Queen's University - Math 844 **Fall 2022**

Problem Set #5

Posted: Tuesday, 22/11/2022

Due: Tuesday, 29/11/2022

1. Let M be a C^∞ $n-$manifold, and let $\omega_1, \cdots, \omega_k \in \Omega^1(M)$. Assume that $\forall p \in M$, $(\omega_1(p), \cdots, \omega_k(p))$ is a linearly independent family of $T_p^\star M$. For each $p \in M$, define the subspace Δ_p of T_pM by:

$$\Delta_p = \cap_{i=1}^k \ker(\omega_i(p)).$$

   (a) Show that the assignment $p \mapsto \Delta_p$ defines a C^∞ distribution Δ of rank $n-k$ on M.

   (b) Show that: Δ involutive $\Leftrightarrow d\omega^i \equiv 0 \mod \omega^1, \cdots, \omega^k$, $\forall i \in \{1, \cdots, k\}$ (i.e. for each i, for each $p \in M$, there exist C^∞ 1-forms $\eta_1^i, \cdots, \eta_k^i$ defined in some open neighborhood U of p such that $d\omega^i = \sum_{j=1}^k \eta_j^i \wedge \omega^j$ in U.)

2. Let M be a C^∞ manifold and let $\omega \in \Omega^k(M)$ $(k \geq 1)$. We wish to show that $d\omega : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to C^\infty(M)$ is alternating and $C^\infty(M)-$multilinear (and hence defines an element of $\Omega^{k+1}(M)$). Recall that $\mathcal{X}(M)$ denotes the $C^\infty(M)-$module of C^∞ vector fields on M, and that for any C^∞ vector fields $X_1, X_2, \ldots, X_{k+1}$ on M, we have:

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \\ &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}). \end{aligned}$$

   Clearly $d\omega$ is $\mathbb{R}-$multilinear.

   (a) Show that for any $i \in \{2, 3, \cdots, k+1\}$, we have:

$$d\omega(X_1, X_2, \ldots, X_k) = -d\omega(X_i, X_2, \ldots, X_{i-1}, X_1, X_{i+1}, \ldots, X_{k+1}).$$

   (b) Deduce from (a) that $d\omega$ is alternating.

   (c) Let $f \in C^\infty(M)$; show that

$$d\omega(fX_1, X_2, \ldots, X_{k+1}) = f d\omega(X_1, X_2, \ldots, X_{k+1}).$$

   (d) Deduce from (b) and (c) that $d\omega : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to C^\infty(M)$ is $C^\infty(M)-$multilinear.

3. (Poincaré's lemma) Let $U \subset \mathbb{R}^n$ be open and star-shaped with respect to 0, and let $p \in \mathbb{N}^\star$. Consider the $\mathbb{R}-$linear map $K : \Omega^p(U) \to \Omega^{p-1}(U)$, defined on a C^∞ monomial $p-$form $a dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ by

$$K(a dx^{i_1} \wedge \cdots \wedge dx^{i_p})|_{\mathbf{x}} = \left(\int_0^1 a(t\mathbf{x}) t^{p-1} dt \right) \sum_{j=1}^p (-1)^{j+1} x^{i_j} dx_{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_p},$$

   which we extend by $\mathbb{R}-$linearity to $\Omega^p(U)$. Show that

$$d \circ K(\omega) + K \circ d(\omega) = \omega, \quad \forall \omega \in \Omega^p(U)$$

4. Let G be a Lie group, and let $\mathfrak{g} = T_eG$ be its Lie algebra. For each $g \in G$, define the mapping $\omega(g) : T_gG \to \mathfrak{g}$ by $\omega(g)(v) = T_gL_{g^{-1}}(v)$. We define the mapping $\omega : \mathcal{X}(G) \to C^\infty(G; \mathfrak{g})$ as follows: $\omega(X)(g) = \omega(g)(X(g))$, $\forall g \in G$.

   (a) Show that $\omega : \mathcal{X}(G) \to C^\infty(G; \mathfrak{g})$ is $C^\infty(G)-$linear, and hence defines a C^∞ $\mathfrak{g}-$valued 1-form on G. ω is called the canonical 1-form of G.

   (b) Show that $\forall g \in G$, $L_g^\star \omega = \omega$; i.e. the canonical $1-$form of G defined above is left-invariant.

   (c) The exterior derivative operator d is extended to $\mathfrak{g}-$valued forms on G as follows: Let $(\mathbf{e}_i)_{i=1}^n$ be a basis for $\mathfrak{g}$, and let η be any $\mathfrak{g}-$valued k form on G. Then there exist uniquely defined smooth differential $k-$forms $\eta^1, \cdots, \eta^n \in \Omega^k(G)$ such that $\eta = \sum_{i=1}^n \eta^i \otimes \mathbf{e}_i$. We then define: $d\eta = \sum_{i=1}^n d\eta^i \otimes \mathbf{e}_i$.
   For any two $\mathfrak{g}-$valued $1-$forms α, β on G, the $\mathfrak{g}-$valued $1-$form $[\alpha, \beta]$ on G is defined by:

$$[\alpha, \beta]|_p(\mathbf{v}, \mathbf{w}) = [\alpha_p(\mathbf{v}), \beta_p(\mathbf{w})] - [\alpha_p(\mathbf{w}), \beta_p(\mathbf{v})],$$

   $\forall p \in G$, $\forall \mathbf{v}, \mathbf{w} \in T_p(G)$.
   Show that the canonical $1-$form ω satisfies:

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

   (Maurer-Cartan structure equation).