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Laurent polynomials and Eulerian numbers

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ABSTRACT

Duistermaat and van der Kallen show that there is no nontrivial complex Laurent polynomial all of whose powers have a zero constant term. Inspired by this, Sturmfels poses two questions: Do the constant terms of a generic Laurent polynomial form a regular sequence? If so, then what is the degree of the associated zero-dimensional ideal? In this note, we prove that the Eulerian numbers provide the answer to the second question. The proof involves reinterpreting the problem in terms of toric geometry.

1. Motivation and statement of theorem

In [6], J.J. Duistermaat and W. van der Kallen establish that, for any Laurent polynomial $f \in \mathbb{C}[z, z^{-1}]$ that is neither a polynomial in z nor z^{-1}, there exists a positive power of f that has a nonzero constant term. Motivated by this result, Sturmfels [15, §2.5] asks for an effective version: Can we enumerate the Laurent polynomials that have the longest possible sequence of powers with zero constant terms?

By rephrasing this question in the language of commutative algebra, Sturmfels also offers a two-step approach for answering it. Specifically, consider the Laurent polynomial

$$f(z) := z^{-m} + x_{-m+1} z^{-m+1} + \cdots + x_{n-1} z^{n-1} + z^n \tag{1}$$

and, for any positive integer i, let $[\![f^i]\!]$ denote the constant coefficient of the i-th power of f. First, Problem 2.11 in [15, §2.5], together with computational evidence, suggests the following:

Conjecture 1. *The coefficients $[\![f^1]\!], [\![f^2]\!], \ldots, [\![f^{m+n}]\!]$ generate the unit ideal in the polynomial ring $\mathbb{C}[x_{-m+1}, \ldots, x_{n-1}]$.*

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Second, assuming this conjecture, Exercise 13 in [15, §2.6] asks for the degree of the ideal $I_{m,n} := \langle [\![f^1]\!], [\![f^2]\!], \ldots, [\![f^{m+n-1}]\!] \rangle$. The zeros of $I_{m,n}$ would be the Laurent polynomials of the form (1) that have the longest possible sequence of powers with vanishing constant terms.

The goal of this article is to complete the second part. Theorem 2 provides the unexpected and attractively simple answer. Following [9, §6.2], the Eulerian number $\left\langle {n \atop k} \right\rangle$ is the number of permutations of $\{1, \ldots, n\}$ with exactly k ascents.

Theorem 2. *If Conjecture* 1 *holds, then the degree of the ideal $I_{m,n}$ is $\left\langle {m+n-1 \atop m-1} \right\rangle$.*

This result is equivalent to saying that the dimension of the $\mathbb{C}$-vector space $\mathbb{C}[x_{-m+1}, \ldots, x_{n-1}]/I_{m,n}$ is $\left\langle {m+n-1 \atop m-1} \right\rangle$.

Notably, Theorem 2 gives a new interpretation for the Eulerian numbers: $\left\langle {m+n-1 \atop m-1} \right\rangle$ enumerates certain Laurent polynomials. Even without Conjecture 1, we show that these Eulerian numbers count the solutions to certain systems of polynomial equations; see Proposition 4. Despite superficial similarities between our work and other appearances of Eulerian numbers in algebraic geometry (e.g. [1–3,11,13,14]), we know of no substantive connection.

Our proof of Theorem 2, given in Section 2, recasts the problem in terms of toric geometry—we construe the degree of $I_{m,n}$ as an intersection number on a toric compactification of the space of Laurent polynomials of the form (1). Building on this idea, Section 3 provides a recursive formula for the degree of ideals similar to $I_{m,n}$ that arise from sparse Laurent polynomials. As a by-product, we give a geometric explanation for a formula expressing $\left\langle {m+n-1 \atop m-1} \right\rangle$ as a sum of nonnegative integers; see (3). We list several questions arising from our work in Section 4.

2. Toric reinterpretation

This section proves Theorem 2 by reinterpreting the degree of $I_{m,n}$ as an intersection number on a projective variety $X(m,n)$. Section 2.1 introduces a homogenization of the ideal $I_{m,n}$, Section 2.2 describes the toric variety $X(m,n)$, and Section 2.3 computes the required intersection number.

