TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 11, November 2018, Pages 7715–7741
http://dx.doi.org/10.1090/tran/7201
Article electronically published on May 30, 2018

TORIC VECTOR BUNDLES AND PARLIAMENTS OF POLYTOPES

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Abstract. We introduce a collection of convex polytopes associated to a torus-equivariant vector bundle on a smooth complete toric variety. We show that the lattice points in these polytopes correspond to generators for the space of global sections and relate edges to jets. Using the polytopes, we also exhibit vector bundles that are ample but not globally generated, and vector bundles that are ample and globally generated but not very ample.

1. Overview of results

The importance and prevalence of toric varieties stems from their calculability and their close relation to polyhedral objects. The challenge is to emulate this success and enlarge the class of varieties with both features. Rather than contemplating spherical varieties or all T-varieties, we extend the theory of toric varieties by studying torus-equivariant vector bundles and their projective bundles. Motivated by the ensuing simplifications in the toric dictionary between line bundles and polyhedra, we concentrate on vector bundles over a smooth complete toric variety. The goal of this paper is to give explicit polyhedral interpretations for properties of these vector bundles.

To accomplish this goal, we fix a smooth complete toric variety X, over $\mathbb{C}$, associated to the fan Σ. Let M denote the character lattice of the dense torus in X, and write $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ for the unique minimal generators of the rays in Σ. A *toric vector bundle* on X is a torus-equivariant locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank r. The celebrated Klyachko classification proves that $\mathscr{E}$ corresponds to a finite-dimensional vector space $E \cong \mathbb{C}^r$ equipped with compatible decreasing filtrations $E \supseteq \cdots \supseteq E^i(j) \supseteq E^i(j+1) \supseteq \cdots \supseteq 0$, where $1 \leqslant i \leqslant n$ and $j \in \mathbb{Z}$; see section 2. This collection of linear subspaces embeds into the lattice of flats for a distinguished matroid $\mathrm{M}(\mathscr{E})$. For each element $\mathbf{e}$ in the ground set of the matroid $\mathrm{M}(\mathscr{E})$, we introduce the convex polytope

$$P_{\mathbf{e}} := \left\{ \mathbf{u} \in M \otimes_{\mathbb{Z}} \mathbb{R} : \langle \mathbf{u}, \mathbf{v}_i \rangle \leqslant \max\bigl(j \in \mathbb{Z} : \mathbf{e} \in E^i(j)\bigr) \text{ for } 1 \leqslant i \leqslant n \right\}.$$

The set of all such polytopes $P_{\mathbf{e}}$ is called the *parliament of polytopes* for $\mathscr{E}$; see section 3. Although the defining half-spaces for the polytopes $P_{\mathbf{e}}$ together with

Received by the editors February 12, 2016, and, in revised form, January 25, 2017, and February 1, 2017.

2010 *Mathematics Subject Classification.* Primary 14M25; Secondary 14J60, 51M20.

Key words and phrases. Toric vector bundle, polytopes, positivity.

The first author was partially supported by the Vetenskapsrådet (VR) grants NT:2010-5563 and NT:2014-4736.

The second author was partially supported by the Göran Gustafsson Stiftelse.

The third author was partially supported by NSERC.

the elements $\mathbf{e}$ in the ground set of $\mathrm{M}(\mathscr{E})$ encode the filtrations, the polytopes themselves may be empty; see Remark 3.6.

The following result gives the first substantive connection between the parliament of polytopes and the toric vector bundle.

Proposition 1.1. *The lattice points in the polytopes of the parliament for $\mathscr{E}$ correspond to the torus-equivariant generators for the space of global sections of $\mathscr{E}$.*

Example 3.5 recovers the polytope associated to a toric line bundle on X. However, when the rank of $\mathscr{E}$ is greater than 1, Example 3.8 demonstrates that the lattice points in the polytopes of the parliament need not yield a basis for the space of global sections. This highlights the key difference between higher-rank toric vector bundles and toric line bundles: toric vector bundles depend on both the combinatorics of the polytopes $P_{\mathbf{e}}$ and the properties of the elements $\mathbf{e}$ in the ground set of the matroid $\mathrm{M}(\mathscr{E})$. For line bundles we may overlook the elements indexing the polytope because linear algebra in a one-dimensional vector space is trivial. Our criterion for deciding whether a toric vector bundle is globally generated underscores this distinction.

To outline this criterion, consider a maximal cone $\sigma \in \Sigma$. The restriction of the toric vector bundle $\mathscr{E}$ to the affine open toric variety U_σ splits equivariantly as a direct sum of toric line bundles. Since toric line bundles on U_σ correspond to lattice points in M, we obtain a multiset $\boldsymbol{u}(\sigma) \subset M$ of associated characters for each maximal cone $\sigma \in \Sigma$; see section 2. With this notation, we have our second result.

Theorem 1.2. *A toric vector bundle is globally generated if and only if, for all maximal cones $\sigma \in \Sigma$, the associated characters in $\boldsymbol{u}(\sigma)$ are vertices of polytopes in the parliament and the elements indexing these polytopes form a basis in the matroid* $\mathrm{M}(\mathscr{E})$.

Example 4.4 demonstrates that global generation is not simply a property of the individual polytopes in the parliament, and Example 5.5 shows that the higher-cohomology groups of a globally generated ample toric vector bundle may be nonzero.

The parliament of polytopes for $\mathscr{E}$ gives new insights into the projective bundle $\mathbb{P}(\mathscr{E})$. This is particularly relevant for the positivity properties of $\mathscr{E}$ defined by the corresponding attribute for the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. For instance, we may picture the restriction of $\mathscr{E}$ to a torus-invariant curve in X as the normalized distances between appropriately matched characters associated to $\mathscr{E}$; see section 4. Hence, Theorem 2.1 in [HMP] allows us to quickly recognize ample and nef toric vector bundles. Exploiting our polyhedral interpretations, Example 5.3 exhibits a toric vector bundle $\mathscr{F}$ on $\mathbb{P}^2$ that is ample but not globally generated, and Example 6.4 exhibits a toric vector bundle $\mathscr{H}$ on $\mathbb{P}^2$ that is ample and globally generated but not very ample. Better still, Proposition 5.4 and Remark 6.8 prove that $\mathscr{F}$ and $\mathscr{H}$ have the minimal rank among all toric vector bundles on $\mathbb{P}^d$ with the given traits. Beyond answering Question 7.5 in [HMP], these examples reinforce the conventional wisdom that versions of positivity that coincide for line bundles diverge for higher-rank vector bundles.

The discrete geometry within the parliament of polytopes nevertheless captures the positivity of jets. In contrast with the conventional wisdom, several forms of higher-order positivity are equivalent for toric vector bundles. A vector bundle

$\mathscr{E}$ separates ℓ-jets for $\ell \in \mathbb{N}$ if, for every closed point $x \in X$ with maximal ideal $\mathfrak{m}_x \subseteq \mathscr{O}_X$, the natural map $H^0(X, \mathscr{E}) \to H^0(X, \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathfrak{m}_x^{\ell+1})$ is surjective; see section 6. As an enhancement of Theorem 1.2, Theorem 6.2 establishes that a toric vector bundle $\mathscr{E}$ separates ℓ-jets if and only if certain edges in the polytopes of the parliament have normalized length at least ℓ. This leads to the following equivalences.

Theorem 1.3. *A toric vector bundle $\mathscr{E}$ separates ℓ-jets if and only if it is ℓ-jet ample. Moreover, a toric vector bundle $\mathscr{E}$ separates* 1*-jets if and only if it is very ample.*

Unlike arbitrary vector bundles on a smooth projective variety, these versions of positivity coincide for toric vector bundles. Specializing to line bundles, we recover the main theorems in [DR]. We also obtain a polyhedral characterization for very ampleness; see Corollary 6.7.

Future directions. The introduction of the parliament of polytopes for a toric vector bundle suggests some new research projects. The most straightforward advances would provide polyhedral interpretations for other properties of toric vector bundles. For example, we suspect that a toric vector bundle is big if and only if some Minkowski sum of the polytopes in the parliament is full dimensional. For a globally generated toric vector bundle $\mathscr{E}$, the complete linear series of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ maps the projective bundle $\mathbb{P}(\mathscr{E})$ into projective space. Can one characterize the homogeneous equations of the image in terms of combinatorial commutative algebra? If so, then one expects a description of the initial ideals via regular triangulations; compare with [S, Section 8]. Since there exists ample, but not globally generated, line bundles on varieties of the form $\mathbb{P}(\mathscr{E})$, this class of varieties makes an interesting testing ground for Fujita's conjecture; see Conjecture 10.4.1 in [L2]. More ambitiously, for an ample toric vector bundle $\mathscr{E}$, one could even ask for an effective polyhedral bound on $m \in \mathbb{N}$ such that $\operatorname{Sym}^m(\mathscr{E})$ is globally generated or very ample. Finally, we wonder if there are natural topological hypotheses on the parliament of polytopes which imply that all of the higher-cohomology groups vanish.

Conventions. Throughout the document, $\mathbb{N}$ denotes the nonnegative integers and X is a smooth complete toric variety over the complex numbers $\mathbb{C}$. The linear subspace generated by the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m$ in a $\mathbb{C}$-vector space is denoted by $\operatorname{span}(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m)$, and the polyhedral cone generated by the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ in an $\mathbb{R}$-vector space is denoted by $\operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$.

2. Background on toric vector bundles

In this section we collect the needed definitions and notation for toric varieties and vector bundles.

Let X be a smooth complete d-dimensional toric variety, over the complex numbers $\mathbb{C}$, determined by the strongly convex rational polyhedral fan Σ in $N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$, where N is a lattice of rank d. The dual lattice is $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, and the dense algebraic torus acting on X is $T := \operatorname{Spec} \mathbb{C}[M]$. For $\sigma \in \Sigma$, the corresponding affine toric variety is $U_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$, where $\sigma^\vee$ denotes the dual cone. The j-dimensional cones of Σ form the set $\Sigma(j)$. For each maximal cone $\sigma \in \Sigma(d)$, the corresponding T-fixed point is $x_\sigma \in X$. We order the one-dimensional cones $\Sigma(1)$ (also known as rays) and, for $1 \leqslant i \leqslant n$, we write $\mathbf{v}_i \in N$ for the unique minimal generator of the ith ray. The ith ray also corresponds to the irreducible T-invariant divisor D_i on X, and the divisors $D_1, D_2, \dots, D_n$ generate the group $\operatorname{Div}_T(X) \cong \mathbb{Z}^n$ of T-invariant divisors. Since X is complete, there is a short exact sequence

$$0 \longrightarrow M \xrightarrow{\operatorname{div}} \operatorname{Div}_T(X) \longrightarrow \operatorname{Pic}(X) \longrightarrow 0,$$

where $\operatorname{div} \mathbf{u} := \langle \mathbf{u}, \mathbf{v}_1 \rangle D_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle D_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle D_n$ and the second map is the projection from the group of divisors to the Picard group. The invertible sheaf or line bundle associated to a divisor $D \in \operatorname{Div}_T(X)$ is denoted by $\mathscr{O}_X(D)$. For more information on toric varieties, see either [CLS] or [F].

A *toric vector bundle* is a locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank r equipped with a T-action that is compatible with the T-action on the toric variety X. In other words, there exists a T-action on the variety $\mathrm{V}(\mathscr{E}) := \operatorname{Spec}(\operatorname{Sym} \mathscr{E})$ such that the projection map $\pi \colon \mathrm{V}(\mathscr{E}) \to X$ is T-equivariant and T acts linearly on the fibers. For all $\sigma \in \Sigma$, there is also an induced T-action on the $\mathbb{C}$-vector spaces of sections $H^0(U_\sigma, \mathscr{E})$, where U_σ is the corresponding affine toric variety. Given a lattice point $\mathbf{u} \in M$, the trivial line bundle $\mathscr{O}_X(\operatorname{div} \mathbf{u})$ has a canonical T-equivariant structure. Explicitly, for all $\sigma \in \Sigma$, we have

$$H^0\big(U_\sigma, \mathscr{O}_X(\operatorname{div} \mathbf{u})\big) = \bigoplus_{\mathbf{u}' \in \sigma^\vee \cap M} \mathbb{C} \cdot \chi^{\mathbf{u}' - \mathbf{u}} \subset T\,,$$

where $\chi^{\mathbf{u}'}, \chi^{\mathbf{u}}$ are the characters associated to the lattice points $\mathbf{u}', \mathbf{u} \in M$; the identity in this semigroup is $\chi^{-\mathbf{u}}$. As in [HMP], we follow the standard convention in invariant theory for the action of the group on the ring of functions, even though the opposite sign convention is more common in the toric literature.

Every toric line bundle on the affine toric variety U_σ is equivariantly isomorphic to $\mathscr{O}_X(\operatorname{div} \mathbf{u})|_{U_\sigma}$, where the class $\overline{\mathbf{u}}$ of the lattice point $\mathbf{u}$ in $M_\sigma := M/(\sigma^\perp \cap M)$ is uniquely determined. In addition, any toric vector bundle on an affine toric variety splits T-equivariantly as a direct sum of toric line bundles whose underlying line bundles are trivial; see Proposition 2.2 in [P1]. Hence, for all $\sigma \in \Sigma$, there is a unique multiset $\boldsymbol{u}(\sigma) \subset M_\sigma$ such that $\mathscr{E}|_{U_\sigma} \cong \bigoplus_{\overline{\mathbf{u}} \in \boldsymbol{u}(\sigma)} \mathscr{O}_X(\operatorname{div} \mathbf{u})|_{U_\sigma}$, where $\mathbf{u} \in M$ is any lift of $\overline{\mathbf{u}}$. If σ is a maximal cone, then the multiset $\boldsymbol{u}(\sigma) \subset M$ is uniquely determined by the toric vector bundle $\mathscr{E}$ and the d-dimensional cone σ. We call the multisets $\boldsymbol{u}(\sigma)$, for all $\sigma \in \Sigma(d)$, the *associated characters* of the toric vector bundle $\mathscr{E}$.

Toric vector bundles are classified in Theorem 0.1.1 of [Kl1] by canonical filtrations. To summarize this classification, let E be the fiber of $\mathscr{E}$ over the identity of the torus T, so E is a $\mathbb{C}$-vector space isomorphic to $\mathbb{C}^r$. The category of toric vector bundles on X is naturally equivalent to the category of finite-dimensional $\mathbb{C}$-vector spaces E with separated exhaustive decreasing filtrations $\{E^i(j)\}_{j \in \mathbb{Z}}$, for all $1 \leqslant i \leqslant n$, that satisfy the following compatibility condition.

(CC) for each $\sigma \in \Sigma(d)$, there exists a decomposition $E = \bigoplus_{\mathbf{u} \in \boldsymbol{u}(\sigma)} L_{\mathbf{u}}$ such that $E^i(j) = \sum_{\langle \mathbf{u}, \mathbf{v}_i \rangle \geqslant j} L_{\mathbf{u}}$.

