Z}^{Q_1}$, set $y^u := \prod_{a \in Q_1} y_a^{u_a}$. For any subset $Q'_1 \subseteq Q_1$, let $f(Q'_1) := \sum_{a \in Q'_1} \chi_a$ so that $y^{f(Q'_1)} := \prod_{a \in Q'_1} y_a$. The definition of B_Y given in Proposition 3.8 (b) implies that

$$(I_R\colon B_Y^\infty) = \bigcap_{Q'} (I_R : (y^{f(Q'_1)})^\infty),$$

where the intersection is over all spanning tree $Q' \subseteq Q$ rooted at 0. Because $(I_R\colon (y^{f(Q'_1)})^\infty)$ is a subset of $(I_R\colon (\prod_{a \in Q_1} y_a)^\infty)$, it is enough to show that each spanning tree $Q' \subseteq Q$ rooted at 0 satisfies $(I_R\colon (y^{f(Q'_1)})^\infty) = (I_R\colon (\prod_{a \in Q_1} y_a)^\infty)$.

The technique of proof follows Example 2.3 in [5]. By increasing the b_i from part 1 if necessary, we may assume that there exists $s \in H^0(X, L_{r-1})$ such that the corresponding lattice point in the polytope P associated L_{r-1} lies in the interior. Fix a spanning tree Q' rooted at 0. Let a_s be the unique arrow $a_s \in \mathcal{A} \cap Q'_1$ and let p_s be a path in Q satisfying $\operatorname{div}(p_s) = \operatorname{div}(s) = \operatorname{div}(a_s)$. Observe that y_{a_s} is invertible in $S_Y[y^{-f(Q'_1)}]$. For any path $p \in \mathcal{P}$, we have $p a_s - p_s \Psi(p) \in R$, so $y^{f(p)} - y^{f(p_s)} y_{\Psi(p)} y_{a_s}^{-1}$ belongs to $I_R S_Y[y^{-f(Q'_1)}]$. Hence, for any $p, p' \in \mathcal{P}$ and $a, a' \in \mathcal{A}$,

$$\begin{aligned}
y^{f(p)} y_a - y^{f(p')} y_{a'} &= y^{f(p_s)} y_{\Psi(p)} y_a y_{a_s}^{-1} - y^{f(p_s)} y_{\Psi(p')} y_{a'} y_{a_s}^{-1} \\
&= y^{f(p_s)} y_{a_s}^{-1} (y_{\Psi(p)} y_a - y_{\Psi(p')} y_{a'})
\end{aligned}$$

in $S_Y[y^{-f(Q'_1)}]/I_R S_Y[y^{-f(Q'_1)}]$. Thus, if

$$J := I_R + (y_{a_0} y_{a_1} - y_{a_2} y_{a_3} : (a_0, a_1, a_2, a_4) \in \mathcal{R}),$$

then we have $J S_Y[y^{-f(Q'_1)}] = I_R S_Y[y^{-f(Q'_1)}]$.

By assumption, L_{r-1} is ample, so the vertices of the polytope P are lattice points. Since the lattice point corresponding to s lies in the interior of P, we can express s in the form $\frac{1}{k} \sum_j c_j s_j$ where k, c_j are positive integers and the s_j correspond to the vertices of P. It follows that $z_{a_s}^k - \prod_j z_{a_{s_j}}^{c_j}$ lies in the toric ideal

$$I' = (z_{a_0} z_{a_1} - z_{a_2} z_{a_3} : (a_0, a_1, a_2, a_3) \in \mathcal{R})$$

introduced in part 2. Changing variables from z's to y's implies $y_{a_s}^k - \prod_j y_{a_{s_j}}^{c_j} \in J$. Since y_{a_s} is invertible in $S_Y[y^{-f(Q_1')}]$, it now follows that all the $y_{a_{s_j}}$ are invertible in the quotient ring $S_Y[y^{-f(Q_1')}]/I_R S_Y[y^{-f(Q_1')}]$. Moreover, we see that y_a is invertible in $S_Y[y^{-f(Q_1')}]/I_R S_Y[y^{-f(Q_1')}]$ for all $a \in \mathcal{A}$ because the lattice point of P corresponding to $a \in \mathcal{A}$ is a positive rational combination of some vertices and each variable $y_{a_{s_j}}$ that corresponds to a vertex is invertible in $S_Y[y^{-f(Q_1')}]/I_R S_Y[y^{-f(Q_1')}]$.