2.1. Homogenization

For positive integers m and n, consider the Laurent polynomial

$$\tilde{f} := x_{-m} z^{-m} + x_{-m+1} z^{-m+1} + \cdots + x_{n-1} z^{n-1} + x_n z^n,$$

and, for any positive integer i, let $[\![\tilde{f}^i]\!]$ denote the constant coefficient of the i-th power of $\tilde{f}$. Let S be the polynomial ring $\mathbb{C}[x_{-m}, \ldots, x_n]$ and let J be the S-ideal $\langle [\![\tilde{f}^1]\!], [\![\tilde{f}^2]\!], \ldots, [\![\tilde{f}^{m+n-1}]\!] \rangle$. The $\mathbb{C}$-valued points of $\mathrm{V}(J) \subset \mathbb{A}^{m+n+1}$ are precisely the Laurent polynomials for which the constant term of the first $m+n-1$ powers vanishes. Since J is contained in the reduced monomial ideal $B := \langle x_{-m}, \ldots, x_{-1} \rangle \cap \langle x_0, x_1, \ldots, x_n \rangle$, the $\mathbb{C}$-valued points of $\mathrm{V}(J)$ not contained in $\mathrm{V}(B)$ give rise to Laurent polynomials that are neither polynomials in z nor z^{-1}.

To understand the ideal J more explicitly, let $\boldsymbol{w} := [\, -m \ \cdots \ n \,]^{\mathrm{t}} \in \mathbb{Z}^{m+n+1}$. If $\boldsymbol{u} \in \mathbb{N}^{m+n+1}$, then the multinomial theorem [9, p. 168] implies that

$$[\![\tilde{f}^i]\!] = \sum_{\substack{|\boldsymbol{u}|=i \\ \boldsymbol{w}\cdot\boldsymbol{u}=0}} \binom{i}{\boldsymbol{u}} \boldsymbol{x}^{\boldsymbol{u}} = \sum_{\substack{|\boldsymbol{u}|=i \\ \boldsymbol{w}\cdot\boldsymbol{u}=0}} \binom{i}{u_1, \ldots, u_{m+n+1}} x_{-m}^{u_1} x_{-m+1}^{u_2} \cdots x_n^{u_{m+n+1}}.$$

Hence, for all positive integers i, the polynomial $[\![\tilde{f}^i]\!]$ is homogeneous of degree $\left[{i \atop 0} \right]$ with respect to the $\mathbb{Z}^2$-grading of S induced by setting $\deg(x_j) := \left[{1 \atop j} \right] \in \mathbb{Z}^2$ for all $-m \leqslant j \leqslant n$. In particular, J is invariant under the automorphism of S determined by the map $\tilde{f}(z) \mapsto \lambda \tilde{f}(\xi z)$ where $\lambda, \xi \in \mathbb{C}^*$. Moreover, if x_{-m} and x_n are both nonzero, then there exist scalars $\lambda, \xi \in \mathbb{C}^*$ such that the image of $\tilde{f}$ under this $(\mathbb{C}^*)^2$-action has the form (1).

2.2. Toric variety

When $m+n>2$, let $X(m,n)$ be the toric variety with total coordinate ring S (a.k.a. the Cox ring) and irrelevant ideal B; see [5, §2]. The variety $X(m,n)$ provides a toric compactification for the space of all Laurent polynomials of the form (1). When no confusion is likely, we simply write X in place of $X(m,n)$. Proposition 2.4 in [5] shows that homogeneous S-ideals (up to B-torsion) correspond to closed subschemes of X. Hence, the ideal J determines a closed subscheme $\mathrm{V}_X(J)$ of X. If $x_{-m}x_n$ is a nonzerodivisor on $\mathrm{V}_X(J)$, then Section 2.1 shows that the degree of the ideal $I_{m,n}$ equals the degree of $\mathrm{V}_X(J)$. We prove Theorem 2 by computing the latter degree.