This compatibility condition is equivalent to the T-equivariant splitting into a direct sum of toric line bundles on the affine open toric variety U_σ, for all $\sigma \in \Sigma(d)$; see Theorem 1.3.2 in [Kl2]. Indirectly, the decreasing filtrations provide the gluing data needed to assemble these direct sums into a toric vector bundle. The filtrations being separated and exhaustive, for each $1 \leqslant i \leqslant n$, means that $E^i(j) = 0$ for all $j \gg 0$ and $E^i(j) = E \cong \mathbb{C}^r$ for all $j \ll 0$, so each filtration contains only finitely many distinct linear subspaces. Hence, for a fixed i, we may conveniently describe the filtration $\{E^i(j)\}_{j\in\mathbb{Z}}$ via a labeled basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r$ for $E \cong \mathbb{C}^r$, where each vector $\mathbf{e}_k \in E$ is labeled by an integer and the linear subspace $E^i(j)$ is simply the span of the basis vectors with labels greater than or equal to j. For a self-contained exposition of this classification, we recommend [P1, Subsection 2.3]; [P1, Subsection 2.4] also provides a brief historical summary.

Given a toric vector bundle $\mathscr{E}$, the filtrations $\{E^i(j)\}_{j\in\mathbb{Z}}$ have a couple of different geometric interpretations. For all cones $\sigma \in \Sigma$ and all lattice points $\mathbf{u} \in M$, evaluating sections at the identity of the torus T gives an injective map $H^0(U_\sigma, \mathscr{E})_{\mathbf{u}} \hookrightarrow E$. The image of this map is the linear subspace $E^\sigma_{\mathbf{u}} \subseteq E$. Following [P2, Subsection 4.2], we define a linear subspace $E^{\mathbf{v}}(j) \subseteq E$ for all $\mathbf{v} \in N$ and all $j \in \mathbb{Z}$. Since X is complete, there exists a unique cone $\sigma \in \Sigma$ containing the lattice point $\mathbf{v}$ in its relative interior. Set $E^{\mathbf{v}}(j) := \sum_{\langle \mathbf{u},\mathbf{v}\rangle \geqslant j} E^\sigma_{\mathbf{u}}$. For any lattice point $\mathbf{v} \in N$, the family of linear subspaces $\{E^{\mathbf{v}}(j)\}_{j\in\mathbb{Z}}$ give a separated exhaustive decreasing filtration of E. When the lattice point $\mathbf{v}$ equals $\mathbf{v}_i$ for some $1 \leqslant i \leqslant n$, we obtain the filtration $\{E^i(j)\}_{j\in\mathbb{Z}}$.

For the second interpretation of the filtrations, consider a cone $\sigma \in \Sigma$ and suppose that $\mathscr{E}|_{U_\sigma} \cong \bigoplus_{\overline{\mathbf{u}}\in \boldsymbol{u}(\sigma)} \mathscr{O}_X(\operatorname{div}\mathbf{u})|_{U_\sigma}$. If the linear subspace $L_{\mathbf{u}} \subseteq E$ is the fiber of $\mathscr{O}_X(\operatorname{div}\mathbf{u})$ over the identity of the torus T, then we obtain a decomposition $E = \bigoplus_{\overline{\mathbf{u}}\in\boldsymbol{u}(\sigma)} L_{\mathbf{u}}$. Hence, the linear subspace $E^\sigma_{\mathbf{u}'}$ is spanned by the linear subspaces $L_{\mathbf{u}}$ for which $\mathbf{u} - \mathbf{u}' \in \sigma^\vee$ and $E^{\mathbf{v}}(j) = \bigoplus_{\langle\mathbf{u},\mathbf{v}\rangle\geqslant j} L_{\mathbf{u}}$. For each maximal cone $\sigma \in \Sigma$, there exists a subset $\underline{\boldsymbol{u}}(\sigma) \subset M$ and a decomposition $E = \bigoplus_{\mathbf{u}\in\underline{\boldsymbol{u}}(\sigma)} E_{\mathbf{u}}$ such that, for all $\mathbf{v} \in \sigma$ and for all $j \in \mathbb{Z}$, we have $E^{\mathbf{v}}(j) = \bigoplus_{\langle\mathbf{u},\mathbf{v}\rangle\geqslant j} E_{\mathbf{u}}$. It follows that $E_{\mathbf{u}} = \bigoplus_{\mathbf{u}\in\boldsymbol{u}(\sigma)} L_{\mathbf{u}}$, so $\dim E_{\mathbf{u}}$ equals the multiplicity of $\mathbf{u}$ in the multiset $\boldsymbol{u}(\sigma)$ and $\underline{\boldsymbol{u}}(\sigma)$ is the underlying set of $\boldsymbol{u}(\sigma)$.

3. Global sections and lattice polytopes

This section introduces explicit T-equivariant generators for the global sections of the toric vector bundle that correspond to the lattice points in a collection of polytopes. Each toric line bundle $\mathscr{L}$ on X corresponds to a rational convex polytope in $M \otimes_{\mathbb{Z}} \mathbb{R}$. We generalize this correspondence by associating a finite collection of convex polytopes to a toric vector bundle $\mathscr{E}$. The polytopes in this collection are indexed by the elements in the ground set of a matroid associated to $\mathscr{E}$.

To describe this matroid, we first observe that the vector bundle $\mathscr{E}$ determines the finite poset $\mathrm{L}(\mathscr{E})$, consisting of all the linear subspaces $V := \bigcap_{i=1}^n E^i(j_i) \subseteq E$, where $(j_1, j_2, \dots, j_n) \in \mathbb{Z}^n$, ordered by inclusion. Since the filtrations $\{E^i(j)\}_{j\in\mathbb{Z}}$ are separated, exhaustive, and decreasing, we see that $0 \in \mathrm{L}(\mathscr{E})$, $E \in \mathrm{L}(\mathscr{E})$, and $\mathrm{L}(\mathscr{E})$ is closed under intersection. Hence, the pair $\big(\mathrm{L}(\mathscr{E}), \cap\big)$ forms a meet-semilattice. The next result shows that $\mathrm{L}(\mathscr{E})$ embeds into the lattice of flats for a distinguished representable matroid.

Proposition 3.1. *For a toric vector bundle $\mathscr{E}$, there exists a unique matroid $\mathrm{M}(\mathscr{E})$, representable over $\mathbb{C}$, such that*

(M1) *the poset $\mathrm{L}(\mathscr{E})$ is isomorphic to a meet-subsemilattice in the lattice of flats;*
(M2) *among all matroids satisfying* (M1)*, the number of elements in the ground set is minimal; and*

(M3) *among all matroids satisfying* (M1) *and* (M2), *the number of circuits is minimal.*

In the language of linear subspace arrangements (and ordering the subspaces by reversed inclusion), Proposition 3.1 is equivalent to Theorem I.4.9 in [Z].

Algorithm 3.2 (Construction of matroid associated to $\mathscr{E}$)**.**

Input: The poset $\mathrm{L}(\mathscr{E})$ of linear subspaces associated to the toric vector bundle $\mathscr{E}$.

Output: The canonical matroid $\mathrm{M}(\mathscr{E})$ associated to $\mathscr{E}$.

Set r to be the dimension of the largest linear subspace E in $\mathrm{L}(\mathscr{E})$;
Initialize G to be a set consisting of a basis vector for each one-dimensional subspace in $\mathrm{L}(\mathscr{E})$;
For each integer k from 2 to r do
  For each k-dimensional linear subspace V in $\mathrm{L}(\mathscr{E})$ do
    Set G' to be the subset of elements in G that lie in V;
    If the linear subspace $\operatorname{span}(G')$ is a proper subspace in V then
      Append to G a basis for a complementary subspace to $\operatorname{span}(G')$ in V;
Return the linear matroid defined by the vectors in G.

Proof. We verify that Algorithm 3.2 returns a representable matroid M with the desired conditions. By construction, each linear subspace V in $\mathrm{L}(\mathscr{E})$ is generated by a subset of vectors in the ground set of the matroid M. The subset of the ground set consisting of all elements contained in V is the flat F_V in M corresponding to V. It follows that $\operatorname{span}(F_V) = V$, $\operatorname{rank}(F_V) = \dim(V)$, and the induced injective map from the poset $\mathrm{L}(\mathscr{E})$ into lattice of flats for M is compatible with intersections. Thus, the matroid M satisfies the condition (M1).

For any matroid, the lattice of flats is relatively complemented; see Proposition 3.4.4 in [W]. It follows that, for any linear subspace V in $\mathrm{L}(\mathscr{E})$ and any matroid satisfying condition (M1), there exists a flat F' such that the join of F_V and F' is F_E and the meet of F_V and F' is $F_{\{0\}}$. By iterating from the smallest to the largest linear subspaces in $\mathrm{L}(\mathscr{E})$, Algorithm 3.2 finds a minimal set of complementary subspaces for $\mathrm{L}(\mathscr{E})$. Adjoining these to $\mathrm{L}(\mathscr{E})$, we obtain a new meet-semilattice L' such that the complementary subspaces are minimal among the nonzero subspaces, and every linear subspace is generated by some collection of minimal nonzero subspaces. Using the terminology from [W, Section 3.4], we see that the atoms in L' are the one-dimensional linear subspaces in $\mathrm{L}(\mathscr{E})$ together with the adjoined complementary subspaces. Moreover, L' is the minimal atomistic meet-semilattice containing $\mathrm{L}(\mathscr{E})$.

Finally, we claim that the matroid M is the free expansion of L'; see Proposition 10.2.3 in [W]. By construction, the ground set of M consists of a basis for each atom in L', so the number of elements in the ground set of M equals the number of elements in the ground set of the free expansion of L'. Moreover, the conditional statement in Algorithm 3.2 implies that a flat D in the matroid M is dependent if and only if there exists a linear subspace $W \in L'$ such that $|D \cap F_W| > \dim(W)$. We conclude that M is the free expansion of L'. Therefore, Proposition 10.2.2 and

Proposition 10.2.6 in [W] establish that the matroid M satisfies conditions (M2) and (M3), respectively. ☐

Remark 3.3. Since $E \in \mathrm{L}(\mathscr{E})$, Algorithm 3.2 shows that the number of elements in the ground set of the matroid $\mathrm{M}(\mathscr{E})$ is at least the rank r of $\mathscr{E}$. To have equality, there must be a basis for E such that every linear subspace in $\mathrm{L}(\mathscr{E})$ is a direct sum of coordinate subspaces. Hence, the number of elements in the ground set of the matroid $\mathrm{M}(\mathscr{E})$ equals r if and only if the toric vector bundle $\mathscr{E}$ splits T-equivariantly into a direct sum of toric line bundles.

For each maximal cone $\sigma \in \Sigma(d)$, the compatibility condition (CC) is equivalent to saying that the subposet of $\mathrm{L}(\mathscr{E})$ consisting of the linear subspaces $\bigcap_{\mathbf{v}_i \in \sigma} E^i(j_i)$, where $j_i \in \mathbb{Z}$, is a distributive lattice; see Remark 2.2.2 in [Kl1]. Equivalently, the matroid $\mathrm{M}(\mathscr{E})$ contains a *compatible basis* $\mathcal{B}_\sigma$ such that each component $E^i(j)$, for $\mathbf{v}_i \in \sigma$ and $j \in \mathbb{Z}$, is a direct sum of the corresponding coordinate subspaces. Example 4.4 demonstrates that, for a given maximal cone σ, there may be more that basis in $\mathrm{M}(\mathscr{E})$ with this property.

Remark 3.4. For an element $\mathbf{e}$ in the ground set of the matroid $\mathrm{M}(\mathscr{E})$ and a linear subspace $V \in \mathrm{L}(\mathscr{E})$, the relation $\mathbf{e} \in V$ depends only on the matroid $\mathrm{M}(\mathscr{E})$ and not on the choice of a representation for $\mathrm{M}(\mathscr{E})$. Nevertheless, Algorithm 3.2 does produce a particular representation for $\mathrm{M}(\mathscr{E})$. This is analogous to a minimal free presentation for a finitely generated graded module over a polynomial ring: the ranks of the free modules are intrinsic invariants, but the matrix representing the map depends on the choice of bases; compare with [E, Section 1B].

For each element $\mathbf{e}$ in the ground set of the matroid $\mathrm{M}(\mathscr{E})$, the associated convex polytope is

$$P_{\mathbf{e}} := \big\{ \mathbf{u} \in M \otimes_{\mathbb{Z}} \mathbb{R} : \langle \mathbf{u}, \mathbf{v}_i \rangle \leqslant \max\big(j \in \mathbb{Z} : \mathbf{e} \in E^i(j)\big) \text{ for all } 1 \leqslant i \leqslant n \big\}.$$

Using a traditional term of venery (namely, the collective noun for owls), we call the collection of all such polytopes $P_{\mathbf{e}}$ the *parliament of polytopes* for the toric vector bundle $\mathscr{E}$. The number of polytopes in the parliament for $\mathscr{E}$ is at least the rank of $\mathscr{E}$ and equals the rank of $\mathscr{E}$ precisely when $\mathscr{E}$ splits into a direct sum of toric line bundles; see Remark 3.3.

Extending the classic theorem [CLS, Theorem 4.3.3] for line bundles on a toric variety, we have the following interpretation for the lattice points in a parliament of polytopes.

Proposition 1.1. *The lattice points in the polytopes of the parliament for $\mathscr{E}$ correspond to the T-equivariant generators for the space of global sections of $\mathscr{E}$,*

$$H^0(X, \mathscr{E}) \cong \sum_{\mathbf{e}} \operatorname{span}(\mathbf{e} \otimes \chi^{-\mathbf{u}} : \mathbf{u} \in P_{\mathbf{e}} \cap M) \subset E \otimes_{\mathbb{C}} T\,,$$

where the sum is over all elements $\mathbf{e}$ in the ground set of the matroid $\mathrm{M}(\mathscr{E})$.

