Queen's University - Math 844 **Fall 2022**

Problem Set #3

Posted: Wednesday, 26/10/2022

Due: Tuesday, 1/11/2022

1. Let X be a set, and let $(X_\alpha)_{\alpha\in\Lambda}$ be a family of subsets of X with $X = \cup_\alpha X_\alpha$; assume $\forall\alpha \in \Lambda$ a topology τ_α is defined on X_α. Let τ be the set of all subsets $V \subset X$ for which $V \cap X_\alpha \in \tau_\alpha$ $\forall\alpha$.

   (i) Show that τ is a topology on X.

   (ii) Give a necessary and sufficient condition for X_α to be open in (X, τ) $\forall\alpha$.

   (iii) Let M be a C^∞ $n-$manifold, and let $\{(U_\alpha, \phi_\alpha)\}$ be a C^∞ atlas on M; For each α, we endowed $\coprod_{q\in U_\alpha} T_qM$ with the unique topology which made the bijection $\Phi_\alpha : \coprod_{q\in U_\alpha} T_qM \to U_\alpha \times \mathbb{R}^n$ a homeomorphism, and we then endowed $TM = \coprod_{q\in M} T_qM$ with the topology induced from the topologies in the $\coprod_{q\in U_\alpha} T_qM$, as in (i) above. Show that in this construction, the condition obtained in (ii) is satisfied, and hence each subset $\coprod_{q\in U_\alpha} T_qM$ is open in TM.

2. Let X, Y, Z be C^∞ manifolds; assume X is a submanifold of Y and Y a submanifold of Z. Show then that X is a submanifold of Z.

3. Let $n \in \mathbb{N}^\star$, and consider S^n, the unit sphere of $\mathbb{R}^{n+1}$, with its usual topology (i.e. the one induced from $\mathbb{R}^{n+1}$ by inclusion). We constructed two C^∞ structures on S^n: (i) using stereographic projections, and (ii) defining S^n as the preimage of 1 under the constant rank map $\mathbf{x} \mapsto \|\mathbf{x}\|$, which immediately established S^n as a submanifold of $\mathbb{R}^{n+1} \setminus \{0\}$, and hence, of $\mathbb{R}^{n+1}$ (as follows from Problem (2) above). Prove that these two C^∞ structures are identical.

4. A Lie group G is a group $(G, \cdot)$ together with a C^∞ manifold structure such that the group "multiplication" $m : G \times G \to G,\ (x, y) \mapsto m(x, y) = x \cdot y$ and the group inversion operation $i : G \to G,\ x \mapsto i(x) = x^{-1}$ are C^∞ maps (i.e. the group structure is "compatible" with the C^∞ structure).

   $\forall a \in G$, we define the maps $L_a : G \to G,\ x \mapsto L_a(x) = a \cdot x$ ("left-translation by a") and $R_a : G \to G,\ x \mapsto R_a(x) = x \cdot a$ ("right-translation by a").

   If G_1, G_2 are two Lie groups, we say a mapping $u : G_1 \to G_2$ is a Lie group homomorphism if it is both a group homomorphism and C^∞.

   (a) Let H be a subgroup of G such that H is also a submanifold of G. Show that H is then a Lie group for the group structure and manifold structure induced by G (H is called a "Lie subgroup" of G).

   (b) Show that $\forall x, y \in G$, $T_yL_x : T_yG \to T_{xy}G$ is a vector space isomorphism.

   (c) Let G_1, G_2 be Lie groups. Deduce from (b) that every Lie group homomorphism $u : G_1 \to G_2$ has constant rank.

   (d) Deduce from (c) (with the same notation) that $\ker(u)$ is a Lie subgroup of G_1.

   (e) Show that $\forall n \geq 1$, $GL(n, \mathbb{R})$, with the usual group structure (the group operation being given by matrix multiplication) and the canonical C^∞ manifold structure (as an open subset of the vector space of real $n \times n$ matrices) is a Lie group.

   (f) Show that $SL(n, \mathbb{R})$, defined as the set of all matrices in $GL(n, \mathbb{R})$ with determinant 1, is a Lie subgroup of $GL(n, \mathbb{R})$.

   (g) Let e denote the identity element of $GL(n, \mathbb{R})$ (i.e. the $n \times n$ identity matrix). Show that $T_eGL(n, \mathbb{R})$ can be canonically identified with the vector space $M_n(\mathbb{R})$ of real $n \times n$ matrices.