More concretely, X is the toric variety associated to the following strongly convex rational polyhedral fan Σ; see [7, §1.4]. The lattice of one-parameter subgroups is $N=\mathbb{Z}^{m+n-1}$ and the rays (i.e. one-dimensional cones) in the fan Σ are generated by the columns of the matrix:

$$\begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 2 & -3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m+n-1 & -m-n & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{2}$$

With the column ordering, we label the rays in Σ by $\rho_{-m},\dots,\rho_n$. For integers $1\leqslant i\leqslant m$ and $0\leqslant j\leqslant n$, let $\sigma_{i,j}$ be the cone in $\mathbb{R}^{m+n-1}=N\otimes_{\mathbb{Z}}\mathbb{R}$ spanned by all the rays except ρ_{-i} and ρ_j. The fan Σ is defined by taking these $\sigma_{i,j}$ as the maximal cones. By construction, X is a singular simplicial projective toric variety of dimension $m+n-1$.

2.3. Intersection theory

Since X is a simplicial toric variety, its rational Chow ring $A^*(X)_{\mathbb{Q}}$ has an explicit presentation; see [7, §5.2]. Specifically, if D_j is the torus-invariant Weil divisor associated to the ray ρ_j for all $-m\leqslant j\leqslant n$, then we have

$$A^*(X)_{\mathbb{Q}}=\frac{\mathbb{Q}[D_{-m},\dots,D_n]}{M+L}$$

where the monomial ideal $M:=\langle D_{-m}D_{-m+1}\cdots D_{-1}, D_0D_1\cdots D_{n-1}D_n\rangle$ is the Alexander dual of B, and the linear ideal

$$L:=\left\langle iD_{-m}-(i+1)D_{-m+1}+D_{-m+i+1}\colon\ 1\leqslant i\leqslant m+n-1\right\rangle$$

encodes the rows of the matrix (2).

Choosing a shelling for the fan Σ yields a distinguished basis for $A^*(X)_{\mathbb{Q}}$; again see [7, §5.2]. With this in mind, we order the maximal cones of Σ by $\sigma_{i,j}>\sigma_{k,\ell}$ if $i+j>k+\ell$ or $i+j=k+\ell$ and $j>\ell$. Let $\tau_{i,j}$ be the subcone of $\sigma_{i,j}$ obtained by intersecting the maximal cone $\sigma_{i,j}$ with all cones $\sigma_{k,\ell}$ satisfying $\sigma_{k,\ell}>\sigma_{i,j}$ and $\dim\sigma_{i,j}\cap\sigma_{k,\ell}=m+n-2$. We obtain a shelling for Σ (i.e. condition $(*)$ in [7, p. 101] is satisfied) because $\dim\sigma_{i,j}\cap\sigma_{k,\ell}=m+n-2$ if and only if $i=k$ and $j\neq\ell$ or $i\neq k$ and $j=\ell$, so $\tau_{i,j}=\sigma_{i,j}\cap(\bigcap_{k>i}\sigma_{k,j})\cap(\bigcap_{\ell>j}\sigma_{i,\ell})$. Hence, the collection $\{[\mathrm{V}(\tau_{i,j})]\}$ forms a basis for $A^*(X)_{\mathbb{Q}}$.

Set $D_{(-i,j)}:=D_{-i+1}\cdots D_{-1}\cdot D_0\cdots D_{j-1}$; the empty product $D_{(-1,0)}=1$ is the unit in $A^*(X)_{\mathbb{Q}}$. The generators of M imply that $D_{(-i,j)}=0$ in $A^*(X)_{\mathbb{Q}}$ if $i>m$ or $j>n$. Since $\tau_{i,j}$ is spanned by the rays ρ_ℓ with $-i<\ell<j$, it follows that $[\mathrm{V}(\tau_{i,j})]=D_{(-i,j)}$. Thus, $D_{(-i,j)}$ for $1\leqslant i\leqslant m$ and $0\leqslant j\leqslant n$ forms a basis for $A^*(X)_{\mathbb{Q}}$. The degree of a zero-dimensional subscheme Y of X, denoted $\deg(Y)$, is the rational number such that $[Y]=\deg(Y)D_{(-m,n)}$ in $A^{m+n-1}(X)_{\mathbb{Q}}$.