Proof of Proposition 1.1. The T-action on the space of global sections yields a decomposition into isotypical components $H^0(X, \mathscr{E})_{\mathbf{u}}$, where $\mathbf{u} \in M$. The regular T-eigenfunction $\chi^{-\mathbf{u}}$ is an element of $H^0(X, \mathscr{E})_{\mathbf{u}}$ and $H^0(X, \mathscr{E}) = \bigoplus_{\mathbf{u} \in M} H^0(X, \mathscr{E})_{\mathbf{u}}$. Since X is complete, at most finitely many of the isotypical components are nonzero. Following Corollary 4.1.3 in [Kl1], evaluation at the identity of the torus T gives a canonical isomorphism

$$H^0(X,\mathscr{E})_{\mathbf{u}} = \bigcap_{\sigma\in\Sigma(d)} H^0(U_\sigma,\mathscr{E})_{\mathbf{u}} \xrightarrow{\cong} \bigcap_{\sigma\in\Sigma(d)} E^\sigma_{\mathbf{u}} = \bigcap_{i=1}^{n} E^i(\langle \mathbf{u},\mathbf{v}_i\rangle)\,.$$

Since the linear subspace $V_{\mathbf{u}} := \bigcap_{i=1}^n E^i(\langle \mathbf{u},\mathbf{v}_i\rangle)$ belongs to the poset $\mathrm{L}(\mathscr{E})$, Proposition 3.1 shows that there is a flat F in the matroid $\mathrm{M}(\mathscr{E})$ such that $\operatorname{span}(F) = V_{\mathbf{u}}$. Hence, we obtain the isomorphism $H^0(X,\mathscr{E})_{\mathbf{u}} \cong \sum_{\mathbf{e}} \operatorname{span}(\mathbf{e}\otimes\chi^{-\mathbf{u}} : \mathbf{e}\in V_{\mathbf{u}})$. Because we have

$$\begin{aligned}
\mathbf{e}\in V_{\mathbf{u}} \quad &\iff \quad \mathbf{e}\in E^i(\langle \mathbf{u},\mathbf{v}_i\rangle) && \text{for all } 1\leqslant i\leqslant n\\
&\iff \quad \langle \mathbf{u},\mathbf{v}_i\rangle \leqslant \max\bigl(j\in\mathbb{Z} : \mathbf{e}\in E^i(j)\bigr) && \text{for all } 1\leqslant i\leqslant n\\
&\iff \quad \mathbf{u}\in P_{\mathbf{e}}\cap M\,,
\end{aligned}$$

we conclude that $H^0(X,\mathscr{E})_{\mathbf{u}} \cong \sum_{\mathbf{e}} \operatorname{span}(\mathbf{e}\otimes\chi^{-\mathbf{u}} : \mathbf{u}\in P_{\mathbf{e}}\cap M)$. □

As expected, we recover the description for the global sections of a line bundle.

Example 3.5. Every line bundle $\mathscr{L}$ on a smooth toric variety X equals $\mathscr{O}_X(D)$ for some T-invariant divisor $D = a_1D_1 + a_2D_2 + \cdots + a_nD_n$. Theorem 6.1.7 in [CLS] establishes that the Cartier divisor D is determined by a collection $\{\mathbf{u}_\sigma \in M : \sigma\in\Sigma(d)\}$, so we obtain $\boldsymbol{u}(\sigma) = \{\mathbf{u}_\sigma\}$ for all $\sigma\in\Sigma(d)$. The associated continuous piecewise linear function $\varphi_D : N_{\mathbb{R}} \to \mathbb{R}$ satisfies $\varphi_D(\mathbf{v}_i) = -a_i$ and $\varphi_D(\mathbf{v}) = \langle \mathbf{u}_\sigma,\mathbf{v}\rangle$ for all $\mathbf{v}\in\sigma$. Following [Kl1, Subsection 2.3.1], the decreasing filtrations corresponding to $\mathscr{L}$ are

$$E^i(j) := \begin{cases} \mathbb{C} & \text{if} \quad j\leqslant a_i \\ 0 & \text{if} \quad j > a_i \end{cases} \qquad \text{for all } 1\leqslant i\leqslant n.$$

If $\mathbf{e}$ is any nonzero vector in $E=\mathbb{C}$, then the ground set of the matroid $\mathrm{M}(\mathscr{L})$ is $\{\mathbf{e}\}$ and the unique polytope in the parliament is $P_{\mathbf{e}} = \{\mathbf{u}\in M\otimes_{\mathbb{Z}}\mathbb{R} : \langle \mathbf{u},\mathbf{v}_i\rangle \leqslant a_i\}$. It follows that $E_{\mathbf{u}_\sigma} = E = \mathbb{C}$ for all $\sigma\in\Sigma(d)$, so $H^0(X,\mathscr{L})_{\mathbf{u}} = \mathbb{C}$ when $\langle \mathbf{u},\mathbf{v}_i\rangle \leqslant a_i$ for all $1\leqslant i\leqslant n$ and $H^0(X,\mathscr{L})_{\mathbf{u}} = 0$ otherwise. Therefore, we conclude that $H^0(X,\mathscr{L}) = \bigoplus_{\mathbf{u}\in P_{\mathbf{e}}\cap M} \operatorname{span}(\mathbf{e}\otimes\chi^{-\mathbf{u}})$. Be aware that we use the opposite sign convention when compared to either [CLS, Section 6.1] or [F, Section 3.4]. ♢

The polytopes in the parliament also have an attractive reinterpretation as toric line bundles.

Remark 3.6. For each flat F in the matroid $\mathrm{M}(\mathscr{E})$, the associated T-invariant divisor on X is defined to be $D_F := a_1(F)\,D_1 + a_2(F)\,D_2 + \cdots + a_n(F)\,D_n$, where $a_i(F) := \max\{j\in\mathbb{Z} : \operatorname{span}(F)\subseteq E^i(j)\}$. In particular, each flat F gives rise to an toric line bundle $\mathscr{O}_X(D_F)$. When a flat is defined by a single element $\mathbf{e}$ in the ground set of $\mathrm{M}(\mathscr{E})$, the polytope corresponding to $\mathscr{O}_X(D_{\mathbf{e}})$ is simply the polytope $P_{\mathbf{e}}$ from the parliament for $\mathscr{E}$. By construction, there is a natural map from the filtrations of the toric vector bundle $\bigoplus_{\mathbf{e}} \mathscr{O}_X(D_{\mathbf{e}})$ onto the filtrations for the toric vector bundle $\mathscr{E}$. Hence, the equivalence of categories yields a canonical surjective homomorphism

$$\eta\colon \bigoplus_{\mathbf{e}} \mathscr{O}_X(D_{\mathbf{e}}) \to \mathscr{E}\,,$$

where the sum is over all elements $\mathbf{e}$ in the ground set of the matroid $\mathrm{M}(\mathscr{E})$. Rephrasing Proposition 1.1, we see that the map η induces a surjection on global sections.

Our second example shows that the ground set of the matroid $\mathrm{M}(\mathscr{E})$ may be strictly larger than the union $\bigcup_{\sigma\in\Sigma(d)} \mathcal{B}_\sigma$ of the bases for E that split the filtrations over the maximal cones.

Example 3.7. To describe a toric vector bundle $\mathscr{E}$ of rank 3 on $\mathbb{P}^1 \times \mathbb{P}^1$, we first specify the fan: the unique minimal lattice points generating the rays are $\mathbf{v}_1 = (1,0)$, $\mathbf{v}_2 = (0,1)$, $\mathbf{v}_3 = (-1,0)$, $\mathbf{v}_4 = (0,-1)$, and the maximal cones are $\sigma_{1,2} = \mathrm{pos}(\mathbf{v}_1,\mathbf{v}_2)$, $\sigma_{2,3} = \mathrm{pos}(\mathbf{v}_2,\mathbf{v}_3)$, $\sigma_{3,4} = \mathrm{pos}(\mathbf{v}_3,\mathbf{v}_4)$, $\sigma_{1,4} = \mathrm{pos}(\mathbf{v}_1,\mathbf{v}_4)$. If $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ denotes the standard basis of $E = \mathbb{C}^3$, then the decreasing filtrations defining $\mathscr{E}$ are

$$E^1(j) = \begin{cases} E & \text{if} \quad j \leqslant -1, \\ \mathrm{span}(\mathbf{e}_1,\mathbf{e}_2) & \text{if} \quad -1 < j \leqslant 0, \\ \mathrm{span}(\mathbf{e}_1+\mathbf{e}_2) & \text{if} \quad 0 < j \leqslant 1, \\ 0 & \text{if} \quad 1 < j, \end{cases}$$

$$E^2(j) = \begin{cases} E & \text{if} \quad j \leqslant 0, \\ \mathrm{span}(\mathbf{e}_2,\mathbf{e}_3) & \text{if} \quad 0 < j \leqslant 1, \\ \mathrm{span}(\mathbf{e}_2) & \text{if} \quad 1 < j \leqslant 2, \\ 0 & \text{if} \quad 2 < j, \end{cases}$$

$$E^3(j) = \begin{cases} E & \text{if} \quad j \leqslant -1, \\ \mathrm{span}(\mathbf{e}_1,\mathbf{e}_3) & \text{if} \quad -1 < j \leqslant 0, \\ \mathrm{span}(\mathbf{e}_1+\mathbf{e}_3) & \text{if} \quad 0 < j \leqslant 1, \\ 0 & \text{if} \quad 1 < j, \end{cases}$$

$$E^4(j) = \begin{cases} E & \text{if} \quad j \leqslant 0, \\ \mathrm{span}(\mathbf{e}_2,\mathbf{e}_3) & \text{if} \quad 0 < j \leqslant 1, \\ \mathrm{span}(\mathbf{e}_2) & \text{if} \quad 1 < j \leqslant 2, \\ 0 & \text{if} \quad 2 < j. \end{cases}$$

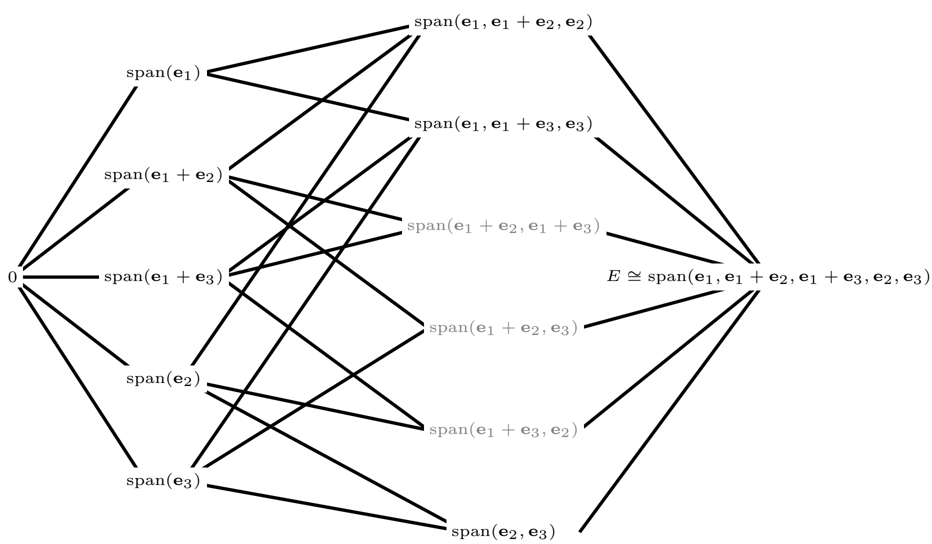

FIGURE 1. Hasse diagram for the lattice of flats

It follows that $\{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3\}$ is the ground set of matroid $\mathrm{M}(\mathscr{E})$. Figure 1 represents its lattice of flats; the flats appearing in $\mathrm{M}(\mathscr{E})$ but not in $\mathrm{L}(\mathscr{E})$ are shaded gray. On each maximal cone, the associated characters and the unique choice of compatible basis are

$$\begin{aligned}
\boldsymbol{u}(\sigma_{1,2}) &= \{(1,0),(0,2),(-1,1)\}\,, & \mathcal{B}_{\sigma_{1,2}} &= \{\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_2,\mathbf{e}_3\}\,,\\
\boldsymbol{u}(\sigma_{2,3}) &= \{(-1,0),(1,2),(0,1)\}\,, & \mathcal{B}_{\sigma_{2,3}} &= \{\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_3\}\,,\\
\boldsymbol{u}(\sigma_{3,4}) &= \{(-1,0),(1,-2),(0,-1)\}\,, & \mathcal{B}_{\sigma_{3,4}} &= \{\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_3\}\,,\\
\boldsymbol{u}(\sigma_{1,4}) &= \{(0,-2),(1,0),(-1,-1)\}\,, & \mathcal{B}_{\sigma_{1,4}} &= \{\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3\}\,.
\end{aligned}$$

The parliament for $\mathscr{E}$ consists of the following convex polytopes: $P_{\mathbf{e}_1} = \operatorname{conv}\big((0,0)\big)$, $P_{\mathbf{e}_1+\mathbf{e}_2} = \operatorname{conv}\big((1,0)\big)$, $P_{\mathbf{e}_1+\mathbf{e}_3} = \operatorname{conv}\big((-1,0)\big)$, $P_{\mathbf{e}_2} = \varnothing$, and $P_{\mathbf{e}_3} = \varnothing$. Although we have $\mathbf{e}_1 \notin \bigcup_{\sigma\in\Sigma(2)} \mathcal{B}_\sigma$, we also have $\operatorname{span}(\mathbf{e}_1) = E^1(0) \cap E^3(0)$. $\diamondsuit$

The lattice points in the parliament of polytopes for a toric vector bundle correspond to a basis if and only if, for all $\mathbf{u} \in M$, the subset $\{\mathbf{e} \in E : \mathbf{u} \in P_{\mathbf{e}}\}$ is linearly independent. The next example illustrates how a single lattice point can correspond to a dependent collection of global sections.