If $p' \in \mathcal{P}$ is the unique path supported on the set Q_1', then $y^{f(p')}$ is invertible in $S_Y[y^{-f(Q_1')}]/I_R S_Y[y^{-f(Q_1')}]$. Let $a' \in \mathcal{A}$ is the unique arrow satisfying $\operatorname{div}(p') = \operatorname{div}(a')$. The previous paragraph shows that $y_{a'}$ in invertible in the ring $S_Y[y^{-f(Q_1')}]/I_R S_Y[y^{-f(Q_1')}]$. For any path $p \in \mathcal{P}$, we have $pa' - p'\Psi(p) \in R$ and the identity $y^{f(p)} = y^{f(p')} y_{\Psi(p)} y_{a'}^{-1}$ in $I_R S_Y[y^{-f(Q_1')}]$. Each monomial on the right side of this identity is invertible in $S_Y[y^{-f(Q_1')}]/I_R S_Y[y^{-f(Q_1')}]$ which implies that every variable that divides $y^{f(p)}$ is also invertible. Since the path $p \in \mathcal{P}$ was arbitrary, we conclude that, for all $a \in Q_1$, y_a is invertible in $S_Y[y^{-f(Q_1')}]/I_R S_Y[y^{-f(Q_1')}]$. Therefore, Corollary 2.6 in [5] implies that $(I_R : (y^{f(Q_1')})^\infty) = (I_R : (\prod_{a \in Q_1} y_a)^\infty)$.

Since part (1) shows that Q is very ample, and combining parts (2) and (3) proves that Q satisfies condition (a) in Theorem 5.4, we conclude that $\mathcal{L}$ is fine. $\qquad\square$

Proof of Theorem 1.2. By combining Theorem 5.5 and Theorem 5.4, it follows that there are many list $\mathcal{L} = (\mathcal{O}_X, L_1, \ldots, L_r)$ of line bundles on X such that the induced morphism $\varphi_{|\mathcal{L}|} : X \longrightarrow |\mathcal{L}|$ identifies X with the fine moduli space $\mathcal{M}_\vartheta(Q, R)$. Theorem 4.15 implies that the tautological bundles on $\mathcal{M}_\vartheta(Q, R)$ coincide with the line bundles $\mathcal{O}_X, L_1, \ldots, L_r$. $\qquad\square$

Remark 5.6. A priori, I_Q depends on the divisors labelling the arrows in Q. However, if $\mathcal{L}$ is fine then I_Q depends only on I_R and hence only the bound quiver (Q, R).

Example 5.7. For $X = \mathbb{P}^1$, Example 5.1 shows that $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(2))$ is not fine. To obtain a fine list, we add an appropriate line bundle: $\mathcal{L}' := (\mathcal{O}_X, \mathcal{O}_X(2), \mathcal{O}_X(4))$. The complete quiver of sections for $\mathcal{L}'$ appears in Figure 3. Corollary 4.10 implies that $|\mathcal{L}'|$ is very ample. Since

$$I_Q = (y_2^2 - y_1 y_3, y_5^2 - y_4 y_6, y_3 y_5 - y_2 y_6, y_2 y_5 - y_1 y_6, y_3 y_4 - y_1 y_6, y_2 y_4 - y_1 y_5)$$
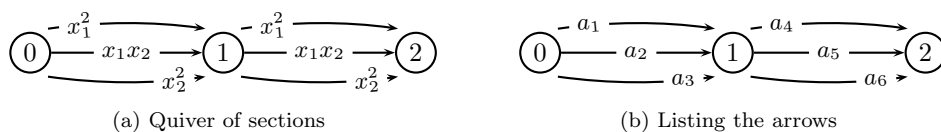

(a) Quiver of sections   (b) Listing the arrows

Figure 3. A fine collection on $\mathbb{P}^1_{\Bbbk}$.
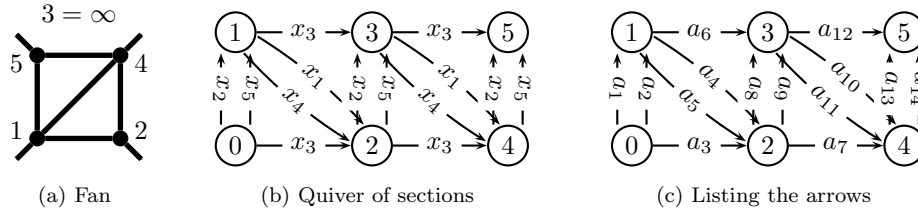
(a) Fan   (b) Quiver of sections   (c) Listing the arrows