The following calculation is the key to proving Theorem 2.

Lemma 3. *For $1\leqslant k\leqslant m+n-1$, we have*

$$k!\,D_0^k=\sum_{i=1}^{k}\left\langle {k \atop i-1}\right\rangle D_{(-i,k-i+1)}\quad \text{in } A^*(X)_{\mathbb{Q}}.$$

Proof. In the polynomial ring $\mathbb{Q}[z]$, Worpitzky's identity is $z^k = \sum_i \left\langle {k \atop i} \right\rangle \binom{z+i}{k}$; see Eq. (6.37) in [9, p. 255] or for a combinatorial proof see [4, §7]. Rearranging, reindexing, and homogenizing this identity give the equation

$$k! z^k = \sum_{i=1}^{k} \left\langle {k \atop i-1} \right\rangle \big(z + (i-1)y\big)\big(z + (i-2)y\big) \cdots \big(z + (i-k)y\big)$$

in the $\mathbb{Z}$-graded polynomial ring $\mathbb{Q}[z, y]$ with $\deg(z) = \deg(y) = 1$. Under the substitution $z \mapsto D_0$ and $y \mapsto D_1 - D_0$, we obtain the equation

$$k!\, D_0^k = \sum_{i=1}^{k} \left\langle {k \atop i-1} \right\rangle \big(\big(1-(i-1)\big)D_0 - (i-1)D_1\big) \cdots \big(\big(1-(i-k)\big)D_0 - (i-k)D_1\big)$$

in $A^*(X)_{\mathbb{Q}}$. To complete the proof, we observe that the ideal L contains the linear relation $D_i = (1-i)D_0 - iD_1$ for all $-m \leqslant i \leqslant n$. □

Using this lemma, we can compute the degree of certain complete intersections in X.

Proposition 4. *Let $g_1, \ldots, g_{m+n-1}$ be homogeneous elements of S such that $\deg(g_j) = \left[{j \atop 0} \right]$ for $1 \leqslant j \leqslant m+n-1$. If $\mathrm{V}_X(g_1, \ldots, g_{m+n-1})$ is a zero-dimensional subscheme of X, then its degree is $\left\langle {m+n-1 \atop m-1} \right\rangle$.*

Proof. Each homogeneous polynomial g_j defines a hypersurface in X. This Cartier divisor is rationally equivalent to $j D_0$ because we have $\deg(g_j) = \left[{j \atop 0} \right]$ for $1 \leqslant j \leqslant m+n-1$. The subscheme $Z := \mathrm{V}_X(g_1, \ldots, g_{m+n-1})$ has dimension zero if and only if it is a complete intersection. Hence, the degree of Z equals the appropriate intersection number, namely the coefficient of $D_{(-m,n)}$ in $\prod_{j=1}^{m+n-1} j D_0$; see Proposition 7.1 in [8]. Since $D_{(-i,k-i+1)} = 0$ for $i > m$ or $k - i + 1 > n$, Lemma 3 yields

$$\prod_{j=1}^{m+n-1} j D_0 = (m+n-1)!\, D_0^{m+n-1} = \left\langle {m+n-1 \atop m-1} \right\rangle D_{(-m,n)}. \qquad \square$$

Proof of Theorem 2. Applying Conjecture 1 for the pairs of positive integers $(m, n-1)$ and $(m-1, n)$, we see that $\mathrm{V}_X(J) \cap \mathrm{V}_X(x_{-m}x_n) = \emptyset$. It follows that $[\mathrm{V}_X(J)]$ belongs to the socle of $A^*(X)_{\mathbb{Q}}$ and thus $\mathrm{V}_X(J)$ has dimension zero. Since $x_{-m}x_n$ is a nonzerodivisor on $\mathrm{V}_X(J)$, we see that $\deg(I_{m,n})$ equals $\deg \mathrm{V}_X(J)$; see Section 2.2. Therefore, applying Proposition 4 completes the proof. □

3. Sparse Laurent polynomials

In this section, we compute the degree of subschemes of $X(m,n)$ corresponding to certain sparse Laurent polynomials. Given the recurrence relation that these degrees satisfy, they may be regarded as a generalized form of Eulerian numbers. This computation also generates a decomposition of $\left\langle {m+n-1 \atop m-1} \right\rangle$ as a sum of nonnegative integers; see (3).