Example 3.8. Consider the tangent bundle $\mathscr{T}_{\mathbb{P}^d}$ on $\mathbb{P}^d$. The minimal lattice points $\mathbf{v}_i$ generating the ith ray in the fan of $\mathbb{P}^d$ equals the ith standard basis vector in $\mathbb{C}^d$ for $1 \leqslant i \leqslant d$, and the additional ray is generated by $\mathbf{v}_{d+1} := -\mathbf{v}_1 - \mathbf{v}_2 - \cdots - \mathbf{v}_d$. The maximal cones are $\sigma_i := \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \dots, \mathbf{v}_{d+1})$ for $1 \leqslant i \leqslant d+1$; compare with [CLS, Example 3.1.10] or [F, Section 1.4]. Following [Kl1, Subsection 2.3.5], we identify the fibre E of $\mathscr{T}_{\mathbb{P}^d}$ over the identity of the torus T with $N \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^d$. Hence, the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d$ also form the standard basis for $E = \mathbb{C}^d$ and the decreasing filtrations defining $\mathscr{T}_{\mathbb{P}^d}$ are

$$E^i(j) = \begin{cases} E & \text{if} \quad j \leqslant 0 \\ \operatorname{span}(\mathbf{v}_i) & \text{if} \quad j = 1 \\ 0 & \text{if} \quad j > 1 \end{cases} \qquad \text{for } 1 \leqslant i \leqslant d+1.$$

Writing $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d$ for the dual basis of M corresponding to the basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d \in N$, we have

$$\begin{aligned}\boldsymbol{u}(\sigma_i) = \{&\mathbf{w}_1 - \mathbf{w}_i, \mathbf{w}_2 - \mathbf{w}_i, \dots, \mathbf{w}_{i-1} - \mathbf{w}_i,\\ &-\mathbf{w}_i, \mathbf{w}_{i+1} - \mathbf{w}_i, \mathbf{w}_{i+2} - \mathbf{w}_i, \dots, \mathbf{w}_d - \mathbf{w}_i\}\end{aligned}$$

for $1 \leqslant i \leqslant d$, and $\boldsymbol{u}(\sigma_{d+1}) = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d\}$. Hence, the ground set of the matroid $\mathrm{M}(\mathscr{T}_{\mathbb{P}^d})$ is $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$ and the convex polytopes in the parliament for $\mathscr{T}_{\mathbb{P}^d}$ are

$$P_{\mathbf{v}_i} = \big\{\mathbf{u} \in M \otimes_{\mathbb{Z}} \mathbb{R} : \langle \mathbf{u}, \mathbf{v}_i \rangle \leqslant 1 \text{ and } \langle \mathbf{u}, \mathbf{v}_j \rangle \leqslant 0 \text{ for all } j \neq i\big\}\,.$$

The lattice points in the parliament of polytopes for $\mathscr{T}_{\mathbb{P}^d}$ correspond to the following $(d+1)^2$ global sections: $\mathbf{v}_i \otimes \chi^{\mathbf{w}_j - \mathbf{w}_i}$ for $1 \leqslant i,j \leqslant d$, $\mathbf{v}_i \otimes \chi^{-\mathbf{w}_i}$ for $1 \leqslant i \leqslant d$, $\mathbf{v}_{d+1} \otimes \chi^{\mathbf{w}_i}$ for $1 \leqslant i \leqslant d$, and $\mathbf{v}_{d+1} \otimes \chi^{\mathbf{0}}$. The origin $\mathbf{0} \in M$ is contained in all $d+1$ polytopes, which yields $d+1$ global sections in a d-dimensional vector space. Following Remark 3.6, the flat $\{\mathbf{v}_i\}$ in the matroid $\mathrm{M}(\mathscr{T}_{\mathbb{P}^d})$ corresponds to the toric line bundle $\mathscr{O}_{\mathbb{P}^d}(D_i)$ for $1 \leqslant i \leqslant d+1$, and the flat given by the unique circuit $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$ in $\mathrm{M}(\mathscr{T}_{\mathbb{P}^d})$ corresponds to $\mathscr{O}_{\mathbb{P}^d}$. Hence, we obtain the short exact

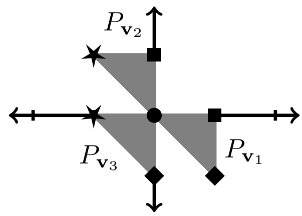

FIGURE 2. The parliament of polytopes for $\mathscr{T}_{\mathbb{P}^2}$

sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^d} \longrightarrow \bigoplus_{i=1}^{d+1} \mathscr{O}_{\mathbb{P}^d}(D_i) \xrightarrow{\ \eta\ } \mathscr{T}_{\mathbb{P}^d} \longrightarrow 0\,,$$

which is dual to the classic Euler sequence; see Theorem 8.1.6 in [CLS].

When $d = 2$, it is possible to visualize the parliament of polytopes. In this case, the associated characters are $\boldsymbol{u}(\sigma_1) = \{(-1, 0), (-1, 1)\}$, $\boldsymbol{u}(\sigma_2) = \{(1, -1), (0, -1)\}$, $\boldsymbol{u}(\sigma_3) = \{(1, 0), (0, 1)\}$, and the convex polytopes are

$$\begin{aligned} P_{\mathbf{v}_1} &= \operatorname{conv}\big((0,0), (1,0), (1,-1)\big)\,, \\ P_{\mathbf{v}_2} &= \operatorname{conv}\big((0,0), (0,1), (-1,1)\big)\,, \\ P_{\mathbf{v}_3} &= \operatorname{conv}\big((0,0), (-1,0), (0,-1)\big)\,. \end{aligned}$$

In Figure 2, the associated characters are represented by stars, diamonds, and squares, respectively. The polytopes are represented by shaded triangles, and the other lattice point lying in the polytopes is represented by a circle. ♢

4. GLOBALLY GENERATED TORIC VECTOR BUNDLES

In this section, we establish our criterion for deciding whether a toric vector bundle is globally generated. To detect the global generation from the parliament of polytopes, we need a local description for a global section in coordinates near a T-fixed point.

To achieve this, consider a maximal cone $\sigma \in \Sigma(d)$ and the T-fixed point x_σ. By reordering the rays in the fan (if necessary), we assume that $\sigma = \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d)$. Since X is a smooth toric variety, the unique minimal generators $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d$ of the dual cone $\sigma^\vee$ form a $\mathbb{Z}$-basis for M. By indexing the underlying set $\underline{\boldsymbol{u}}(\sigma)$ of associated characters, we have $\underline{\boldsymbol{u}}(\sigma) = \{\mathbf{u}_{\sigma,1}, \mathbf{u}_{\sigma,2}, \ldots, \mathbf{u}_{\sigma,s}\} \subset M$, for some integer s satisfying $1 \leqslant s \leqslant r$. Following [Kl1, Section 6.3], we identify the fiber of $\mathscr{E}$ over the T-fixed point x_σ with the $\mathbb{C}$-vector space

$$\bigoplus_{\mathbf{u} \in \underline{\boldsymbol{u}}(\sigma)} \frac{E^\sigma_{\mathbf{u}}}{E^\sigma_{>\mathbf{u}}} = \bigoplus_{k=1}^{s} \frac{E^\sigma_{\mathbf{u}_{\sigma,k}}}{E^\sigma_{>\mathbf{u}_{\sigma,k}}} \cong \mathscr{E}_{x_\sigma} \cong \mathbb{C}^r\,,$$

where $E^\sigma_{\mathbf{u}} := \bigcap_{i=1}^{d} E^i(\langle \mathbf{u}, \mathbf{v}_i \rangle)$ and

$$E^\sigma_{>\mathbf{u}} := \sum_{\mathbf{0} \neq \mathbf{u}' - \mathbf{u} \in \sigma^\vee} E^\sigma_{\mathbf{u}'} = \sum_{i=1}^{d} E^\sigma_{\mathbf{u}+\mathbf{w}_i}\,;$$

see section 2. The linear subspaces $E^\sigma_{\mathbf{u}}$ and $E^\sigma_{>\mathbf{u}}$ correspond to flats in the matroid $\mathrm{M}(\mathscr{E})$ that are generated by subsets of any compatible basis $\mathcal{B}_\sigma$ in $\mathrm{M}(\mathscr{E})$. For a given compatible basis $\mathcal{B}_\sigma$, this decomposition of the fiber yields a partition of the set $\mathcal{B}_\sigma$. Specifically, we have the disjoint union $\mathcal{B}_\sigma = \mathcal{B}_{\sigma,1} \sqcup \mathcal{B}_{\sigma,2} \sqcup \cdots \sqcup \mathcal{B}_{\sigma,s}$, where the subset $\mathcal{B}_{\sigma,k}$ consists of all $\mathbf{e} \in \mathcal{B}_\sigma$ such that $\mathbf{e} \in E^\sigma_{\mathbf{u}_{\sigma,k}} \setminus E^\sigma_{>\mathbf{u}_{\sigma,k}}$ for $1 \leqslant k \leqslant s$. By construction we see that, for all $1 \leqslant k \leqslant s$, the quotient space $E^\sigma_{\mathbf{u}_{\sigma,k}} / E^\sigma_{>\mathbf{u}_{\sigma,k}}$ is identified with the linear subspace $\operatorname{span}(\mathcal{B}_{\sigma,k}) \subseteq E$, and the multiplicity of $\mathbf{u}_{\sigma,k}$ in the multiset $\boldsymbol{u}(\sigma)$ of associated characters equals the number of elements in $\mathcal{B}_{\sigma,k}$. With these preliminaries, we have the following technical lemma.

Lemma 4.1. *Let $\sigma = \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d)$ be a maximal cone in the fan Σ, let $\underline{\boldsymbol{u}}(\sigma) = \{\mathbf{u}_{\sigma,1}, \mathbf{u}_{\sigma,2}, \ldots, \mathbf{u}_{\sigma,s}\}$ be the underlying set of associated characters, and let $\mathcal{B}_\sigma = \mathcal{B}_{\sigma,1} \sqcup \mathcal{B}_{\sigma,2} \sqcup \cdots \sqcup \mathcal{B}_{\sigma,s}$ be the corresponding partition of a compatible basis in $\mathrm{M}(\mathscr{E})$. For each $\mathbf{e} \in \mathcal{B}_\sigma$, consider the continuous piecewise linear function on Σ defined by*

$$\varphi_{\mathbf{e}}(\mathbf{v}_i) := \max\{j \in \mathbb{Z} : \mathbf{e} \in E^i(j)\}$$

for all $1 \leqslant i \leqslant n$. If $\mathbf{e} \in \mathcal{B}_{\sigma,k}$ for some $1 \leqslant k \leqslant s$, then we have $\varphi_{\mathbf{e}}(\mathbf{v}_i) = \langle \mathbf{u}_{\sigma,k}, \mathbf{v}_i \rangle$ for all $1 \leqslant i \leqslant d$. In particular, if $\mathbf{e} \in \mathcal{B}_{\sigma,k}$ and $\mathbf{u}_{\sigma,k} \in P_{\mathbf{e}}$, then the lattice point $\mathbf{u}_{\sigma,k}$ is a vertex of the polytope $P_{\mathbf{e}}$.

Proof. Fix an index k such that $1 \leqslant k \leqslant s$ and an element $\mathbf{e} \in \mathcal{B}_{\sigma,k}$. Since we have $\mathbf{e} \in \mathcal{B}_{\sigma,k} \subset E^\sigma_{\mathbf{u}_{\sigma,k}}$, it follows that $\mathbf{e} \in E^i(\langle \mathbf{u}_{\sigma,k}, \mathbf{v}_i \rangle)$ for all $1 \leqslant i \leqslant d$, so $\max\{j \in \mathbb{Z} : \mathbf{e} \in E^i(j)\} \geqslant \langle \mathbf{u}_{\sigma,k}, \mathbf{v}_i \rangle$ for all $1 \leqslant i \leqslant d$. Suppose that, for some index i satisfying $1 \leqslant i \leqslant d$, we have

$$\max\{j \in \mathbb{Z} : \mathbf{e} \in E^i(j)\} > \langle \mathbf{u}_{\sigma,k}, \mathbf{v}_i \rangle \,.$$

It would follow that $\mathbf{e} \subset E^\sigma_{\mathbf{u}_{\sigma,k} + \mathbf{w}_\ell}$ for some minimal generator $\mathbf{w}_\ell$ of the dual cone $\sigma^\vee$. However, this would imply that $\mathbf{e} \in E^\sigma_{>\mathbf{u}_{\sigma,k}}$ which contradicts the definition of $\mathcal{B}_{\sigma,k}$. Therefore, we conclude that

$$\max\{j \in \mathbb{Z} : \mathbf{e} \in E^i(j)\} = \langle \mathbf{u}_{\sigma,k}, \mathbf{v}_i \rangle$$

for all $1 \leqslant i \leqslant d$. Moreover, when $\mathbf{e} \in \mathcal{B}_{\sigma,k}$, the piecewise linear function $\varphi_{\sigma,k}$ is simply the support function for the polytope $P_{\mathbf{e}}$ in the parliament for $\mathscr{E}$. Thus, if $\mathbf{e} \in \mathcal{B}_{\sigma,k}$ and $\mathbf{u}_{\sigma,k} \in P_{\mathbf{e}}$, then we see that the lattice point $\mathbf{u}_{\sigma,k}$ is a vertex of this polytope. $\square$

We can now give a local description for a global section around the T-fixed point x_σ. The affine semigroup ring $\mathbb{C}[\sigma^\vee \cap M]$ is the coordinate ring for the affine open set $U_\sigma \subset X$ and is isomorphic to the polynomial ring $\mathbb{C}[y_1, y_2, \ldots, y_d]$ where $y_i := \chi^{-\mathbf{w}_i}$ for $1 \leqslant i \leqslant d$. For any compatible basis $\mathcal{B}_\sigma$ and any vector $\mathbf{e}' \in E$, there exists unique scalars $\lambda_{\mathbf{e}} \in \mathbb{C}$, for all $\mathbf{e} \in \mathcal{B}_\sigma$, such that $\mathbf{e}' = \sum_{\mathbf{e} \in \mathcal{B}_\sigma} \lambda_{\mathbf{e}} \, \mathbf{e}$. By Proposition 1.1, a T-equivariant global section of $\mathscr{E}$ has the form $\mathbf{e}' \otimes \chi^{-\mathbf{u}}$, where $\mathbf{e}' \in E$ and $\mathbf{u} \in M$. Hence, the section $\mathbf{e}' \otimes \chi^{-\mathbf{u}}$ is given in local coordinates near x_σ by

$$(*) \qquad \sum_{\mathbf{e} \in \mathcal{B}_\sigma} \lambda_{\mathbf{e}} \left(\mathbf{e} \otimes \prod_{i=1}^{d} y_i^{-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\mathbf{e}}(\mathbf{v}_i)} \right) .$$

Using this local description, we characterize the global generation of a toric vector bundle via its parliament of polytopes.

Theorem 1.2. *A toric vector bundle $\mathscr{E}$ is globally generated if and only if, for all $\sigma \in \Sigma(d)$, the associated characters $\boldsymbol{u}(\sigma)$ are vertices of polytopes in the parliament and the elements indexing these polytopes form a basis in the matroid $\mathrm{M}(\mathscr{E})$.*

Proof of Theorem 1.2. As Proposition 1.1 shows, the toric vector bundle $\mathscr{E}$ has a T-equivariant basis of global sections. Hence, the locus in the toric variety X on which all global sections vanish is closed and T-invariant. Since X is complete, it follows that the toric vector bundle $\mathscr{E}$ is globally generated if and only if it is globally generated at every T-fixed point.

