Queen's University - Math 844 **Fall 2022**

Problem Set #2 Posted: 7/10/2022

Due: Tuesday, 20/10/2022 (in class)

1. In this problem, we prove the existence of a C^∞ partition of unity subordinated to any locally finite open cover on a smooth manifold. Let then M be a $\mathbf{C}^\infty$ $n-$manifold, and let $(U_i)_{i\in\mathbb{N}}$ a locally finite open cover of M. In all that follows, $B(p;r)$ denotes the open ball of center $p \in \mathbb{R}^n$ and radius $r > 0$ in $\mathbb{R}^n$.
   (a) Let $a, b \in \mathbb{R}$, with $a < b$. We define $h_{(a,b)} : \mathbb{R} \to \mathbb{R}$ by $h_{(a,b)}(t) = \exp(-\frac{1}{(t-b)^2} - \frac{1}{(t-a)^2})$ for $a < t < b$, and $h_{(a,b)}(t) = 0$ otherwise. Note that $h_{(a,b)}(t) \geq 0$, $\forall t \in \mathbb{R}$, and $h_{(a,b)}(t) > 0$, $\forall t \in]a, b[$. Show that $h_{(a,b)}$ is C^∞ on $\mathbb{R}$.
   (b) Let $\eta_{(a,b)} : \mathbb{R} \to \mathbb{R}$ be defined by $\eta_{(a,b)}(t) = \frac{\int_{-\infty}^{t} h_{(a,b)}(u)du}{\int_{-\infty}^{\infty} h_{(a,b)}(u)du}$. Show that $\eta_{(a,b)}$ is C^∞ on $\mathbb{R}$, $\eta_{(a,b)}(t) = 0$ for $t \leq a$, $\eta_{(a,b)}(t) = 1$ for $t \geq b$, $\eta_{(a,b)}(t) \in]0, 1[$ $\forall t \in]a, b[$, and $\eta_{(a,b)}$ is strictly monotonically increasing on $]a, b[$.
   (c) Let now $K \subset \Omega \subset \mathbb{R}^n$, with K compact and Ω open. Show that there exists a finite open cover $(B_{(p_i,a_i,b_i)})_{i=1}^{N}$ of K by open sets such that:
      - $0 < a_i < b_i, \quad \forall i,$
      - $B_{(p_i,a_i,b_i)} = B(p_i; a_i), \quad \forall i,$
      - $\overline{B(p_i; b_i)} \subset \Omega, \quad \forall i.$
   (d) Continuing with the above, define $g_i : \mathbb{R} \to \mathbb{R}$ for each $i \in \{1, \cdots, N\}$ by $g_i(x) = \eta_{(a_i,b_i)}(\|x - p_i\|)$, and define $g : \mathbb{R} \to \mathbb{R}$ by $g = 1 - \Pi_{i=1}^{N} g_{p_i,a_i,b_i}$. Show that g is C^∞ on $\mathbb{R}^n$, $g = 1$ on K, and $\text{supp}(g) \subset \Omega$.
   (e) Let now $K \subset \Omega \subset M$, with K compact and Ω open in M. Show, using (d), that there exists $f : M \to \mathbb{R}$ C^∞ such that $f = 1$ on K, and $\text{supp}(f) \subset \Omega$.
   (f) Construct a locally finite open cover $(V_j)_j$ of M with each V_j relatively compact, such that $\forall j$, $\exists i$ such that $\overline{V}_j \subset U_i$.
   (g) Let now,

$$
\begin{aligned}
J_0 &= \{j \in \mathbb{N} \mid \overline{V}_j \subset U_0\},\\
J_1 &= \{j \in \mathbb{N} \setminus J_0 \mid \overline{V}_j \subset U_1\},\\
&\vdots\\
J_k &= \{j \in \mathbb{N} \setminus (J_0 \cup J_1 \cup \cdots \cup J_{k-1}) \mid \overline{V}_j \subset U_k\},\\
&\vdots
\end{aligned}
$$

   Show that $\bigcup_{j=1}^{\infty} J_j = \mathbb{N}$.
   (h) $\forall j \in \mathbb{N}$, let $f_j : M \to \mathbb{R}$ C^∞ with $f_j = 1$ on $\overline{V}_j$ and $\text{supp}(f_j) \subset U_i$ (the existence of which was proved in (e)), where i is the unique integer for which $\overline{V}_j \subset U_i$. Define $\forall i \in \mathbb{N}$:

$$\mu_i = \frac{\sum_{j\in J_i} f_j}{\sum_i \sum_{j\in J_i} f_j}$$

   Show that $(\mu_i)_{i\in\mathbb{N}}$ is a C^∞ partition of unity on M subordinated to the open cover $(U_i)_{i\in\mathbb{N}}$ of M.
2. Let M be a C^∞ manifold of dimension m, and let (U, ϕ) be a local chart around $p \in M$. Let V be an open subset of $\phi(U) \subset \mathbb{R}^m$, and let h be a diffeomorphism of V onto some open subset of $\mathbb{R}^m$. Show that $(\phi^{-1}(V), h \circ \phi))$ is a chart for the C^∞ structure of M.
3. Find $f : \mathbb{R} \to \mathbb{R}^2$ C^∞ such that $f(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 | \sup(|x|, |y|) = 1\}$. Can f be an immersion ?
4. Let $n \in \mathbb{N}^\star$, and let $f : S^n \to \mathbb{R}^n$ be a C^∞ map. Show that f can be neither an immersion nor a submersion.
5. Let $p \in \mathbb{N}^\star$; construct a local diffeomorphism $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ such that the pre-image of any point in $\mathbb{R}^2 \setminus \{0\}$ has p distinct points (Hint: Consider first a suitable holomorphic function $\mathbb{C}^\star \to \mathbb{C}^\star$ ...).
6. Find a C^∞ mapping $f : M \to N$ such that:
   (a) f is injective but not an immersion.
   (b) f is an immersion but not injective.
   (c) f is surjective but not a submersion.
   (d) f is a submersion but not surjective.