Fix a pair of positive integers (m,n) and let d be a positive integer dividing $m+n$. Consider the closed subscheme X_d of X corresponding to Laurent polynomials of the form

$$x_{-m}\, z^{-m} + x_{-m+d}\, z^{-m+d} + \cdots + x_{n-d}\, z^{n-d} + x_n\, z^n.$$

In other words, X_d is the subscheme of X defined by the monomial ideal generated by the variables not belonging to $\{x_{-m}, x_{-m+d}, \ldots, x_{n-d}, x_n\}$. When $d = 1$, we have $X_d = X$.

For $1 \leqslant j \leqslant m+n-1$, let g_j be a generic polynomial in S of degree $\left[{j \atop 0} \right]$. These generic polynomials cut out the subscheme $Z := \mathrm{V}_X(g_1, \ldots, g_{m+n-1})$. Consider $Z_d := Z \cap X_d$. To compute the degree of Z_d, we introduce the following notation. If $0 \leqslant \ell \leqslant d-1$, then we define

$$\left\langle {d-1 \atop \ell} \right\rangle_d := \begin{cases} 0 & \text{if } \gcd(\ell+1,d) \neq 1, \\ 1 & \text{if } \gcd(\ell+1,d) = 1, \end{cases}$$

and we extend the definition of $\langle {k \atop \ell} \rangle_d$ for all triples (k,ℓ,d) such that d divides $k+1$ via

$$\left\langle {k \atop \ell} \right\rangle_d := (\ell+1)\left\langle {k-d \atop \ell} \right\rangle_d + (k-\ell)\left\langle {k-d \atop \ell-d} \right\rangle_d.$$

It follows that $\langle {k \atop \ell} \rangle = \langle {k \atop \ell} \rangle_1$.

Proposition 5. *The scheme Z_d has dimension zero and degree $\langle {m+n-1 \atop m-1} \rangle_d$ when $\gcd(d,n) = 1$; otherwise the scheme Z_d is empty.*

Before proving this proposition, we record a technical lemma. Let W_i be the vector space of all polynomials in S of degree $\left[{i \atop 0} \right]$ with support contained in $\{x_{-m}, x_{-m+d}, \dots, x_{n-d}, x_n\}$. Given a subset $\mathcal{S} \subseteq \{d, 2d, \dots, m+n-d\}$, let $D(\mathcal{S})$ be the subscheme of X_d defined by the ideal generated by W_i for all $i \in \mathcal{S}$.

Lemma 6. *If $\mathcal{S} \subseteq \{d, 2d, \dots, m+n-d\}$, then* $\dim D(\mathcal{S}) \leqslant \frac{m+n}{d} - 1 - |\mathcal{S}|$.

Proof. It suffices to show that $D(\mathcal{S})$ is contained in a finite union of subschemes with dimension $\frac{m+n}{d} - 1 - |\mathcal{S}|$. To a point $P = [p_{-m} : p_{-m+d} : \dots : p_n]$ in the subscheme $D(\mathcal{S})$, we associate the support sets $\mathcal{E}_+ := \{i \geqslant 0 \mid p_i \neq 0\}$ and $\mathcal{E}_- := \{i > 0 \mid p_{-i} \neq 0\}$. From the definition of X_d, we deduce that $\mathcal{E}_+ \subseteq \{m, m-d, \dots\}$ and $\mathcal{E}_- \subseteq \{n, n-d, \dots\}$. Observe that P lies in the subspace defined by the ideal $\langle x_i \mid i \in \{-m, -m+d, \dots, n\} \setminus (\mathcal{E}_+ \cup \mathcal{E}_-)\rangle$ and that this subspace has dimension $|\mathcal{E}_+| + |\mathcal{E}_-| - 2$. Hence, it is enough to prove $|\mathcal{E}_+| + |\mathcal{E}_-| - 2 \leqslant \frac{m+n}{d} - 1 - |\mathcal{S}| = |\mathcal{S}^{\complement}|$ where $\mathcal{S}^{\complement} := \{d, 2d, \dots, m+n-d\} \setminus \mathcal{S}$. To accomplish this, we consider the set