Fix a maximal cone $\sigma = \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d)$ in the fan Σ, let

$$\underline{\boldsymbol{u}}(\sigma) = \{\mathbf{u}_{\sigma,1}, \mathbf{u}_{\sigma,2}, \dots, \mathbf{u}_{\sigma,s}\}$$

be the underlying set of associated characters, and let

$$\mathcal{B}_\sigma = \mathcal{B}_{\sigma,1} \sqcup \mathcal{B}_{\sigma,2} \sqcup \cdots \sqcup \mathcal{B}_{\sigma,s}$$

be the corresponding partition of a compatible basis in the matroid $\mathrm{M}(\mathscr{E})$. The toric vector bundle $\mathscr{E}$ is globally generated at the T-fixed point x_σ if and only if the evaluation map

$$\operatorname{ev}_\sigma \colon H^0(X, \mathscr{E}) \to H^0(X, \mathscr{E} \otimes \mathscr{O}_X/\mathfrak{m}_{x_\sigma}) \cong \operatorname{span}(\mathcal{B}_\sigma)$$

is surjective. Since a T-equivariant global section $\mathbf{e}' \otimes \chi^{-\mathbf{u}}$ is given in local coordinates near x_σ by $(*)$, its evaluation at the T-fixed point x_σ is given by

$$\sum_{\mathbf{e} \in \mathcal{B}_\sigma} \lambda_{\mathbf{e}} \left(\mathbf{e} \otimes \prod_{i=1}^{d} y_i^{-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\mathbf{e}}(\mathbf{v}_i)} \right) \Bigg|_{y_1 = y_2 = \cdots = y_d = 0} .$$

The $\mathbf{e}$th summand in this expression has neither a zero nor a pole at $(y_1, y_2, \dots, y_d) = (0, 0, \dots, 0)$ if and only if $-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\mathbf{e}}(\mathbf{v}_i) = 0$ for all $1 \leqslant i \leqslant d$. By Lemma 4.1, it follows that there exists an index k such that $\mathbf{e} \in \mathcal{B}_{\sigma,k}$ and $\mathbf{u} = \mathbf{u}_{\sigma,k}$. In this case, the lattice point $\mathbf{u}_{\sigma,k}$ is also a vertex of the polytope $P_{\mathbf{e}}$ in the parliament for $\mathscr{E}$. Hence, the image of a T-equivariant global section under the evaluation map ev_σ is nonzero in the fiber at x_σ if and only if the global section has the form $\sum_{\mathbf{e} \in \mathcal{B}_{\sigma,k}} \lambda_{\mathbf{e}} (\mathbf{e} \otimes \chi^{-\mathbf{u}_{\sigma,k}})$, for some $1 \leqslant k \leqslant s$, which evaluates to $\sum_{\mathbf{e} \in \mathcal{B}_{\sigma,k}} \lambda_{\mathbf{e}}\, \mathbf{e} \in \mathscr{E}_{x_\sigma}$. Therefore, the evaluation map ev_σ is surjective if and only if there exists a compatible basis $\mathcal{B}_\sigma$ such that each $\mathbf{e} \otimes \chi^{-\mathbf{u}_{\sigma,k}}$, for $\mathbf{e} \in \mathcal{B}_{\sigma,k}$ and $1 \leqslant k \leqslant s$, is a global section. □

Using Theorem 1.2, we create a low-rank toric vector bundle on $\mathbb{P}^2$ that is not globally generated; Example 5.3 will show that this low-rank toric vector bundle is also ample.

Example 4.2. To describe a second toric vector bundle $\mathscr{F}$ of rank 3 on $\mathbb{P}^2$, we use the notation from Example 3.8. Specifically, the minimal lattice points generating the rays in the fan are $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (0, 1)$, $\mathbf{v}_3 = (-1, -1)$, and the maximal cones are

$$\sigma_1 = \operatorname{pos}(\mathbf{v}_2, \mathbf{v}_3)\,, \qquad \sigma_2 = \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_3)\,, \qquad \sigma_3 = \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2)\,.$$

If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denotes the standard basis of $E = \mathbb{C}^3$, then the decreasing filtrations defining $\mathscr{F}$ are

$$E^1(j) = \begin{cases} E & \text{if} \quad j \leqslant -1, \\ \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2) & \text{if} \quad -1 < j \leqslant 0, \\ \operatorname{span}(\mathbf{e}_1) & \text{if} \quad 0 < j \leqslant 4, \\ 0 & \text{if} \quad 4 < j, \end{cases}$$
$$E^2(j) = \begin{cases} E & \text{if} \quad j \leqslant -2, \\ \operatorname{span}(\mathbf{e}_2, \mathbf{e}_3) & \text{if} \quad -2 < j \leqslant 0, \\ \operatorname{span}(\mathbf{e}_3) & \text{if} \quad 0 < j \leqslant 3, \\ 0 & \text{if} \quad 3 < j, \end{cases}$$
$$E^3(j) = \begin{cases} E & \text{if} \quad j \leqslant -1, \\ \operatorname{span}(\mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2) & \text{if} \quad -1 < j \leqslant 2, \\ \operatorname{span}(\mathbf{e}_1 - \mathbf{e}_2) & \text{if} \quad 2 < j \leqslant 3, \\ 0 & \text{if} \quad 3 < j. \end{cases}$$

It follows that $\{\mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3\}$ is the ground set of the matroid $\mathrm{M}(\mathscr{F})$. On each maximal cone, the associated characters and the unique choice of compatible bases are

$$\begin{aligned} \boldsymbol{u}(\sigma_1) &= \{(-1,-2), (-2,0), (-2,3)\}\,, & \mathcal{B}_{\sigma_1} &= \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3\}\,, \\ \boldsymbol{u}(\sigma_2) &= \{(4,-3), (0,-3), (-1,-1)\}\,, & \mathcal{B}_{\sigma_2} &= \{\mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3\}\,, \\ \boldsymbol{u}(\sigma_3) &= \{(4,-2), (0,0), (-1,3)\}\,, & \mathcal{B}_{\sigma_3} &= \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\,, \end{aligned}$$

so the convex polytopes in the parliament for $\mathscr{F}$ are

$$\begin{aligned} P_{\mathbf{e}_1} &= \operatorname{conv}\big((3,-2), (4,-2), (4,-3)\big)\,, \\ P_{\mathbf{e}_1 - \mathbf{e}_2} &= \operatorname{conv}\big((-1,-2), (0,-2), (0,-3)\big)\,, \\ P_{\mathbf{e}_2} &= \varnothing\,, \\ P_{\mathbf{e}_2 - \mathbf{e}_3} &= \operatorname{conv}\big((-2,0), (-1,0), (-1,-1)\big)\,, \\ P_{\mathbf{e}_3} &= \operatorname{conv}\big((-2,3), (-1,3), (-1,2)\big)\,. \end{aligned}$$

In Figure 3 the associated characters are represented by stars, diamonds, and squares, respectively. The polytopes are represented by shaded triangles and the other lattice points lying in the polytopes are represented by circles. The square with empty interior represents the unique associated character $(0,0)$ not in any of the polytopes. Therefore, Theorem 1.2 shows that $\mathscr{F}$ is not globally generated. $\diamondsuit$

Remark 4.3. Our diagrams for parliaments of polytopes, such as the one appearing in Figure 3, have at least some superficial similarities to the twisted polytopes appearing in [KT, Section 6]. It would be interesting to develop a more substantive connection.

If all the polytopes in the parliament for a toric vector bundle $\mathscr{E}$ correspond to globally generated line bundles, then the toric vector bundle $\mathscr{E}$ itself is globally generated. However, the converse is false. We close this section with a globally generated toric vector bundle in which some members of the parliament of polytopes do not correspond to globally generated line bundles.

Example 4.4. To describe our toric vector bundle $\mathscr{G}$ of rank 2 on the first Hirzebruch surface $X = \mathbb{P}\big(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)\big)$, we first specify the fan. The minimal lattice

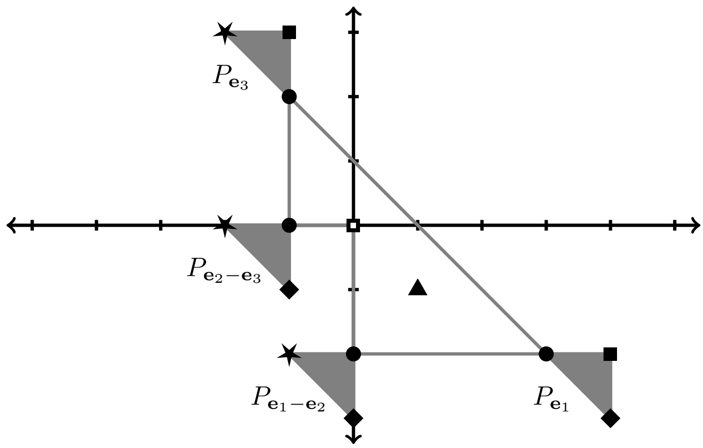

FIGURE 3. The parliament of polytopes for $\mathscr{F}$

points generating the rays in the fan are $\mathbf{v}_1 = (1,0)$, $\mathbf{v}_2 = (0,1)$, $\mathbf{v}_3 = (-1,1)$, $\mathbf{v}_4 = (0,-1)$, and the maximal cones are $\sigma_{1,2} = \operatorname{pos}(\mathbf{v}_1,\mathbf{v}_2)$, $\sigma_{2,3} = \operatorname{pos}(\mathbf{v}_2,\mathbf{v}_3)$, $\sigma_{3,4} = \operatorname{pos}(\mathbf{v}_3,\mathbf{v}_4)$, $\sigma_{1,4} = \operatorname{pos}(\mathbf{v}_1,\mathbf{v}_4)$. If $\mathbf{e}_1,\mathbf{e}_2$ denotes the standard basis of $E = \mathbb{C}^2$, then the decreasing filtrations defining $\mathscr{G}$ are

$$E^1(j) = \begin{cases} E & \text{if} \quad j \leqslant -2, \\ \operatorname{span}(\mathbf{e}_1) & \text{if} \quad -2 < j \leqslant 4, \\ 0 & \text{if} \quad 4 < j, \end{cases}$$

$$E^2(j) = \begin{cases} E & \text{if} \quad j \leqslant 2, \\ \operatorname{span}(\mathbf{e}_1) & \text{if} \quad 2 < j \leqslant 3, \\ 0 & \text{if} \quad 3 < j, \end{cases}$$

$$E^3(j) = \begin{cases} E & \text{if} \quad j \leqslant 0, \\ \operatorname{span}(\mathbf{e}_2) & \text{if} \quad 0 < j \leqslant 5, \\ 0 & \text{if} \quad 5 < j, \end{cases}$$

$$E^4(j) = \begin{cases} E & \text{if} \quad j \leqslant -1, \\ \operatorname{span}(\mathbf{e}_1+\mathbf{e}_2) & \text{if} \quad -1 < j \leqslant 3, \\ 0 & \text{if} \quad 3 < j. \end{cases}$$

It follows that the ground set of the matroid $\mathrm{M}(\mathscr{G})$ is $\{\mathbf{e}_1, \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2\}$. On the maximal cones, the associated characters and a choice of compatible bases are

$$\begin{aligned} \boldsymbol{u}(\sigma_{1,2}) &= \{(-2,2),(4,3)\}, & \mathcal{B}_{\sigma_{1,2}} &= \{\mathbf{e}_2,\mathbf{e}_1\}, \\ \boldsymbol{u}(\sigma_{2,3}) &= \{(-3,2),(3,3)\}, & \mathcal{B}_{\sigma_{2,3}} &= \{\mathbf{e}_2,\mathbf{e}_1\}, \\ \boldsymbol{u}(\sigma_{3,4}) &= \{(-4,1),(-3,-3)\}, & \mathcal{B}_{\sigma_{3,4}} &= \{\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2\}, \\ \boldsymbol{u}(\sigma_{1,4}) &= \{(-2,-3),(4,1)\}, & \mathcal{B}_{\sigma_{1,4}} &= \{\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_1\}. \end{aligned}$$

The convex polytopes in the parliament for $\mathscr{G}$ are

$$\begin{aligned} P_{\mathbf{e}_1} &= \operatorname{conv}\big((1,1),(3,3),(4,3),(4,1)\big), \\ P_{\mathbf{e}_1+\mathbf{e}_2} &= \operatorname{conv}\big((-3,-3),(-2,-2),(-2,-3)\big), \\ P_{\mathbf{e}_2} &= \operatorname{conv}\big((-4,1),(-3,2),(-2,2),(-2,1)\big). \end{aligned}$$

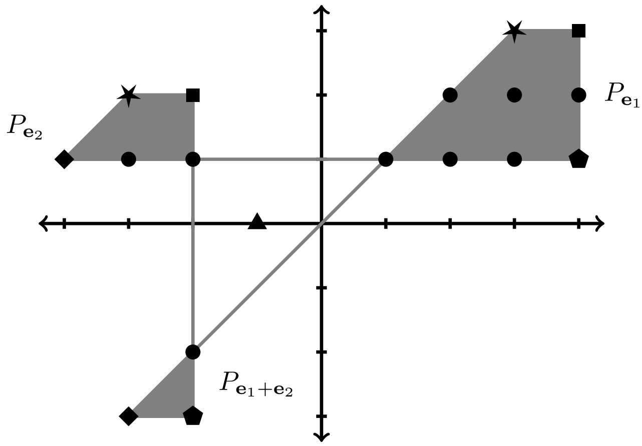

FIGURE 4. The parliament of polytopes for $\mathscr{G}$

The set $\{\mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_1\}$ also forms a compatible basis on $\sigma_{1,2}$, but the character $(-2,2)$ does not belong to the polytope $P_{\mathbf{e}_1+\mathbf{e}_2}$. In Figure 4, the associated characters are represented by squares, stars, diamonds, and pentagons, respectively. The polytopes are represented by shaded regions, and the other lattice points lying in the polytopes are represented by circles. We see that each associated character lies in a unique polytope in the parliament. Moreover, for each maximal cone, the elements indexing polytopes containing the associated characters are equal to our chosen compatible bases, so Theorem 1.2 shows that $\mathscr{G}$ is globally generated. Remark 3.6 shows that the elements $\{\mathbf{e}_1, \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2\}$ correspond to the toric line bundles

$$\begin{aligned}&\mathscr{O}_X(4D_1+3D_2-D_4)\,,\\&\mathscr{O}_X(-2D_1+2D_2+3D_4)\,,\\&\mathscr{O}_X(-2D_1+2D_2+5D_3-D_4)\,.\end{aligned}$$

The first two line bundles are very ample, but the third is not even globally generated. The third line bundle is globally generated at the T-fixed points $x_{\sigma_{3,4}}$ and $x_{\sigma_{1,4}}$, but not at the other T-fixed points. $\Diamond$

5. CONTRASTING NOTIONS OF POSITIVITY

In this section we distinguish the ampleness of a toric vector bundle from other algebraic notions of positivity. Following Definition 6.1.1 in [L2], a vector bundle $\mathscr{E}$ on X is ample or nef if the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ on the projectivized bundle $\mathbb{P}(\mathscr{E})$ is ample or nef, respectively. Theorem 2.1 in [HMP] states that "a toric vector bundle on a complete toric variety is ample if and only if its restriction to every torus-invariant curve is ample". This provides the key tool for recognizing ample toric vector bundles.