Figure 4. Projective threefold admitting a flop.

and

$$I_R = (y_3y_5 - y_2y_6, y_3y_4 - y_2y_5, y_2y_5 - y_1y_6, y_2y_4 - y_1y_5)$$

$$= I_Q \cap (y_1, y_2, y_3) \cap (y_4, y_5, y_6),$$

it follows that $\mathcal{L}$ is fine.

Example 5.8. Let X be the smooth toric threefold determined by the following fan Σ_X in $\mathbb{R}^3$: the rays $\Sigma_X(1)$ are generated by the vectors $v_1 := (1, 0, 0)$, $v_2 := (0, 1, 0)$, $v_3 := (-1, -1, -1)$, $v_4 := (0, 1, 1)$, $v_5 := (1, 0, 1)$ and the minimal nonfaces correspond to $\{v_1, v_3, v_4\}$ and $\{v_2, v_5\}$. The induced triangulation of the 2-sphere is given in Figure 4 (a). There is a flop $X \dashrightarrow X'$ where the toric variety X' is the determined by the triangulation of $\Sigma_X(1)$ with minimal nonfaces $\{v_1, v_4\}$ and $\{v_2, v_3, v_5\}$. For $(k, \ell) \in \mathbb{Z}^2$, write $\mathcal{O}_X(k, \ell) := \mathcal{O}_X(kD_3 + \ell D_2) \in \text{Pic}(X)$. The complete quiver of sections for $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(0, 1), \mathcal{O}_X(1, 0), \mathcal{O}_X(1, 1), \mathcal{O}_X(2, 0), \mathcal{O}_X(2, 1))$ appears in Figure 4 (b).

Since we have

$$I_Q = \begin{pmatrix} y_9y_{13} - y_8y_{14}, y_2y_{13} - y_1y_{14}, y_9y_{12} - y_7y_{14}, y_8y_{12} - y_7y_{13}, \\ y_5y_{10} - y_4y_{11}, y_2y_8 - y_1y_9, y_5y_7 - y_6y_{11}, y_4y_7 - y_6y_{10}, \\ y_2y_6 - y_3y_9, y_1y_6 - y_3y_8, y_2y_5y_{12} - y_3y_{11}y_{14}, \\ y_1y_5y_{12} - y_3y_{11}y_{13}, y_2y_4y_{12} - y_3y_{10}y_{14}, y_1y_4y_{12} - y_3y_{10}y_{13} \end{pmatrix}$$

$$I_R = \begin{pmatrix} y_9y_{12} - y_7y_{14}, y_8y_{12} - y_7y_{13}, y_5y_7 - y_6y_{11}, y_4y_7 - y_6y_{10}, y_2y_6 - y_3y_9, \\ y_1y_6 - y_3y_8, y_9y_{11}y_{13} - y_8y_{11}y_{14}, y_9y_{10}y_{13} - y_8y_{10}y_{14}, \\ y_5y_9y_{10} - y_4y_9y_{11}, y_5y_8y_{10} - y_4y_8y_{11}, y_2y_5y_8 - y_1y_5y_9, y_2y_4y_8 - y_1y_4y_9 \end{pmatrix}$$

$$= I_Q \cap (y_{12}, y_{11}, y_{10}, y_7, y_2y_8 - y_1y_9, y_2y_6 - y_3y_9, y_1y_6 - y_3y_8)$$

$$\cap (y_6, y_5, y_4, y_3, y_9y_{13} - y_8y_{14}, y_9y_{12} - y_7y_{14}, y_8y_{12} - y_7y_{13})$$

$$\cap (y_{12}, y_{11}, y_{10}, y_7, y_6, y_5, y_4, y_3) \cap (y_9, y_8, y_7, y_6),$$

it follows that $\mathcal{L}$ fine. Observe that $\mathbb{V}(I_Q)$ has four components contained in coordinate hyperplanes of $\mathbb{A}^{Q_1} = \mathbb{A}^{14}$ and the points in these components correspond to representations of disconnected subquivers of Q.

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