$$\mathcal{P} := \{i + j \mid i \in \mathcal{E}_+,\ j \in \mathcal{E}_-,\ \text{and } i + j \leqslant m+n-d\} \subseteq \{d, 2d, \dots, m+n-d\}.$$

To conclude, one verifies that $\mathcal{P} \subseteq \mathcal{S}^{\complement}$ and that $|\mathcal{E}_+| + |\mathcal{E}_-| - 2 \leqslant |\mathcal{P}|$. $\square$

Sketch of the proof for Proposition 5. To begin, we assume that $\gcd(d,n) = 1$. Let $\mathbb{P}(W) := \mathbb{P}(W_d) \times \mathbb{P}(W_{2d}) \times \dots \times \mathbb{P}(W_{m+n-d})$ and consider the incidence variety

$$U := \{(P, (h_d, \dots, h_{m+n-d})) \mid h_d(P) = \dots = h_{m+n-d}(P) = 0\} \subseteq X_d \times \mathbb{P}(W)$$

with canonical projection maps $\pi_1 : U \to X_d$ and $\pi_2 : U \to \mathbb{P}(W)$. We claim that $\dim U \leqslant \dim \mathbb{P}(W)$. To see this, observe that a general point Q in X_d does not belong to the base locus of any W_i, so the fiber $\pi_1^{-1}(Q)$ has dimension $\dim \mathbb{P}(W) - \frac{m+n}{d} + 1$. One must also consider the dimensions of the various $\pi_1^{-1}(D(\mathcal{S}))$, but Lemma 6 shows that none of these preimages has dimension greater than $\dim \mathbb{P}(W)$. Since Z_d equals the fiber of π_2 over a general point of $\mathbb{P}(W)$, the inequality $\dim U \leqslant \dim \mathbb{P}(W)$ implies that Z_d has dimension zero. The appropriate modifications to the proofs of Lemma 3 and Proposition 4 show that the degree of Z_d is $\langle {m+n-1 \atop m-1} \rangle_d$.

Assume that $e := \gcd(d,n) > 1$. If $m' := m/e$, $n' := n/e$, and $d' := d/e$, then there is an isomorphism $X_d = X(m,n)_d \xrightarrow{\cong} X(m',n')_{d'} = X'_{d'}$. Under this identification, Z_d is determined by the ideal $\langle g_{d'}, g_{2d'}, \dots, g_{e(n'+m')-d'} \rangle$. Let U' be the incidence variety for the parameters (m', n', d'). From the proof of Lemma 6, we deduce that $x_{-m'}x_{n'}$ is a nonzerodivisor on the top dimensional components of U'. Hence, the generic polynomial $g_{m'+n'}$ is also a nonzerodivisor on U', so the intersection of the general fibre of $\pi'_2 : U' \to \mathbb{P}(W)$ with the hypersurface defined by $g_{m'+n'}$ is empty. Therefore, we have $Z_d = \emptyset$. $\square$

To obtain a decomposition for the Eulerian numbers, we stratify the generic complete intersection Z by singularity type. Let X_d° be the open subscheme of X_d consisting of all singularities of type

$B(\mathbb{Z}/d\mathbb{Z})$ in X. Each point in Z belongs to X_d for some d that divides $m+n$. Setting $Z_d^\circ := Z \cap X_d^\circ$, we obtain

$$\left\langle \begin{matrix} m+n-1 \\ m-1 \end{matrix} \right\rangle = \deg(Z) = \sum_{d|m+n} \deg\bigl(Z_d^\circ\bigr). \tag{3}$$

Moreover, Möbius inversion and Proposition 5 yield

$$\deg\bigl(Z_d^\circ\bigr) = \sum_{c|(m+n)/d} \mu(c) \left\langle \begin{matrix} m+n-1 \\ m-1 \end{matrix} \right\rangle_{cd},$$

where μ is the classical Möbius function; see Eqs. (4.55) and (4.56) in [9, p. 136].