To be more precise, consider a T-invariant curve C in X corresponding to the cone $\tau \in \Sigma(d-1)$. Since X is complete, there are two maximal cones σ and σ' in $\Sigma(d)$ that contain τ and $C \cong \mathbb{P}^1$. Given two lattice points $\mathbf{u}$ and $\mathbf{u}'$ in M that agree as linear functionals on τ, the toric line bundle $\mathscr{L}_{\mathbf{u},\mathbf{u}'}$ on the union $U_\sigma \cup U_{\sigma'}$ is constructed by gluing $\mathscr{L}_{\mathbf{u}}|_{U_\sigma}$ and $\mathscr{L}_{\mathbf{u}'}|_{U_{\sigma'}}$ via the transition function

$\chi^{\mathbf{u}-\mathbf{u}'}$, which is regular and invertible on U_τ. If the lattice vector $\mathbf{v}_\tau \in \sigma$ is dual to the primitive generator of $\tau^\perp$, then the line bundle $\mathscr{L}_{\mathbf{u},\mathbf{u}'}|_C$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(\langle \mathbf{u}, \mathbf{v}_\tau\rangle D_1 - \langle \mathbf{u}', \mathbf{v}_\tau\rangle D_2)$ where D_1 and D_2 are the irreducible T-invariant divisors on $\mathbb{P}^1$. Corollary 5.5 and Corollary 5.10 in [HMP] show that the restriction $\mathscr{E}|_C$ splits T-equivariantly into a sum of line bundles

$$\mathscr{L}_{\mathbf{u}_1,\mathbf{u}_1'}|_C \oplus \mathscr{L}_{\mathbf{u}_2,\mathbf{u}_2'}|_C \oplus \cdots \oplus \mathscr{L}_{\mathbf{u}_r,\mathbf{u}_r'}|_C$$

and the pairs $(\mathbf{u}_i, \mathbf{u}_i')$ are unique up to reordering. This pairing can be visualized as line segments parallel to $\tau^\perp$ joining the associated characters in $\boldsymbol{u}(\sigma)$ and $\boldsymbol{u}(\sigma')$. Edges in the parliament of polytopes of $\mathscr{E}$ are contained in such line segments, but these line segments may connect disjoint polytopes. For each individual summand, we have $\mathscr{L}_{\mathbf{u},\mathbf{u}'}|_C \cong \mathscr{O}_{\mathbb{P}^1}(a)$ where $\mathbf{u}-\mathbf{u}'$ is a times the primitive generator of $\tau^\perp$ that is positive on σ. Pictorially, the integer a is the normalized lattice distance between the associated characters in the one-dimensional lattice $(\tau^\perp + \mathbf{u}) \cap M$.

To demonstrate this apparatus, we re-establish that the tangent bundle on projective space is ample; compare with Remark 2.4 and Example 5.6 in [HMP].

Example 5.1. Using the notation from Example 3.8, the characters associated to the tangent bundle $\mathscr{T}_{\mathbb{P}^d}$ are

$$\begin{aligned}\boldsymbol{u}(\sigma_i) = \{\mathbf{w}_1 - \mathbf{w}_i, \mathbf{w}_2 - \mathbf{w}_i, \dots, \mathbf{w}_{i-1} - \mathbf{w}_i,& \\ -\mathbf{w}_i, \mathbf{w}_{i+1} - \mathbf{w}_i, \mathbf{w}_{i+2} - \mathbf{w}_i, \dots, \mathbf{w}_d - \mathbf{w}_i\}&\end{aligned}$$

for $1 \leqslant i \leqslant d$, and $\boldsymbol{u}(\sigma_{d+1}) = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d\}$. On the T-invariant curve $C_{i,j}$ corresponding to the cone $\tau_{i,j} := \sigma_i \cap \sigma_j \in \Sigma(d-1)$ where $1 \leqslant i < j \leqslant d$, the characters in $\boldsymbol{u}(\sigma_i)$ and $\boldsymbol{u}(\sigma_j)$ are paired as follows: $(-\mathbf{w}_i, \mathbf{w}_i - \mathbf{w}_j)$, $(\mathbf{w}_j - \mathbf{w}_i, -\mathbf{w}_j)$, and $(\mathbf{w}_k - \mathbf{w}_i, \mathbf{w}_k - \mathbf{w}_j)$ for all $k \neq i$ or j. Thus, we deduce that

$$\mathscr{T}_{\mathbb{P}^d}|_{C_{i,j}} = \mathscr{O}_{\mathbb{P}^1}(D_1 + D_2) \oplus \Big(\bigoplus_{j=1}^{d-1} \mathscr{O}_{\mathbb{P}^1}(D_2)\Big) \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \Big(\bigoplus_{j=1}^{d-1} \mathscr{O}_{\mathbb{P}^1}(1)\Big).$$

A similar calculation for the curve $C_{i,d+1}$, which corresponds to the $(d-1)$-dimensional cone $\tau_{i,d+1} := \sigma_i \cap \sigma_{d+1} \in \Sigma(d-1)$ where $1 \leqslant i \leqslant d$, yields

$$\mathscr{T}_{\mathbb{P}^d}|_{C_{i,d+1}} \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \Big(\bigoplus_{j=1}^{d-1} \mathscr{O}_{\mathbb{P}^1}(1)\Big).$$

When $d = 2$, we also see from Figure 2 that the normalized lattice distance between matched pairs of associated characters is either 1 or 2. Since the restriction to every T-invariant curve is ample, we conclude that $\mathscr{T}_{\mathbb{P}^d}$ is ample. $\Diamond$

With these tools, we can also prove directly that the cotangent bundle on a smooth toric variety is never ample; compare with [L2, Section 6.3B].

Example 5.2. Let Ω_X be the cotangent bundle on a smooth toric variety X, and let Σ be the fan of X. Identifying the fiber E over the identity of the torus T with $M \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^d$ as done in [Kl1, Section 2.3.5], the decreasing filtrations for Ω_X are

$$E^i(j) = \begin{cases} E & \text{if} \quad j \leqslant -1 \\ \mathbf{v}_i^\perp & \text{if} \quad j = 0 \\ 0 & \text{if} \quad j > 0 \end{cases} \qquad \text{for all } 1 \leqslant i \leqslant n.$$

Consider two adjacent cones $\sigma, \sigma' \in \Sigma(d)$. Since X is smooth, we have $\sigma = \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d)$ where $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d$ is a basis for N. We may assume that $\sigma' = \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d-1}, \mathbf{v}_{d+1})$ where

$$\mathbf{v}_{d+1} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{d-1}\mathbf{v}_{d-1} - \mathbf{v}_d$$

for some $a_j \in \mathbb{Z}$. If $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d \in M$ form the dual basis to $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d$, then the associated characters are

$$\begin{aligned} \boldsymbol{u}(\sigma) &= \{-\mathbf{w}_1, -\mathbf{w}_2, \dots, -\mathbf{w}_d\}\,, \\ \boldsymbol{u}(\sigma') &= \{-\mathbf{w}_1 - a_1\mathbf{w}_d, -\mathbf{w}_2 - a_2\mathbf{w}_d, \dots, -\mathbf{w}_{d-1} - a_{d-1}\mathbf{w}_{d-1}, \mathbf{w}_d\}\,. \end{aligned}$$

Along the T-invariant curve C corresponding to the cone $\tau = \sigma \cap \sigma'$, the characters are paired as $(-\mathbf{w}_d, \mathbf{w}_d)$ and $(-\mathbf{w}_i, -\mathbf{w}_i - a_i\mathbf{w}_d)$ for $1 \leqslant i \leqslant d-1$. Therefore, we obtain

$$\Omega_X|_C \cong \mathscr{O}_{\mathbb{P}^1}(-2) \oplus \Big(\bigoplus_{i=1}^{d-1} \mathscr{O}_{\mathbb{P}^1}(a_i) \Big),$$

which implies that Ω_X is not ample. $\diamondsuit$

More significantly, we next exhibit an ample toric vector bundle on a smooth toric variety that is not globally generated. In particular, this supersedes Examples 4.15–4.17 in [HMP] and answers the second part of Question 7.5 in [HMP].

Example 5.3. Consider the toric vector bundle $\mathscr{F}$ on $\mathbb{P}^2$ appearing in Example 4.2. Having already established that $\mathscr{F}$ is not globally generated, it remains to show that $\mathscr{F}$ is ample. Let C_k denote the T-invariant curve in $\mathbb{P}^2$ corresponding to the one-dimensional cone $\tau_{i,j} := \sigma_i \cap \sigma_j$, where $\{i,j,k\} = \{1,2,3\}$. From the line segments in Figure 3 joining diamonds to squares, we see that the characters in $\boldsymbol{u}(\sigma_2)$ and $\boldsymbol{u}(\sigma_3)$ are paired on C_1 as $\big((-1,3),(-1,-1)\big)$, $\big((0,0),(0,-3)\big)$, $\big((4,-2),(4,-3)\big)$, so we obtain

$$\begin{aligned} \mathscr{F}|_{C_1} &= \mathscr{O}_{\mathbb{P}^1}(3D_1 + D_2) \oplus \mathscr{O}_{\mathbb{P}^1}(3D_2) \oplus \mathscr{O}_{\mathbb{P}^1}(-2D_1 + 3D_2) \\ &\cong \mathscr{O}_{\mathbb{P}^1}(4) \oplus \mathscr{O}_{\mathbb{P}^1}(3) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\,. \end{aligned}$$

Similar calculations give both $\mathscr{F}|_{C_2} \cong \mathscr{O}_{\mathbb{P}^1}(5) \oplus \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)$ and $\mathscr{F}|_{C_3} \cong \mathscr{O}_{\mathbb{P}^1}(6) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)$. Since the restriction to every T-invariant curve is ample, the toric vector bundle $\mathscr{F}$ is ample. $\diamondsuit$

The vector bundle $\mathscr{F}$ has minimal rank among all ample toric vector bundles on $\mathbb{P}^2$ that are not globally generated. More than that, the ensuing proposition proves that, for low-rank toric vector bundles on $\mathbb{P}^d$, nef is equivalent to globally generated.

Proposition 5.4. *If $\mathscr{E}$ is a toric vector bundle on $\mathbb{P}^d$ with rank at most d, then $\mathscr{E}$ is globally generated if and only if it is nef.*

Proof. As follows from Example 1.4.5 in [L1], every globally generated vector bundle is nef, so it suffices to prove the converse implication. Moreover, a line bundle on a complete toric variety is nef if and only if it is globally generated; see Theorem 6.3.13 in [CLS]. Hence, the proposition follows immediately when $\mathscr{E}$ splits as a direct sum of line bundles. If the rank of $\mathscr{E}$ is less than d, then Corollary 3.5 in [K] or Corollary 6.1.5 in [Kl1] imply that $\mathscr{E}$ splits into a direct sum of line bundles. Therefore, we may assume that $\mathscr{E}$ is indecomposable and has rank equal to d.

Under these hypotheses, Theorem 4.6 in [K] establishes that $\mathscr{E}$ is isomorphic to either $\mathscr{Q}(\ell)$ or $\mathscr{Q}^*(\ell)$ for some $\ell \in \mathbb{Z}$, where $\mathscr{Q}$ is defined by the short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^d} \xrightarrow{\left[\begin{smallmatrix} y_1^{a_1} & y_2^{a_2} & \cdots & y_{d+1}^{a_{d+1}} \end{smallmatrix}\right]} \bigoplus_{k=1}^{d+1} \mathscr{O}_{\mathbb{P}^d}(a_k D_k) \longrightarrow \mathscr{Q} \longrightarrow 0\,,$$

where $D_1, D_2, \dots, D_{d+1}$ are the T-invariant divisors on $\mathbb{P}^d$, and $a_1, a_2, \dots, a_{d+1}$ are positive integers. Using the notation from Example 5.1, let $C_{i,j}$ denote the T-invariant curve corresponding to the cone $\tau_{i,j} = \sigma_i \cap \sigma_j$ in $\Sigma(d-1)$ where $1 \leqslant i < j \leqslant d+1$. Restricting the short exact sequence to the curve $C_{i,j}$, we obtain

$$\mathscr{Q}|_{C_{i,j}} \cong \mathscr{O}_{\mathbb{P}^1}(a_i + a_j) \oplus \Big(\bigoplus_{k=1, k\neq i,j}^{d+1} \mathscr{O}_{\mathbb{P}^1}(a_k) \Big).$$

If $\mathscr{E} = \mathscr{Q}(\ell)$ and $\mathscr{E}$ is nef, then we have $a_k + \ell \geqslant 0$ for all $1 \leqslant k \leqslant d+1$ which means that the vector bundle $\mathcal{S} := \bigoplus_{k=1}^{d+1} \mathscr{O}_{\mathbb{P}^d}(a_k + \ell)$ is globally generated. Since $\mathscr{E}$ is a quotient of $\mathcal{S}$, we conclude that $\mathscr{E}$ is also globally generated; see Example 6.1.4 in [L2]. If $\mathscr{E} = \mathscr{Q}^*(\ell)$ and $\mathscr{E}$ is nef, then we have $\ell - a_k \geqslant 0$ for all $1 \leqslant k \leqslant d+1$ and $\ell - a_i - a_j \geqslant 0$ for all $1 \leqslant i < j \leqslant d+1$. The functorial properties of the dual imply that $\mathscr{Q}^*(\ell) \hookrightarrow \bigoplus_{k=1}^{d+1} \mathscr{O}_{\mathbb{P}^d}(\ell - a_k)$ and

$$\mathscr{Q}^*(\ell) \cong \Big(\bigwedge^{d-1} \mathscr{Q}^*(\ell) \Big)^* \otimes \det\big(\mathscr{Q}^*(\ell)\big)\,.$$

It follows that $\mathscr{E}$ is a quotient of the vector bundle

$$\mathcal{S}' := \Big(\bigwedge^{d-1} \big(\bigoplus_{k=1}^{d+1} \mathscr{O}_{\mathbb{P}^d}(\ell - a_k) \big) \Big)^* \otimes \det\big(\mathscr{Q}^*(\ell)\big)\,.$$

Since we have

$$\bigwedge^{d-1} \big(\bigoplus_{k=1}^{d+1} \mathscr{O}_{\mathbb{P}^d}(\ell - a_k) \big) \cong \bigoplus_{1 \leqslant k_1 < k_2 < \cdots < k_{d-1} \leqslant d+1} \mathscr{O}_{\mathbb{P}^d}\big((d-1)\ell - a_{k_1} - a_{k_2} - \cdots - a_{k_{d-1}} \big)$$

and $\det\big(\mathscr{Q}^*(\ell)\big) \cong \mathscr{O}_{\mathbb{P}^d}(d\ell - a_1 - a_2 - \cdots - a_{d+1})$, we see that $\mathcal{S}'$ is a direct sum of line bundles of the form $\mathscr{O}_{\mathbb{P}^d}(\ell - a_j - a_k)$ which implies that both $\mathcal{S}'$ and $\mathscr{E}$ are globally generated. $\square$

To complement Examples 4.9–4.10 in [HMP], we end this section by illustrating that the higher cohomology groups of a globally generated ample toric vector bundle on a smooth toric variety may be nonzero.

Example 5.5. Consider the globally generated toric vector bundle $\mathscr{G}$ appearing in Example 4.4. Restricting to the T-invariant curves gives

$$\mathscr{G}|_{C_1} \cong \mathscr{O}_{\mathbb{P}^1}(5) \oplus \mathscr{O}_{\mathbb{P}^1}(2)\,, \qquad \mathscr{G}|_{C_3} \cong \mathscr{O}_{\mathbb{P}^1}(6) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\,,$$
$$\mathscr{G}|_{C_2} \cong \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\,, \qquad \mathscr{G}|_{C_4} \cong \mathscr{O}_{\mathbb{P}^1}(8) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\,,$$

and shows that $\mathscr{G}$ is ample. Furthermore, Theorem 4.2.1 in [Kl1] establishes that the T-equivariant Euler characteristic of $\mathscr{G}$ is

$$\begin{aligned}\chi(\mathscr{G}) &= \sum_i (-1)^i \dim H^i(X,\mathscr{G})_{\mathbf{u}} \cdot t^{\mathbf{u}} \\ &= \tfrac{t_1^{-2}t_2^2+t_1^4t_2^3}{(1-t_1)(1-t_2)} + \tfrac{t_1^{-3}t_2^2+t_1^3t_2^3}{(1-t_1)(1-t_1^{-1}t_2^{-1})} + \tfrac{t_1^{-4}t_2+t_1^{-3}t_2^{-3}}{(1-t_1)(1-t_1t_2)} + \tfrac{t_1^{-2}t_2^{-3}+t_1^4t_2}{(1-t_1^{-1})(1-t_2)} \\ &= t_1^4t_2^3 + t_1^4t_2^2 + t_1^4t_2 + t_1^3t_2^3 + t_1^3t_2^2 + t_1^3t_2 \\ &\quad + t_1^2t_2^2 + t_1^2t_2 + t_1t_2 - t_1^{-1} + t_1^{-2}t_2^2 + t_1^{-2}t_2 \\ &\quad + t_1^{-2}t_2^{-2} + t_1^{-2}t_2^{-3} + t_1^{-3}t_2^2 + t_1^{-3}t_2 + t_1^{-3}t_2^{-3} + t_1^{-4}t_2\,,\end{aligned}$$

so we have $H^1(X,\mathscr{G})_{(-1,0)} \neq 0$. Using Theorem 4.1.1 in [Kl1], a longer calculation confirms that we have $H^1(X,\mathscr{G})_{\mathbf{u}} \cong \mathbb{C}$ when $\mathbf{u} = (-1,0)$ and $H^1(X,\mathscr{G})_{\mathbf{u}} = 0$ when $\mathbf{u} \neq (-1,0)$. In Figure 4, the triangle represents the unique character for which the higher cohomology groups do not vanish. $\diamondsuit$

Remark 5.6. Using the techniques from Example 5.5 or Example 4.3.5 in [Kl1], we see that $H^1(\mathbb{P}^2,\mathscr{F})_{\mathbf{u}} \neq 0$ where $\mathbf{u} = (1,-1)$ and $\mathscr{F}$ is the toric vector bundle appearing in Example 4.2. In Figure 3, the triangle represents the unique character for which the higher cohomology groups do not vanish.

6. Higher-order jets

This final section relates positivity of higher-order jets to properties of the associated parliament of polytopes. In particular we determine which results for jets of line bundles on smooth toric varieties extend to higher-rank toric vector bundles. For toric vector bundles we also provide an explicit polyhedral characterization for very ampleness.

Fix $\ell \in \mathbb{N}$. A vector bundle $\mathscr{E}$ *separates ℓ-jets* if, for every closed point $x \in X$ with maximal ideal $\mathfrak{m}_x \subseteq \mathscr{O}_X$, the map

$$J_x^\ell \colon H^0(X,\mathscr{E}) \to H^0(X, \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathfrak{m}_x^{\ell+1})\,,$$

which evaluates a global section and its derivatives of order at most ℓ at x, is surjective; compare with Definition 5.1.15 in [L1]. When X is a toric variety, this map is T-equivariant, because differentiation is $\mathbb{C}$-linear. As a special case, we see that a vector bundle separates 0-jets if and only if it is globally generated. A vector bundle that separates ℓ-jets is also called *ℓ-jet spanned*.

As a stronger attribute, we say that a vector bundle $\mathscr{E}$ is *ℓ-jet ample* if, for all distinct closed points $x_1, x_2, \dots, x_t \in X$ and for all positive integers $\ell_1, \ell_2, \dots, \ell_t$ satisfying $\sum_{i=1}^t \ell_i = \ell + 1$, the natural map

$$\psi \colon H^0(X,\mathscr{E}) \to \bigoplus_{i=1}^t H^0(X, \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathfrak{m}_{x_i}^{\ell_i})$$

is surjective. Hence, an ℓ-jet ample vector bundle does separate ℓ-jets, and a vector bundle separates 0-jets if and only if it is 0-jet ample. Proposition 4.2 in [BDRS] proves that every 1-jet ample vector bundle on a smooth projective variety is very ample, and Example 4.3 in [BDRS] shows that the converse does not always hold. If $0 \leqslant m \leqslant \ell$, then a vector bundle that separates ℓ-jets also separates m-jets, and a vector bundle that is ℓ-jet ample is also m-jet ample.

We start by placing ampleness into this hierarchy of positivity properties on a smooth toric variety.

Lemma 6.1. *Every toric vector bundle that separates* 1*-jets is ample.*

Proof. Let $\mathscr{E}$ be a toric vector bundle that separates 1-jets. For any T-invariant curve C, the restriction $\mathscr{E}|_C$ separates 1-jets and splits T-equivariantly into sum of line bundles. For a line bundle on a toric variety, Theorem 4.2 in [DR] shows that separating 1-jets is equivalent to being ample. Hence, if $\mathscr{E}|_C \cong \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \mathscr{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(a_r)$, then each line bundle $\mathscr{O}_{\mathbb{P}^1}(a_i)$ is ample. Therefore, the restriction to every T-invariant curve is ample, which ensures that $\mathscr{E}$ is ample; see Theorem 2.1 in [HMP]. $\square$

We next characterize the toric vector bundles that separate ℓ-jets by enhancing Theorem 1.2.

Theorem 6.2. *A toric vector bundle $\mathscr{E}$ separates ℓ-jets, for $\ell \geqslant 1$, if and only if, for all maximal cones $\sigma \in \Sigma(d)$, the following hold:*

(i) *the associated characters $\mathbf{u}(\sigma)$ are vertices of polytopes in the parliament for $\mathscr{E}$,*
(ii) *the edges adjacent to these vertices correspond to the generators of the dual cone $\sigma^\vee$,*
(iii) *the edges adjacent to these vertices have normalized length at least ℓ, and*
(iv) *the elements indexing these polytopes form a basis in the matroid $\mathrm{M}(\mathscr{E})$.*

If we ignore the conditions on the edges, then we recover Theorem 1.2 which characterizes toric vector bundles that separate 0-jets.

Proof. The locus in the toric variety X, on which

$$H^0(X, \mathscr{E}) \to H^0(X, \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X / \mathfrak{m}_x^{\ell+1})$$

is not surjective, is closed and T-invariant. Since X is complete, it follows that $\mathscr{E}$ separates ℓ-jets if and only if it separates ℓ-jets at the T-fixed points.

Fix a maximal cone $\sigma = \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d)$ in the fan Σ, let

$$\underline{\boldsymbol{u}}(\sigma) = \{\mathbf{u}_{\sigma,1}, \mathbf{u}_{\sigma,2}, \dots, \mathbf{u}_{\sigma,s}\}$$

be the underlying set of associated characters, and let

$$\mathcal{B}_\sigma = \mathcal{B}_{\sigma,1} \sqcup \mathcal{B}_{\sigma,2} \sqcup \cdots \sqcup \mathcal{B}_{\sigma,s}$$

be the corresponding partition of a compatible basis in the matroid $\mathrm{M}(\mathscr{E})$; see section 4. The vector bundle $\mathscr{E}$ separates ℓ-jets at the T-fixed point x_σ if and only if the natural map

$$J^\ell_{x_\sigma} : H^0(X, \mathscr{E}) \to H^0(X, \mathscr{E} \otimes \mathscr{O}_X / \mathfrak{m}_{x_\sigma}^{\ell+1}) \cong \operatorname{span}(\mathcal{B}_\sigma) \otimes_{\mathbb{C}} \mathbb{C}^{\binom{\ell+d}{d}}$$

is surjective, where the standard basis for the vector space $\mathbb{C}^{\binom{\ell+d}{d}}$ corresponds to the partial derivatives of order less than ℓ. Since a T-equivariant global section $\mathbf{e}' \otimes \chi^{-\mathbf{u}}$ is given in local coordinates near x_σ by $(*)$, the map $J^\ell_{x_\sigma}$ sends $\mathbf{e}' \otimes \chi^{-\mathbf{u}}$ to the first ℓ terms of the Taylor expansion about x_σ. Hence, for $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ satisfying $m_1 + m_2 + \cdots + m_d \leqslant \ell$, the $\mathbf{m}$th component of $J^\ell_{x_\sigma}(\mathbf{e}' \otimes \chi^{-\mathbf{u}})$ is given in local coordinates by

$$\sum_{\mathbf{e} \in \mathcal{B}_\sigma} \lambda_{\mathbf{e}} \left(\mathbf{e} \otimes \frac{1}{\mathbf{m}!} \frac{\partial^{m_1+m_2+\cdots+m_d}}{\partial^{m_1} y_1 \partial^{m_2} y_2 \cdots \partial^{m_d} y_d} \prod_{i=1}^{d} y_i^{-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\ell,\sigma}(\mathbf{v}_i)} \right) \Bigg|_{y_1 = y_2 = \cdots = y_d = 0},$$

where $\mathbf{m}! = m_1! m_2! \cdots m_d!$. The $\mathbf{e}$th summand in this expression has neither a zero nor a pole at $(y_1, y_2, \dots, y_d) = (0, 0, \dots, 0)$ if and only if $-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\mathbf{e}}(\mathbf{v}_i) = m_i$ for all $1 \leqslant i \leqslant d$. By Lemma 4.1, it follows that there exists an index k such that $\mathbf{e} \in \mathcal{B}_{\sigma,k}$ and $\mathbf{u} = \mathbf{u}_{\sigma,k} + \mathbf{m}$. In this case, the lattice point $\mathbf{u}_{\sigma,k} - \sum_{i=1}^{d} m_i \mathbf{w}_i$, where $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d$ are the unique minimal generators of the dual cone $\sigma^\vee$, belongs to the polytope $P_{\mathbf{e}}$ in the parliament for $\mathscr{E}$. Hence, the $\mathbf{m}$th component of $J^\ell_{x_\sigma}(\mathbf{e} \otimes \chi^{-\mathbf{u}})$ is nonzero if and only if the global section includes summands of the form $\sum_{\mathbf{e} \in \mathcal{B}_{\sigma,k}} \lambda_{\mathbf{e}} (\mathbf{e} \otimes \chi^{-\mathbf{u}_{\sigma,k} - \mathbf{m}})$, which map to $\sum_{\mathbf{e} \in \mathcal{B}_{\sigma,k}} \lambda_{\mathbf{e}} \, \mathbf{e}$. For the map $J^\ell_{x_\sigma}$ to be surjective, we need each vector $\mathbf{e} \in \mathcal{B}_\sigma$ to appear in each component of factor $\mathbb{C}^{\binom{\ell+d}{d}}$. Therefore, the map $J^\ell_{x_\sigma}$ is surjective if and only if there exists a compatible basis $\mathcal{B}_\sigma$ such that each element $\mathbf{e} \otimes \chi^{-\mathbf{u}_{\sigma,k} - m_1 \mathbf{w}_1 - m_2 \mathbf{w}_2 - \cdots - m_d \mathbf{w}_d}$, for $1 \leqslant k \leqslant s$, $\mathbf{e} \in \mathcal{B}_{\sigma,k}$, and $\mathbf{m} \in \mathbb{N}^d$ satisfying $m_1 + m_2 + \cdots + m_d \leqslant \ell$, is a global section. By convexity, this characterization is equivalent to requiring that the edges through $\mathbf{u}_{\sigma,k}$ in the directions of dual vectors $\mathbf{w}_i$ have normalized length at least ℓ. $\square$

With Theorem 6.2, we easily verify that the tangent bundle on projective space separates 1-jets.

Example 6.3. As computed in Example 3.8, the parliament of polytopes for the tangent bundle $\mathscr{T}_{\mathbb{P}^d}$ consists of

$$P_{\mathbf{v}_i} = \operatorname{conv} \begin{pmatrix} \mathbf{0}, & \mathbf{w}_i - \mathbf{w}_1, & \mathbf{w}_i - \mathbf{w}_2, & \dots, & \mathbf{w}_i - \mathbf{w}_{i-1}, \\ \mathbf{w}_i, & \mathbf{w}_i - \mathbf{w}_{i+1}, & \mathbf{w}_i - \mathbf{w}_{i+1}, & \dots, & \mathbf{w}_i - \mathbf{w}_d \end{pmatrix}$$

for $1 \leqslant i \leqslant d$, and $P_{\mathbf{v}_{d+1}} = \operatorname{conv}(\mathbf{0}, -\mathbf{w}_1, -\mathbf{w}_2, \dots, -\mathbf{w}_d)$. Hence, the associated characters are vertices of polytopes in the parliament, the edges in each polytope have normalized length 1 and point in directions corresponding to generators of the dual cone, and the elements indexing these polytopes equal the unique choice of compatible basis. Therefore, the tangent bundle $\mathscr{T}_{\mathbb{P}^d}$ separates 1-jets. $\diamondsuit$

Since Example 4.2 exhibits an ample toric vector bundle that is not globally generated, the converse to Lemma 6.1 is false. To sharpen this distinction, we present an ample toric vector bundle that is globally generated but does not separate 1-jets.