Eq. (3) has an elegant combinatorial refinement which we learnt from Alexander Postnikov; cf. [12, §6]. To sketch this refinement, we observe that the Eulerian number $\left\langle {n \atop k} \right\rangle$ also counts the circular permutations of $\{0, \dots, n\}$ with $k+1$ circular ascents. The group $\mathbb{Z}/(n+1)\mathbb{Z}$ naturally acts on this subset of circular permutations; add 1 modulo $n+1$ to each element. The cardinalities of the orbits then give rise to (3). More precisely, $\deg(Z_d^\circ)$ equals the product of $(m+n)/d$ and the number of orbits with cardinality $(m+n)/d$. For example, if $m=2$ and $n=3$, then we have $\left\langle {4 \atop 1} \right\rangle = 11$, $\deg(Z_5^\circ) = 1$, and $\deg(Z_1^\circ) = 10 = 2 \cdot 5$. On the other hand, the eleven circular permutations of $\{0, \dots, 4\}$ with two circular ascents are partitioned into three $\mathbb{Z}/5\mathbb{Z}$-orbits, namely $\{0\,3\,2\,4\,1\}$, $\{0\,1\,4\,3\,2, 0\,4\,3\,1\,2, 0\,4\,2\,3\,1, 0\,3\,4\,2\,1, 0\,3\,2\,1\,4\}$, and $\{0\,2\,1\,4\,3, 0\,4\,1\,3\,2, 0\,2\,4\,3\,1, 0\,4\,2\,1\,3, 0\,3\,2\,4\,1\}$.

4. Further questions

4.1. Regular sequence

Theorem 2 underscores the significance of Conjecture 1. To prove this conjecture, it would be enough to show that $\mathrm{V}_X([\![\tilde{f}^1]\!], \dots, [\![\tilde{f}^{m+n}]\!])$ is the empty set. From this perspective, the proof of Proposition 5 could be viewed as evidence supporting this conjecture: for generic elements g_j of S with degree $\left[{j \atop 0} \right]$, the subscheme $\mathrm{V}_X(g_1, \dots, g_{m+n})$ is indeed empty.

On the other hand, Conjecture 1 is false over a field with positive characteristic. For instance, if $f := z^{-1} + z \in \mathbb{F}_2[z, z^{-1}]$, then we have $[\![f^i]\!] = 0$ for all i. Even if Conjecture 1 holds, the $\mathbb{F}_p$-vector space $\mathbb{F}_p[x_{-m+1}, \dots, x_{n-1}]/I_{m,n}$ may fail to have a finite dimension; this happens when $p=2$, $m=1$, and $n=2$.

4.2. Combinatorics

The positivity and simplicity of many formulae in this article suggest that we have uncovered only part of the combinatorial structure. To help orient the search for further structure, we pose two specific questions:

- Can one find an explicit basis for $\mathbb{C}[x_{-m+1}, \dots, x_{n-1}]/I_{m,n}$ together with a bijection to the permutations of $[m+n-1]$ with exactly $m-1$ ascents?
- Does $\sum_{j \geqslant 0} \dim_{\mathbb{C}}\bigl(\frac{S}{\langle g_1, \dots, g_{m+n}\rangle}\bigr)_{\left[{j \atop 0} \right]} = \left\langle {m+n-1 \atop m-1} \right\rangle$ hold for all positive m and n? When $m=3$ and $n=3$, we have

$$\left\langle \begin{matrix} 5 \\ 2 \end{matrix} \right\rangle = 66 = 1+0+2+3+6+7+9+10+9+7+6+3+2+0+1.$$

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