Example 6.4. Using the notation from Examples 4.2 and 5.3, consider the toric vector bundle $\mathscr{H}$ of rank 3 on $\mathbb{P}^2$ defined by the decreasing filtrations

$$E^1(j) = \begin{cases} E & \text{if } \quad j \leqslant -2, \\ \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2) & \text{if } -2 < j \leqslant -1, \\ \operatorname{span}(\mathbf{e}_1) & \text{if } -1 < j \leqslant 2, \\ 0 & \text{if } \quad 2 < j, \end{cases}$$

$$E^2(j) = \begin{cases} E & \text{if } \quad j \leqslant -2, \\ \operatorname{span}(\mathbf{e}_2, \mathbf{e}_3) & \text{if } -2 < j \leqslant 0, \\ \operatorname{span}(\mathbf{e}_3) & \text{if } \quad 0 < j \leqslant 2, \\ 0 & \text{if } \quad 2 < j, \end{cases}$$

$$E^3(j) = \begin{cases} E & \text{if } \quad j \leqslant 1, \\ \operatorname{span}(\mathbf{e}_3 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2) & \text{if } \quad 1 < j \leqslant 3, \\ \operatorname{span}(\mathbf{e}_1 - \mathbf{e}_2) & \text{if } \quad 3 < j \leqslant 4, \\ 0 & \text{if } \quad 4 < j. \end{cases}$$

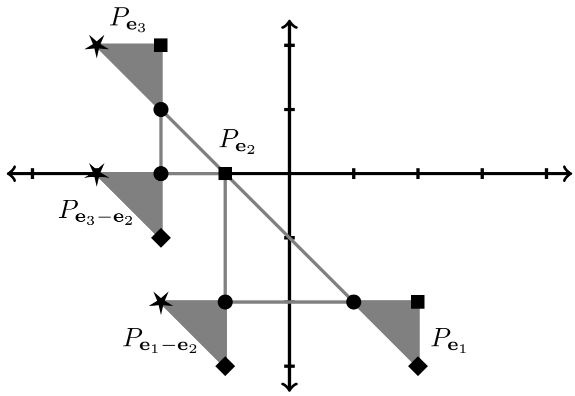

FIGURE 5. The parliament of polytopes for $\mathscr{H}$

It follows that $\{\mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3\}$ is the ground set of the matroid $\mathrm{M}(\mathscr{H})$. On each maximal cone, the associated characters and the unique choice of compatible bases are

$$\begin{aligned} \boldsymbol{u}(\sigma_1) &= \{(-2,-2),(-3,0),(-3,2)\}\,, & \mathcal{B}_{\sigma_1} &= \{\mathbf{e}_1-\mathbf{e}_2, \mathbf{e}_2-\mathbf{e}_3, \mathbf{e}_3\}\,,\\ \boldsymbol{u}(\sigma_2) &= \{(2,-3),(-1,-3),(-2,-1)\}\,, & \mathcal{B}_{\sigma_2} &= \{\mathbf{e}_1, \mathbf{e}_1-\mathbf{e}_2, \mathbf{e}_2-\mathbf{e}_3\}\,,\\ \boldsymbol{u}(\sigma_3) &= \{(2,-2),(-1,0),(-2,2)\}\,, & \mathcal{B}_{\sigma_3} &= \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\,, \end{aligned}$$

so the convex polytopes in the parliament for $\mathscr{H}$ are

$$\begin{aligned} P_{\mathbf{e}_1} &= \operatorname{conv}\big((1,-2),(2,-2),(2,-3)\big)\,,\\ P_{\mathbf{e}_1-\mathbf{e}_2} &= \operatorname{conv}\big((-2,-2),(-1,-2),(-1,-3)\big)\,,\\ P_{\mathbf{e}_2} &= \operatorname{conv}\big((-1,0)\big)\,,\\ P_{\mathbf{e}_2-\mathbf{e}_3} &= \operatorname{conv}\big((-3,0),(-2,0),(-2,-1)\big)\,,\\ P_{\mathbf{e}_3} &= \operatorname{conv}\big((-3,2),(-2,2),(-2,1)\big)\,. \end{aligned}$$

In Figure 5, the associated characters are represented by stars, diamonds, and squares, respectively. The polytopes are represented by shaded regions and the other lattices points lying in the polytopes are represented by circles. Using Theorem 1.2, we see that $\mathscr{H}$ is globally generated. In contrast, Theorem 6.2 implies that $\mathscr{H}$ does not separate 1-jets because $P_{\mathbf{e}_2}$ is simply a point. Lastly, restricting to the T-invariant curves gives

$$\begin{aligned} \mathscr{H}|_{C_1} &\cong \mathscr{O}_{\mathbb{P}^1}(3) \oplus \mathscr{O}_{\mathbb{P}^1}(3) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\,,\\ \mathscr{H}|_{C_2} &\cong \mathscr{O}_{\mathbb{P}^1}(4) \oplus \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\,,\\ \mathscr{H}|_{C_3} &\cong \mathscr{O}_{\mathbb{P}^1}(5) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\,, \end{aligned}$$

so the toric vector bundle is ample. $\Diamond$

On a smooth projective variety, being 1-jet ample is generally a stronger condition than separating 1-jets, as Example 2.3 in [LM] and Example 4.6 in [L] demonstrate for line bundles. For line bundles on a smooth complete toric variety, these conditions are equivalent; see [DR]. Extending this result, we prove that these conditions are equivalent for toric vector bundles on a smooth complete toric variety.

Theorem 6.5. *A toric vector bundle separates ℓ-jets if and only if it is ℓ-jet ample.*

Proof. It suffices to show that every toric vector bundle $\mathscr{E}$ which separates ℓ-jets is ℓ-jet ample. The locus in the toric variety $\prod_{i=1}^{t} X$, on which $H^0(X, \mathscr{E}) \to \bigoplus_{i=1}^{t} H^0(X, \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X / \mathfrak{m}_{x_i}^{\ell_i})$ is not surjective, is closed and T-invariant. Since X is complete, it follows that $\mathscr{E}$ is ℓ-jet ample if and only if it is ℓ-jet ample at the T-fixed points. Thus, it is enough to prove that, for all distinct T-fixed points $x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_t}$ and all positive integers $\ell_1, \ell_2, \ldots, \ell_t$ satisfying $\sum_{i=1}^{t} \ell_i = \ell + 1$, the map $\psi \colon H^0(X, \mathscr{E}) \to \bigoplus_{i=1}^{t} H^0(X, \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X / \mathfrak{m}_{x_{\sigma_i}}^{\ell_i})$ is surjective.

Since $\mathscr{E}$ separates ℓ_i-jets, for all $\ell_i \leqslant \ell$, the map ψ surjects onto each individual summand. Consider a T-equivariant global section $\mathbf{e} \otimes \chi^{-\mathbf{u}}$ where the element $\mathbf{e}$ belongs to the ground set of the matroid $\mathrm{M}(\mathscr{E})$ and $0 \neq J_{x_{\sigma_1}}^{\ell_1 - 1}(\mathbf{e} \otimes \chi^{-\mathbf{u}}) \in H^0(X, \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X / \mathfrak{m}_{x_{\sigma_1}}^{\ell_1})$. To prove that ψ is surjective, it is enough to show that $J_{x_{\sigma_2}}^{\ell_2 - 1}(\mathbf{e} \otimes \chi^{-\mathbf{u}}) = 0$ because we may re-index the T-fixed points. As in the proof of Theorem 6.2, the hypothesis $J_{x_{\sigma_1}}^{\ell_1 - 1}(\mathbf{e} \otimes \chi^{-\mathbf{u}}) \neq 0$ implies that the global section $\mathbf{e} \otimes \chi^{-\mathbf{u}}$ corresponds to the lattice $\mathbf{u} \in P_{\mathbf{e}}$ and the lattice distance from the vertex $\mathbf{u}_{\sigma_1}$ of $P_{\mathbf{e}}$ associated to the maximal cone σ_1 is at most $\ell_1 - 1$. Similarly, if $J_{x_{\sigma_2}}^{\ell_2 - 1}(\mathbf{e} \otimes \chi^{-\mathbf{u}}) \neq 0$, then the lattice distance from $\mathbf{u} \in P_{\mathbf{e}}$ to the vertex $\mathbf{u}_{\sigma_2}$ of $P_{\mathbf{e}}$ associated to the maximal cone σ_2 would also be at most $\ell_2 - 1$. As $\mathscr{E}$ separates ℓ-jets at x_{σ_1}, Theorem 6.2 implies that the lattice length of each edge in $P_{\mathbf{e}}$ emanating from the vertex corresponding to σ_1 is at least ℓ. Since $\ell_1 + \ell_2 - 2 \leqslant \ell - 1$, the convexity of $P_{\mathbf{e}}$ guarantees that the lattice point $\mathbf{u}$ cannot be simultaneously this close to both $\mathbf{u}_{\sigma_1}$ and $\mathbf{u}_{\sigma_2}$. Thus, we conclude that $J_{x_{\sigma_2}}^{\ell_2 - 1}(\mathbf{e} \otimes \chi^{-\mathbf{u}}) = 0$ and ψ is surjective. $\square$

For a line bundle on a smooth toric variety, Theorem 4.2 in [DR] establishes that separating 1-jets is equivalent to being very ample. As a final result, we generalize this equivalence to higher-rank toric vector bundles on a smooth toric variety.

Theorem 6.6. *A toric vector bundle separates 1-jets if and only if it is very ample.*

Proof. It suffices to show that every very ample toric vector bundle $\mathscr{E}$ separates 1-jets at the T-fixed points. Let X be the smooth toric variety determined by the fan Σ. Fix a maximal cone $\sigma_0 \in \Sigma(d)$, and consider the blowing up $\pi \colon X' \to X$ of X at x_{σ_0} with exceptional divisor $D_0 := \pi^{-1}(x_{\sigma_0})$. Since $\mathscr{E}$ is very ample, Corollary 1 in [BSS] establishes that the toric vector bundle $\mathscr{E}' := \pi^*(\mathscr{E}) \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}(-D_0)$ is globally generated.

To complete the proof, we relate the parliament of polytopes for $\mathscr{E}'$ and $\mathscr{E}$. First, we describe the underlying fan for X'. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the primitive lattice vectors generating the rays in Σ. By reordering these rays if necessary, we may assume that $\sigma_0 = \operatorname{pos}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d)$. If $\mathbf{v}_0 := \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_d$ and Σ' is the fan of X', then the primitive lattice vectors generating the rays in Σ' are $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$, and the maximal cones are $\Sigma'(d) = \big(\Sigma(d) \setminus \sigma_0\big) \cup \{\sigma_1, \sigma_2, \ldots, \sigma_d\}$, where

$$\sigma_i := \operatorname{pos}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \ldots, \mathbf{v}_d)$$

for $1 \leqslant i \leqslant d$; compare with Example 3.1.15 in [CLS].

We next specify the linear invariants which determine the toric vector bundle $\mathscr{E}'' := \pi^*(\mathscr{E})$ on X'. The characters associated to $\mathscr{E}''$ are $\boldsymbol{u}_{\mathscr{E}''}(\sigma') = \boldsymbol{u}_{\mathscr{E}}(\sigma')$ for all

$\sigma' \in \Sigma(d) \setminus \sigma_0$ and $\boldsymbol{u}_{\mathscr{E}''}(\sigma_i) = \boldsymbol{u}_{\mathscr{E}}(\sigma_0)$ for all $1 \leqslant i \leqslant d$. The compatible decreasing filtrations corresponding to $\mathscr{E}''$ are identical to those for $\mathscr{E}$ along the rays generated by $\mathbf{v}_i$ for $1 \leqslant i \leqslant n$. Along the new ray generated by $\mathbf{v}_0$, the decreasing filtration for $\mathscr{E}''$ is $E''^{\mathbf{v}_0}(j) = \sum_{\langle \mathbf{u}, \mathbf{v}_0 \rangle \geqslant j} E_{\mathbf{u}}^{\sigma_0}$, where $E_{\mathbf{u}}^{\sigma_0}$ is the linear subspace associated to $\mathscr{E}$; see Section 2. It follows that $\mathcal{B}(\mathscr{E}'')_{\sigma_i} = \mathcal{B}(\mathscr{E})_{\sigma_0}$ for all $1 \leqslant i \leqslant d$.

To analyze $\mathscr{E}'$, set $\mathscr{L} := \mathscr{O}_{X'}(-D_0)$. Writing $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d \in M$ for the minimal generators the dual cone $\sigma^\vee$, Example 3.5 shows that the characters associated to the line bundle $\mathscr{L}$ are $\boldsymbol{u}_{\mathscr{L}}(\sigma') = \{\mathbf{0}\}$ for all $\sigma' \in \Sigma(d) \setminus \sigma_0$ and $\boldsymbol{u}_{\mathscr{L}}(\sigma_i) = \{\mathbf{w}_i\}$ for all $1 \leqslant i \leqslant d$. Combining this data with that for $\mathscr{E}''$, we see that the characters associated to toric vector bundle $\mathscr{E}' = \mathscr{E}'' \otimes_{\mathscr{O}_X} \mathscr{L}$ are $\boldsymbol{u}_{\mathscr{E}'}(\sigma') = \boldsymbol{u}_{\mathscr{E}}(\sigma')$ for all $\sigma' \in \Sigma(d) \setminus \sigma_0$ and $\boldsymbol{u}_{\mathscr{E}'}(\sigma_i) = \{\mathbf{u} + \mathbf{w}_i : \mathbf{u} \in \boldsymbol{u}_{\mathscr{E}}(\sigma_0)\}$ for all $1 \leqslant i \leqslant d$. As $\mathscr{L}$ is a line bundle, we also have $\mathcal{B}(\mathscr{E}')_{\sigma_i} = \mathcal{B}(\mathscr{E}'')_{\sigma_i} = \mathcal{B}(\mathscr{E})_{\sigma_0}$ for all $1 \leqslant i \leqslant d$. If $\mathbf{u}_\ell \in \boldsymbol{u}_{\mathscr{E}}(\sigma_0)$ corresponds to the vector $\mathbf{e}_{\ell,\sigma_0} \in E$, then the element $\mathbf{u}_\ell + \mathbf{w}_i \in \boldsymbol{u}_{\mathscr{E}'}(\sigma_i)$ corresponds to the same vector in E. Since $\mathscr{E}'$ is globally generated and $\mathscr{E}$ is very ample, Theorem 1.2 implies that $\mathbf{u}_\ell + \mathbf{w}_i \in P_{\mathbf{e}_{\ell,\sigma_0}}$ and $\mathbf{u}_\ell \in P_{\ell,\sigma_0}$ for all $\mathbf{u}_\ell \in \boldsymbol{u}_{\mathscr{E}}(\sigma_0)$ and for all $1 \leqslant i \leqslant d$. Applying Theorem 6.2, we conclude that $\mathscr{E}$ separates 1-jets. □

An a consequence, we have a polyhedral characterization for very ample toric vector bundles.

Corollary 6.7. *A toric vector bundle $\mathscr{E}$ is very ample if and only if, for all maximal cones $\sigma \in \Sigma(d)$, the following conditions hold:*

(i) *the associated characters $\boldsymbol{u}(\sigma)$ are vertices of polytopes in the parliament for $\mathscr{E}$,*
(ii) *the edges adjacent to these vertices correspond to the generators of the dual cone $\sigma^\vee$,*
(iii) *the edges adjacent to these vertices have lattice length at least 1, and*
(iv) *the elements indexing these polytopes form a basis in the matroid $\mathrm{M}(\mathscr{E})$.*

Proof. Combine Theorem 6.2 and Theorem 6.6. □

Proof of Theorem 1.3. This follows immediately by combining Theorem 6.5 and Theorem 6.6. □

Remark 6.8. Combining Example 6.4 with Theorem 6.6, we see that the toric vector bundle $\mathscr{H}$ is globally generated and ample but not very ample, answering the first part of Question 7.5 in [HMP]. Modifying the proof of Proposition 5.4 by replacing some nonstrict inequalities with strict inequalities, we also obtain a partial converse to Lemma 6.1: If $\mathscr{E}$ is a toric vector bundle on $\mathbb{P}^d$ with rank at most d, then $\mathscr{E}$ is ample if and only if it separates 1-jets. Hence, $\mathscr{H}$ has minimal rank among all globally generated ample toric vector bundles on $\mathbb{P}^2$ that are not very ample.

Acknowledgments

We thank Alex Fink, Milena Hering, Nathan Ilten, Diane Maclagan, Bernt Ivar Utstøl Nødland, Sam Payne, Vic Reiner, Mike Roth, and Frank Sottile for helpful conversations. We especially thank an anonymous referee for wonderfully constructive feedback and for suggesting Example 3.7.

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